A fibrational account of local states

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***Abstract*—One main challenge of the theory of computational effects is to understand how to combine various notions of effects in a meaningful way. Here, we study the particular case of the local state monad, which we would like to express as the result of combining together a family of global state monads parametrized by the number of available registers. To that purpose, we develop a notion of *indexed monad* inspired by the early work by Street, which refines and generalizes Power's recent notion of indexed Lawvere theory. One main achievement of the paper is to integrate the block structure necessary to encode allocation as part of the resulting notion of indexed state monad. We then explain how to recover the local state monad from the functorial data provided by our notion of indexed state monad. This reconstruction is based on the guiding idea that an algebra of the indexed state monad should be defined as a section of a 2-categorical notion of fibration associated to the indexed state monad by a Grothendieck construction.**

***Index Terms*—Computational effects; algebraic theories; Lawvere theories; state monad; local state monad; 2-categories; fibration**

I. Introduction

Despite years of intensive mathematical study, the semantic nature of memory and state in programming languages has not yet revealed all its secrets, and thus remains particularly interesting and important to investigate today — in particular because it combines and interleaves two entirely different aspects:

- on the one hand, each memory register can be read and written using dedicated lookup and update operations available as effects to the programmer,
- on the other hand, memory registers can be allocated and deallocated at any time, depending on the memory management policy, as well as on the needs and desires of the very same programmer.

These two complementary aspects of memory have been thoroughly studied in the semantic literature. The primary focus on allocation and deallocation mechanisms in higher-order imperative languages like Algol have lead pionneer researchers like Oles and Reynolds to promote the idea that types should be interpreted as presheaves over combinatorial categories describing the memory shapes and resources [17], [23]. This simple idea had an impressive posterity in our field, including separation logic [24], [22] and nominal sets [18]. A traditional example of such a combinatorial category of memory shapes related to separation logic and nominal sets is provided by the category Inj with natural numbers as objects, seen as finite cardinals $[n] = \{0, \ldots, n-1\}$, and injections between them.

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In parallel, and in a somewhat independent strand of research, Moggi realized that it was possible to understand the read and write operations of a memory register as a specific monadic effect living on top of a purely functional language [15]. This monadic account of states works as follows. Suppose that n denotes the number of registers allocated in your computer, and that S denotes the set of states possibly assigned to any of these n registers. In this case, one defines a monad (called the state monad)

$$T_n \quad : \quad Set \quad \longrightarrow \quad Set$$

on the category Set of sets and functions, defined as follows:

$$T_n \quad : \quad A \quad \mapsto \quad S^n \Rightarrow (S^n \times A)$$

where each element of S^n is a finite list

$$(s_0, \ldots, s_{n-1}) \quad \in \quad S^n$$

consisting of the states of each allocated register. A typical instance of such a set of states (or values) is provided by $S = V = \{true, false\}$ where each register is thus meant to contain a bit. On some occasions, we find useful to call *mnemoid* (for set with memory) or more precisely n*-mnemoid* an algebra A of the state monad T_n.

a) The local state monad

For many years, these two semantic approaches to memory and states in higher-order programming language remained largely disconnected. The situation drastically changed when Plotkin and Power, inspired by discussions with O'Hearn, exhibited a monad (called the local state monad)

$$T \quad : \quad [Inj, Set] \quad \longrightarrow \quad [Inj, Set]$$

on the category $[Inj, Set]$ of covariant presheaves over the category Inj of finite sets and injections. This monad is defined on a given presheaf A by the slightly intimidating co-end formula

$$TA \ : \ n \ \mapsto \ S^n \ \Rightarrow \ \left(\int^{p \in Inj} S^p \times A_p \times Inj(n,p) \right)$$

whose purpose is to adapt the definition of the traditional state monad T_n on Set to the presheaf category $[Inj, Set]$. In their seminal, Plotkin and Power established in particular that the local state monad T just defined implements at the same time the lookup, update, and allocation operations expected of a state monad on the presheaf category $[Inj, Set]$. This result is technically established by formulating a purely algebraic description of the algebras A of the local state monad T. This paper by Plotkin and Power together with later elaborations [25], [12] characterize such an algebra A as a family of sets $(A_n)_{n\in\mathbb{N}}$ indexed by natural numbers

$$A_0 \qquad A_1 \qquad A_2 \qquad A_3 \qquad A_4 \quad \dots$$

equipped with five families of operations, described as follows in the case when $S = \{true, false\}$. First of all, for each natural number $n \in \mathbb{N}$ and each location $loc \in [n]$, there is a binary lookup operation

$$\texttt{lookup}_{\langle loc\rangle} : A_n \times A_n \longrightarrow A_n$$

which intuitively behaves like a conditional test, and branches on its left operand when the value of the register loc is true, and on its right operand when the value of the register loc is false. There is also an update operation for each value $val \in \{true, false\}$ possibly assigned to the register $loc \in [n]$:

$$\texttt{update}_{\langle loc,val\rangle} : A_n \longrightarrow A_n$$

Then, for each natural number $n \in \mathbb{N}$, for each location $loc \in [n+1]$ and for each value $val \in \{true, false\}$, there is an operation

$$\texttt{fresh}_{\langle loc,val\rangle} \quad : \quad A_{n+1} \quad \longrightarrow \quad A_n$$

whose intuitive purpose is to allocate among n registers a fresh register at location $loc \in [n+1]$ moreover initialized with the value $val \in \{true, false\}$. Then, for each natural number $n \in \mathbb{N}$ and for each location $loc \in [n+1]$, there is an operation

$$\texttt{collect}_{\langle loc\rangle} \quad : \quad A_n \quad \longrightarrow \quad A_{n+1}$$

whose intuitive purpose is to deallocate or garbage collect the register at location $loc \in [n+1]$. Finally, for each natural number $n \in \mathbb{N}$ and for each pair of locations $loc, loc{+}1 \in [n]$, there is an operation

$$\texttt{permute}_{\langle loc,loc+1\rangle} \quad : \quad A_n \quad \longrightarrow \quad A_n$$

whose intuitive purpose is to permute the two registers at location loc and $loc+1$. These five families of operations are moreover regulated by a natural but also pretty long series of equations carefully enumerated and depicted as string diagrams in [12].

b) The three groups of operations

Once this algebraic presentation of the local state monad T has been achieved, a natural question is to understand what it can teach us about the very construction and nature of the local state monad T itself. A preliminary observation is that the five families of operations can be organized in three groups:

1) for a given natural number n, the operations `lookup` and `update` of degree n provide the set A_n with the structure of n-mnemoid — that is, of an algebra of the state monad T_n on n registers introduced earlier,
2) the operations `collect` and `permute` provide together the presheaf structure of the family of sets $(A_n)_{n\in\mathbb{N}}$ on the category Inj of finite sets and injections,
3) the algebraic purpose of the allocation operation `fresh` remains something of a mystery at this stage — this is a serious conceptual concern, since allocation plays a central role in the definition of the local state monad T, and we thus wish to resolve it in the present paper.

The algebraic presentations by Staton [25] and Melliès [12] of the local state monad were to a large extent designed to clarify how these three groups of operations and equations are intertwined in the local state monad. Here, we would like to revisit this analysis starting from a slightly different angle, offered by the elegant and compelling fibrational point of view recently advocated and developed by Power [20].

c) Indexed Lawvere theories

In his work, Power observes that the groups of operations (1) and (2) may be combined by defining a functor

$$\mathcal{T} \quad : \quad Inj \quad \longrightarrow \quad Law$$

from the category Inj to the category Law of Lawvere theories and finite product preserving functors between them. Such a functor $\mathcal{T}$ is called by Power an *indexed Lawvere theory*. The functor $\mathcal{T}$ transports every natural number n to the algebraic theory $\mathcal{T}_n$ associated to the state monad T_n, which is indeed finitary. This construction relies on the fact that every injection $f : m \to n$ induces a morphism

$$\mathcal{T}_f \quad : \quad \mathcal{T}_m \quad \longrightarrow \quad \mathcal{T}_n \tag{1}$$

defined by relabelling along f the register locations $loc \in [m]$ used by the operations

$$\texttt{lookup}_{\langle loc\rangle} \qquad\qquad \texttt{update}_{\langle loc,val\rangle}$$

of degree m into the corresponding operations

$$\texttt{lookup}_{\langle f(loc)\rangle} \qquad\qquad \texttt{update}_{\langle f(loc),val\rangle}$$

of degree n. This establishes that the read and write operations of the local state monad define something which deserves the name of *presheaf of operations.*

d) Models of Indexed Lawvere theories

The indexed Lawvere theory $\mathcal{T}$ induces a notion of *model* defined by Power as a family of models (in the usual sense)

$$A_0 \qquad A_1 \qquad A_2 \qquad A_3 \qquad A_4 \quad \dots$$

of the Lawvere theory $\mathcal{T}_n$ of the corresponding degree n. Note that each such model A_n may be alternatively seen as a n-mnemoid, that is, as an algebra of the state monad T_n. This family of models is moreover required to satisfy a series of coherence conditions recalled in Section II. Although he does not state it exactly in that way, an important observation by Power is that a model $(A_n)_{n\in\mathbb{N}}$ of the indexed Lawvere theory $\mathcal{T}$ in his sense is the same thing as a family of sets $(A_n)_{n\in\mathbb{N}}$ equipped with the operations and equations of the two groups (1) and (2) of the local state monad T. This result establishes that the indexed Lawvere theory $\mathcal{T}$ provides a precise description of the local state monad T as a combination of the state monads T_n, as long as this monad T is restricted to the two groups (1) and (2) of operations and equations. As such, the indexed Lawvere theory $\mathcal{T}$ is not able to capture the allocation mechanisms at work in the local state monad T.

e) Block structure for allocation

In order to fill the gap with the local state monad T, Power introduces what he calls a *block structure* on the models of his indexed Lawvere theory $\mathcal{T}$. A block structure as defined in [20] is a family of homomorphisms

$$\mathtt{block}_n \quad : \quad A_{n+1} \quad \longrightarrow \quad V \Rightarrow A_n$$

between models of the Lawvere theory $\mathcal{T}_{n+1}$. The very definition of block structure relies on the key observation that the exponentiation

$$V \Rightarrow - \quad : \quad A \quad \mapsto \quad V \Rightarrow A$$

by the set V of values possibly assigned to a register, defines a functor between the categories of models

$$V \Rightarrow - \quad : \quad \mathcal{T}_n\text{-}Mod \quad \longrightarrow \quad \mathcal{T}_{n+1}\text{-}Mod$$

of the Lawvere theories $\mathcal{T}_n$ and $\mathcal{T}_{n+1}$. Another equivalent way to look at it, is to view the exponentiation by V as a functor

$$V \Rightarrow - \quad : \quad T_n\text{-}Alg \quad \longrightarrow \quad T_{n+1}\text{-}Alg$$

between the categories of n-mnemoids (or T_n-algebras) and $(n+1)$-mnemoids (or T_{n+1}-algebras). This family of homomorphisms $\mathtt{block}_n$ between $(n+1)$-mnemoids is moreover required to make two coherence diagrams commute, recalled in Section II. Thanks to this notion of block structure, Power establishes the following striking result:

Theorem (Power 2011) The category of algebras of the local state monad T is equivalent to the category of models of the indexed Lawvere theory $\mathcal{T}$ equipped with a block structure.

This result is important conceptually because it pinpoints the exact algebraic structure missing in order to fill the gap between the indexed Lawvere theory $\mathcal{T}$ which does not handle allocation and the local state monad T which does. From that point of view, the notion of block structure appears as some kind of algebraic "glue" necessary (and at the same time sufficient) in order to combine the various state monads T_n and to obtain the local state monad T in the end. However, Power himself recognizes in his paper that the notion of block structure remains somewhat unsatisfactory in the way it is formulated, and that it thus deserves to be further elaborated. This is precisely what we intend to do in the present paper, by investigating the algebraic and conceptual nature of the notion of block structure, using 2-categorical ideas.

f) Indexed monads

A good starting point for our journey is to think of an "indexed Lawvere theory" as an algebraic refinement of the traditional notion of "indexed category". Recall that a $\mathscr{C}$-indexed category is defined as a (contravariant) pseudo-functor

$$F \quad : \quad \mathscr{C}^{op} \quad \longrightarrow \quad Cat$$

from a basis category $\mathscr{C}$ to the category Cat of small categories. There is a well-known correspondence between such $\mathscr{C}$-indexed categories and cloven fibrations over the category $\mathscr{C}$. This correspondence relies on the Grothendieck construction which associates to every $\mathscr{C}$-indexed category a cloven fibration

$$\pi_F \quad : \quad \int F \quad \longrightarrow \quad \mathscr{C}$$

whose fiber $\pi_F^{-1}(c)$ over an object c of the category $\mathscr{C}$ coincides with the category $F(c)$. The ongoing discussion on the local state monad T and the indexed Lawvere theory $\mathcal{T}$ leads us to introduce the notion of *indexed monad*, defined as a (contravariant) pseudo-functor

$$\mathscr{T} \quad : \quad \mathscr{C}^{op} \quad \longrightarrow \quad Mnd$$

from the base category $\mathscr{C}$ to the 2-category Mnd introduced by Street [10] in his celebrated paper on the formal theory of monads. Recall that the objects of the 2-category Mnd are pairs $(\mathscr{E}, T)$ consisting of a category $\mathscr{E}$ together with a monad on it:

$$T \quad : \quad \mathscr{E} \quad \longrightarrow \quad \mathscr{E}.$$

As expected, the usual 2-category Cat of categories, functors and natural transformations defines a sub-2-category of Mnd where a category $\mathscr{E}$ living in Cat is seen in Mnd as the pair $(\mathscr{E}, Id)$ consisting of the category $\mathscr{E}$ equipped with the identity monad Id. An indexed category F is thus a specific case of indexed monad $\mathscr{T}$. At the same time, there exists a functor

$$\iota \quad : \quad Law^{op} \quad \longrightarrow \quad Mnd$$

which transports every Lawvere theory $\mathcal{T}$ to the pair (Set, T) consisting of the category Set together with the finitary monad T associated to the Lawvere theory $\mathcal{T}$. The contravariant nature of this functor ι has to do with the very definition by Street of the 2-category Mnd. From that follows that every indexed Lawvere theory

$$\mathcal{T} \quad : \quad \mathscr{C} \quad \longrightarrow \quad Law$$

in the sense of Power gives rise to a $\mathscr{C}$-indexed monad

$$\mathscr{T} \quad : \quad \mathscr{C}^{op} \quad \longrightarrow \quad Mnd$$

obtained by composition with ι.

g) Block structures integrated as part of indexed monads

The key observation of the present paper is that the block structure introduced by Power may be smoothly integrated in the conceptual framework offered by indexed monads. This is achieved along the following idea. In our alternative account, the indexed Lawvere theory $\mathcal{T}$ formulated by Power is replaced by an indexed monad

$$\mathscr{T} \quad : \quad Inj^{\,op} \quad \longrightarrow \quad Mnd$$

which transports every natural number n to the pair (Set, T_n) consisting of the category Set and the state monad T_n. Similarly, every injection $f : m \to n$ is transported into a morphism

$$\mathscr{T}(f) \quad : \quad (Set, T_n) \quad \longrightarrow \quad (Set, T_m) \tag{2}$$

of the 2-category Mnd. Note in particular the change of orientation, reflecting the contravariant definition of indexed monads. It is worth observing that this morphism (2) in Mnd is entirely characterized by a monad morphism

$$T_f \quad : \quad T_m \quad \longrightarrow \quad T_n$$

which performs the same relabelling along f as explained previously in (1). So, the key novelty with respect to indexed Lawvere theories is that the notion of morphism in the category Mnd is in fact more expressive than the notion of morphism in the category Law. In particular, it is sufficiently expressive to incorporate the functor

$$V \Rightarrow - \quad : \quad T_n\text{-}Alg \quad \longrightarrow \quad T_{n+1}\text{-}Alg \tag{3}$$

as part of a morphism

$$Block \quad : \quad (Set, T_n) \quad \longrightarrow \quad (Set, T_{n+1})$$

of the 2-category Mnd. This observation leads us to the idea of extending the category Inj with a family of morphisms

$$alloc \quad : \quad [n+1] \quad \longrightarrow \quad [n]$$

which would be transported by the indexed monad to the Mnd-morphism $Block$

$$\mathscr{T}(alloc) = Block \quad : \quad (Set, T_n) \quad \longrightarrow \quad (Set, T_{n+1})$$

in the same as the injections f are transported by $\mathscr{T}$ to the Mnd-morphism T_f. Guided by this intuition, we construct an indexed monad

$$\mathscr{T} \quad : \quad \mathcal{P}Inj^{op} \quad \longrightarrow \quad Mnd$$

which extends the indexed monad $\mathscr{T}$ by shifting from the category Inj to the 2-category $\mathcal{P}Inj$ of partial injections as described in Section IV where, as we have just explained, the canonical morphism

$$alloc \quad : \quad [1] \quad \longrightarrow \quad [0]$$

in the category $\mathcal{P}Inj$ is transported to the morphism $Block$ in the 2-category Mnd.

h) Main theorem of the paper

The main theorem of our paper relies on the extension and adaptation to indexed monads of the usual Grothendieck construction for indexed categories. The resulting Grothendieck construction associates to every $\mathscr{C}$-indexed monad

$$\mathscr{T} \quad : \quad c \quad \mapsto \quad (\mathscr{E}_c, T_c) \quad : \quad \mathscr{C}^{\,op} \quad \longrightarrow \quad Mnd$$

a 2-categorical notion of fibration (or 2-fibration) in the sense of Hermida, whose fiber above a given object c coincides with the category T_c-Alg of algebras associated to the monad T_c over the category $\mathscr{E}_c$. We then define an algebra of such an indexed monad $\mathscr{T}$ as a section of this 2-fibration. In the case of the indexed state monad, one recovers in this way a family of sets

$$A_0 \qquad A_1 \qquad A_2 \qquad A_3 \qquad A_4 \qquad \dots$$

where each A_n is an n-mnemoid, that is, an algebra of the state monad T_n on the category $\mathscr{E}_n = Set$. Our main technical contribution in this paper is to establish that the coherence conditions required on this family of mnemoids in order to define an algebra of the local state monad T in the category $[Inj, Set]$, is in fact entirely handled by the 2-fibrational structure associated to a specific indexed state monad $\mathscr{T}$, fully described in the paper. This enables one to characterize the algebras of the local state monad T as the 2-dimensional

Theorem There is a correspondence between the algebras of the local state monad T and the sections of the 2-fibration

$$\pi_{\mathscr{T}} \quad : \quad \int \mathcal{P}Inj \quad \longrightarrow \quad \mathcal{P}Inj.$$

This correspondence defines moreover an equivalence of categories between the two concepts.

The description of an algebra A of the local state monad T as a section of the 2-fibration $\pi_{\mathscr{T}}$ may be reformulated in an even simpler way, by observing that such a section is the same thing as a 2-dimensional cell

$$\begin{tikzcd} \mathcal{P}Inj^{\,op} \arrow[rr, bend left, "\mathscr{T}"] \arrow[rr, bend right, "\mathbb{1}"'] \arrow[rr, phantom, "\Uparrow A"] & & \mathcal{M}nd \end{tikzcd}$$

in the 2-category of 2-categories, 2-functors, and colax natural transformations. Here, $\mathbb{1}$ denotes the constant 2-functor which transports every natural number n to the terminal category. Since $\mathbb{1}$ is also terminal in the category $[\mathcal{P}Inj^{\,op}, Mnd]$ of colax natural transformations, this diagram enables one to see an algebra A of the indexed monad $\mathscr{T}$ as a global element of the indexed monad itself.

Despite the need for climbing one degree of abstraction, and working with a 2-categorical notion of fibration, we believe that this reconstruction of the local state monad is sufficiently general and conceptual to shed light on its true nature. A number of new ideas and techniques emerged in the course of the construction, which we did our best to describe as meticulously as possible in the course of the paper. The paper is organized in five sections. Section II is devoted to an analysis of the block structure introduced by Power. This analysis leads us to introduce a 2-categorical Grothendieck construction on monads in Section III. Then in Section IV we explain how to apply the 2-categorical framework just defined to the particular case of the local state monad T and establish our main theorem. Finally, we relate in Section V our construction to the algebraic presentation of the local state monad T formulated by Melliès [12], before concluding the paper in the next section.

II. Blocks in Power's work

In this section, we revisit Power's work and show how it leads us to investigate indexed monads . We first present in details Power's notion of model of an indexed Lawvere theories. This is then applied to the particular case of the indexed Lawvere theory $\mathcal{T}$ for local state. As explained in the introduction, Power needed to introduce a notion of block-structure algebra on these models in order to recover the traditional local state monad. A meticulous investigation of this block algebra structure exhibits its fibrational nature.

i) Model of an indexed Lawvere theory

A *model* of an indexed Lawvere theory

$$F \quad : \quad \mathscr{C} \quad \longrightarrow \quad Law$$

is given by a family $(M_c)_{c\in\mathscr{C}}$ of models where M_c is a model of $F(c)$. Moreover, this assignement must be functorial, meaning that for any 1-cell $f : c \to c'$ in $\mathscr{C}$ there is a natural transformation M_f making the following diagram commute

$$\begin{tikzcd} F(c) \arrow[rr, "F(f)"] \arrow[dr, "M_c"'] & \arrow[d, phantom, "\xRightarrow{M_f}"] & F(c') \arrow[dl, "M_{c'}"] \\ & Set & \end{tikzcd}$$

and such that

$$M_{id} = id \qquad\qquad M_{f\circ g} = M_f \cdot M_g$$

A more classical presentation of this notion can be given by considering the functor

$$\mathscr{C}^{op} \xrightarrow{F^{op}} Law^{op} \xrightarrow{Mod} \mathcal{C}at$$

where $Mod : Law \to \mathcal{C}at$ is the functor sending a Lawvere theory to its category of models. Note then that the above diagrams and equations is the same thing as what is sometimes called a *colax cone* over this functor with apex the terminal category $*$:

$$\begin{tikzcd} & * \arrow[dl, "M_c"'] \arrow[dr, "M_{c'}"] \arrow[d, phantom, "\xRightarrow{M_f}"] & \\ Mod \circ F^{op}(c) & & Mod \circ F^{op}(c') \arrow[ll, "- \circ F(f)"] \end{tikzcd}$$

That means that we have an equivalence of category between the category of models of the indexed Lawvere theory F and the category of colax transformation from the terminal functor $\mathbb{1} : \mathscr{C} \to \mathcal{C}at$ to F

$$\begin{tikzcd} \mathscr{C} \arrow[rr, bend left, "F"] \arrow[rr, bend right, "\mathbb{1}"'] \arrow[rr, phantom, "\Uparrow"] & & \mathcal{C}at \end{tikzcd}$$

As such, the category of models of the indexed Lawvere theory F coincides with the category $\int Mod \circ F$, the total space of the fibration

$$\pi \quad : \quad \int Mod \circ F \quad \longrightarrow \quad \mathscr{C}$$

obtained through the Grothendieck construction. The important point here is that a model of F corresponds exactly to a section of π. Indeed, a section s of π is a functor

$$s \quad : \quad \mathscr{C} \quad \longrightarrow \quad \int Mod \circ F$$

such that $\pi \circ s = id_{\mathscr{C}}$. Such a section sends

- an object $c \in \mathscr{C}$ to an element $s(c) \in Mod \circ F(c)$, that is a model of $F(c)$,
- a 1-cell $f : c \to c'$ to a natural transformation

$$s(f) \quad : \quad s(c) \quad \longrightarrow \quad F(f)^*(s(c')) = s(c) \circ F(f)$$

j) The block structure

Let us focus on the case of the indexed Lawvere theory of local state $\mathcal{T}$. A model A of this theory is a family

$$A_0 \qquad A_1 \qquad A_2 \qquad A_3 \qquad A_4 \qquad \ldots$$

where A_i is a model of the Lawvere theory $\mathcal{T}_n$ of state on n registers. Moreover, for each injection $f : [m] \to [n]$, there is a morphism of $\mathcal{T}_m$-model

$$A_f \quad : \quad A_m \quad \longrightarrow \quad A_n \circ \mathcal{T}_f$$

With this data, we can interpret :

- the operations `lookup` and `update` in each A_n,
- the operations `collect` and `permute` as injections acting through the coercions between the fibers.

For instance, the injection $i_n \in Inj([n], [n+1])$ sending each natural number to itself allow to interpret the operation

$$\texttt{collect}_{\langle n\rangle} = A_{i_n} \quad : \quad A_n \quad \longrightarrow \quad A_{n+1}$$

and the bijection $s_n^{n+1} \in Inj([n+2], [n+2])$ only swapping n and $n+1$ enable to interpret the operation

$$\texttt{permute}_{\langle n,n+1\rangle} = A_{s_n^{n+1}} \quad : \quad A_{n+2} \quad \longrightarrow \quad A_{n+2}$$

Now, in order to obtain all the operations present in the local state monad, we are left with interpreting the family of operations `fresh`. For this purpose, Power introduced the notion of a block structure on a model A of the indexed Lawvere theory for local state $\mathcal{T}$. This model A is a block algebra when it is equipped with a family

$$\texttt{block}_{\langle n\rangle} \quad : A_{n+1} \quad \longrightarrow \quad V \Rightarrow A_n$$

of $\mathcal{T}_{n+1}$-homomorphims which interact soundly with the two families of operations `collect` and `permute`, that is such that the two following diagrams commute :

$$\begin{tikzcd}
 & A_{n+1} \arrow[r, "\texttt{block}_{\langle n \rangle}"] & V \Rightarrow A_n \\
A_n \arrow[ur, "A_{i_n}"] \arrow[rr, bend right, "id_{A_n}"'] & & A_n \arrow[u, "\texttt{const}_{A_n}"']
\end{tikzcd} \tag{4}$$

$$\begin{tikzcd}
 & V \Rightarrow A_{n+1} \arrow[rr, "V \Rightarrow \texttt{block}_{\langle n \rangle}"] & & V \Rightarrow (V \Rightarrow A_n) \\
A_{n+2} \arrow[ur, "\texttt{block}_{\langle n+1 \rangle}"] \arrow[dr, "A_{s_n^{n+1}}"'] & & & \\
 & A_{n+2} \arrow[r, "\texttt{block}_{\langle n+1 \rangle}"'] & V \Rightarrow A_{n+1} \arrow[r, "V \Rightarrow \texttt{block}_{\langle n \rangle}"'] & V \Rightarrow (V \Rightarrow A_n) \arrow[uu, "\texttt{swap}^*"']
\end{tikzcd} \tag{5}$$

Here, the arrow $\texttt{const}_{A_n} : A_n \to V \Rightarrow A_n$ is the natural transformation sending an element x to the constant function with value x and $\texttt{swap}^*$ is the natural transformation swapping the two V parameters.

As explained in the introduction, the definition of the block structure relies on the crucial fact that the functor $(V \Rightarrow -)$ transports n-mnemoids (models of $\mathcal{T}_n$) to $(n+1)$-mnemoids (models of $\mathcal{T}_{n+1}$).

k) A fibrational speculation

Power shows that the category of models of $\mathcal{T}$ equipped with a block structure is equivalent to the category of algebra of the local state monad [20]. However, we advocate that this block-algebra structure ought to be derived directly from the indexing category of the indexed Lawvere theory. Let's step back a little and suppose speculatively that we are working with a modified version of the category Inj containing "enough" morphisms to induce the familly of morphisms $(\texttt{block}_{\langle n \rangle})_{n \in \mathbb{N}}$. Let's call b_n the (hypothetical) morphism of the indexing category such that

$$\texttt{block}_{\langle n \rangle} = A_{b_n} \quad : \quad A_{n+1} \quad \longrightarrow \quad A_n^V$$

Such an arrow does not correspond to anything existing in the framework of models of indexed Lawvere theory and that is one of our motivations to the fibrational framework introduced in Section III. But for now let's suppose we can work with such morphisms and see what we can deduce from the two preceding diagrams :

$$\begin{aligned} A_{b_n} \circ A_{i_n} &= \texttt{const}_{A_n} \circ A_{id_n} \\ (V \Rightarrow A)_{b_n} \circ A_{b_{n+1}} &= \texttt{swap}^* \circ (V \Rightarrow A)_{b_n} \circ A_{b_{n+1}} \circ A_{s_n^{n+1}} \end{aligned}$$

These two equations relates the image of morphisms of the indexing category via some action on the $(V \Rightarrow -)$ functors. This suggest that the morphisms themselves should be related in some way in the base category :

$$\begin{tikzcd}
 & {[n+1]} \arrow[dr, "b_n"] \arrow[d, phantom, "\Downarrow"] & \\
{[n]} \arrow[ur, "i_n"] \arrow[rr, bend right, "id_{[n]}"'] & {} & {[n]}
\end{tikzcd}$$

$$\begin{tikzcd}
 & {[n+2]} \arrow[r, "b_{n+1}"] & {[n+1]} \arrow[dr, "b_n"] & \\
{[n+2]} \arrow[ur, "s_n^{n+1}"] \arrow[drr, phantom, "\Downarrow"] \arrow[r, bend right, "b_{n+1}"'] & {[n+1]} \arrow[rr, bend right, "b_n"'] & & {[n]}
\end{tikzcd}$$

We advocate in the next section that 2-categories are the right setting for obtaining such relations between morphisms.

III. An algebraic and 2-categorical Grothendieck construction

In the precedent section, we outlined how the functor $(V \Rightarrow -)$ enable us to handle allocation by bridging the gap between n-mnemoids and $(n+1)$-mnemoids. Thus the right setting to continue our work should be a category of monads where the monad morphisms act accross some functor. It turns out that this correspond exactly to the definition given by Street in [10] of the category of monads in $\mathcal{C}at$.

l) The 2-category of monads

First, recall that a monad can be defined in a any 2-category $\mathcal{C}$ as a 1-cell $T : c \to c$ on some object $c \in \mathcal{C}$ equipped with 2-cells

$$\eta^T : Id \longrightarrow T \qquad\qquad \mu^T : T \circ T \longrightarrow T$$

verifying the usual equations for unit and multiplication. This definition enables one to formally define a 2-category $\mathcal{M}nd(\mathcal{C})$ of monads in $\mathcal{C}$. Since we are only interrested in monads in $\mathcal{C}at$, we will note $\mathcal{M}nd = \mathcal{M}nd(\mathcal{C}at)$.The category $\mathcal{M}nd$ has as objects pairs $\langle \mathscr{C}, T \rangle$ of a category $\mathscr{C}$ and a monad T on $\mathscr{C}$. A morphism

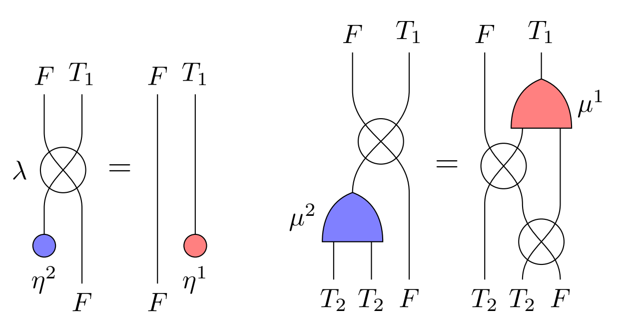
$$\langle F, \lambda \rangle \quad : \quad \langle \mathscr{C}_1, T_1 \rangle \quad \longrightarrow \quad \langle \mathscr{C}_2, T_2 \rangle$$

is then a pair of a functor and a natural transformation

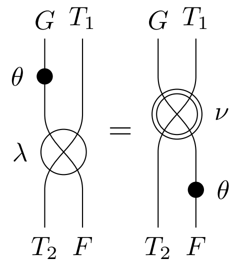
$$F \ : \ \mathscr{C}_1 \ \longrightarrow \ \mathscr{C}_2 \qquad\qquad \lambda \ : \ T_2 F \ \longrightarrow \ F T_1$$

satisfying the two following properties

$$\lambda \circ \eta^2_F = F\eta^1 \qquad\qquad \lambda \circ \mu^2_F = F\mu^1 \circ \lambda_{T_1} \circ T_2 \lambda \tag{6}$$



Moreover between two morphism (F, λ) and (G, ν) are natural transformations $\theta : F \to G$ such that



$$\theta_{T_1} \circ \lambda = \nu \circ T_2\theta \tag{7}$$

In particular, $\mathcal{M}nd$ contains objects (Set, T_n) corresponding to the state monad on n register on Set and morphisms of the form $(V \Rightarrow -, \lambda)$ from T_n to T_{n+1} for each $n \in \mathbb{N}$ that we will use extensively in the next section.

m) A 2-Grothendieck construction

Let us fix some arbitrary 2-category $\mathcal{B}$ and a 2-functor

$$\mathcal{F} \quad : \quad \mathcal{B}^{op(1,2)} \quad \longrightarrow \quad \mathcal{M}nd$$

Here $\mathcal{B}^{op(1,2)}$ denotes the category obtained from $\mathcal{B}$ by formally reversing 1-cells as well as 2-cells. Note that the category $\mathcal{M}nd$ that we introduced in the last section contains a terminal object $* = \langle *, id_* \rangle$ which consists of the identity monad on the terminal category. In the same way as the original Grothendieck construction, we can use this object $*$ by considering the *colax slice category* over $\mathcal{F}$ with apex $*$. This 2-category consists roughly of points over the $\mathcal{B}$-shaped diagram $\mathcal{F}$ and triangles commuting up to natural transformations. In details, the colax slice category over $\mathcal{F}$ is defined by the following data :

- its objects are pairs $(b, \langle A, \lambda \rangle)$ of an object $b \in \mathcal{B}$ and a 1-cell $\langle A, \lambda \rangle \in \mathcal{M}nd(*, \mathcal{F}(b))$. Posing $\mathcal{F}(b) = \langle \mathscr{C}_b, T_b \rangle$, we can identify the functor

$$A \quad : \quad * \quad \longrightarrow \quad \mathscr{C}_b$$

with a single object $A \in \mathscr{C}_b$ and the natural transformation

$$\lambda \quad : \quad T_b A \quad \longrightarrow \quad A$$

with a T_b-algebra structure on A. Indeed, the two commuting diagrams (6) induce exactly the required equations for an algebra map.

- its 1-cells between $(b, \langle A, \alpha \rangle)$ and $(b', \langle B, \beta \rangle)$ are pairs (f, φ) of a morphism $f \in \mathcal{B}(b, b')$ which induces a pair $\mathcal{F}(f) = (F_f, \lambda_f)$ and a natural transformation $\varphi : A \to F_f \circ B$

$$\begin{tikzcd} & * \arrow[dl, bend right, "\langle A,\alpha\rangle"'] \arrow[dr, bend left, "\langle B,\beta\rangle"] & \\ \mathcal{F}(b) \arrow[rr, phantom, "\overset{\varphi}{\Longrightarrow}" near start, shift left=3] & & \mathcal{F}(b') \arrow[ll, "\mathcal{F}(f)"] \end{tikzcd} \qquad \begin{tikzcd} T_b A \arrow[r, "T_b\varphi"] \arrow[dd, "\alpha"'] & T_b F_f B \arrow[d, "\lambda_B"] \\ & F_f T_{b'} B \arrow[d, "F_f\beta"] \\ A \arrow[r, "\varphi"] & F_f B \end{tikzcd}$$

The commuting diagram on the right-hand side obtained from (7) shows that φ is can be identified with a T_b-algebra morphism.

- its 2-morphisms between (f, φ) and (g, ψ) are 2-morphisms $\alpha \in \mathcal{B}(f, g)$ such that

$$\begin{tikzcd} & * \arrow[dl, bend right] \arrow[dr, bend left] & \\ \mathcal{F}(c) \arrow[rr, phantom, "\overset{\varphi}{\Longrightarrow}", shift left=3] & & \mathcal{F}(c') \arrow[ll, bend left, "\mathcal{F}(f)"] \end{tikzcd} = \begin{tikzcd} & * \arrow[dl, bend right] \arrow[dr, bend left] & \\ \mathcal{F}(c) \arrow[rr, phantom, "\overset{\psi}{\Longrightarrow}", shift left=5] & & \mathcal{F}(c') \arrow[ll, bend right, "\mathcal{F}(g)"'] \arrow[ll, bend left, "\mathcal{F}(f)"] \arrow[ll, phantom, "\Downarrow \mathcal{F}(\alpha)"] \end{tikzcd}$$

Let us call this 2-category $\int \mathcal{F}$ by analogy with the Grothendieck construction. From the description of $\int \mathcal{F}$, there is an obvious forgetful functor

$$\pi_{\mathcal{F}} \quad : \quad \int \mathcal{F} \quad \longrightarrow \quad \mathcal{B}$$

which projects out the first component from objects, morphisms and 2-morphisms onto $\mathcal{B}$. This 2-functor $\pi_{\mathcal{F}}$ is a 2-fibration in the sense of [8].

Indeed, given a 1-cell in $\mathcal{B}$

$$f \quad : \quad b' \quad \longrightarrow \quad b$$

and an element $\langle b, X_b \rangle$ in the fiber $\pi_{\mathcal{F}}^{-1}(b)$ of b, we have an arrow covering f

$$\varphi = \langle f, id_{F(f) \circ X_b} \rangle \quad : \quad \langle b, X_b \rangle \quad \longrightarrow \quad \langle b', F(f) \circ X_b \rangle$$

which is 2-cartesian, in the sense that the following commuting diagram square is a (strict) pullback square :

$$\begin{tikzcd} \int \mathcal{F}(\langle b'', Y_{b''} \rangle, \langle b', \mathcal{F}(f) \circ X_b \rangle) \arrow[r, "\varphi\circ-"] \arrow[d, "\pi_{\mathcal{F}}"'] & \int \mathcal{F}(\langle b'', Y_{b''} \rangle, \langle b, X_b \rangle) \arrow[d, "\pi_{\mathcal{F}}"] \\ \mathcal{B}(b'', b') \arrow[r, "f\circ-"] & \mathcal{B}(b'', b) \end{tikzcd}$$

Moreover, the restriction of $\pi_{\mathcal{F}}$ on the $\mathcal{H}om$-categories are (1-)fibrations and precomposition by a 1-cell of the total space is a morphism of fibration.

Let us note that the fiber over some object $c \in \mathcal{B}$ is the category of $\mathcal{F}(c)$-algebras. Thus, a section of the fibration $\pi_{\mathcal{F}}$, that is a 2-functor $s : \mathcal{B} \to \int \mathcal{F}$ such that

$$s \circ \pi_{\mathcal{F}} = id_{\mathcal{B}}$$

picks out a family $s(c)$ of $\mathcal{F}(c)$-algebras indexed by the objects of $\mathcal{B}$ and coherently with respect to the 1-cells and 2-cells present in the base category $\mathcal{B}$.

IV. The 2-category $\mathcal{P}Inj$

In this section we instanciate the fibrational framework presented in Section III to obtain a new presentation of the local state monad. In order to do so, we first construct a 2-category $\mathcal{P}Inj$ of resources which will serve as a base category for the fibration. Then we describe a 2-functor from this resource category $\mathcal{P}Inj$ into the category of monads $\mathcal{M}nd$.

n) The 2-category of partial injections

We stressed out in Section II that the ideal category of resources can be thought as completing the category Inj with “missing” arrows. Informally speaking, the category Inj is generated by the two following morphisms

$$\begin{array}{rcccc} dealloc & : & [0] & \longrightarrow & [1] \\ permute & : & [2] & \longrightarrow & [2] \end{array}$$

We want to add some operation

$$alloc \quad : \quad [1] \quad \longrightarrow \quad [0]$$

with the following 2-cells between morphisms

$$\begin{tikzcd}
 & {[1]} \arrow[dr, bend left, "alloc"] \arrow[d, phantom, "\Downarrow"] & \\
{[0]} \arrow[ur, bend left, "dealloc"] \arrow[rr, bend right, "id_{[0]}"'] & {} & {[0]}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & {[2]} \arrow[rr, "id_{[1]}\otimes alloc"] & {} \arrow[dd, phantom, "\Downarrow"] & {[1]} \arrow[dr, "alloc"] & \\
{[2]} \arrow[ur, "permute"] \arrow[drr, "id_{[1]}\otimes alloc"'] & & & & {[0]} \\
 & & {[1]} \arrow[urr, bend right, "alloc"'] & &
\end{tikzcd}$$

We can obtain this exact setting by considering the category $\mathcal{P}Inj$ of partial injections. Its set of objects consists of finite cardinals $[n]$ for $n \in \mathbb{N}$. A partial injection between $[m]$ and $[n]$ is a pair $\langle p, f\rangle$ of an integer $p \in \mathbb{N}$ and an injection $f : [m] \hookrightarrow [n+p]$.

Two partial injections $\langle p, f\rangle$ and $\langle q, g\rangle$ between $[m]$ and $[n]$ can be related by an injection $\alpha : [q] \hookrightarrow [p]$ such that

$$\begin{tikzcd}
 & {[m]} \arrow[dl, "g"'] \arrow[dr, "f"] & \\
{[n+q]} \arrow[rr, "id_{[n]}+\alpha"] & & {[n+p]}
\end{tikzcd}$$

$$f = (\alpha \otimes id_{[m]}) \circ g \tag{8}$$

Thus we have a 2-category $\mathcal{P}Inj$ of finite cardinals, partial injections and injections relating the partial part as 2-cells. In this setting the morphisms $dealloc$ and $permute$ are inherited from Inj and correspond respectively to the pairs (where we i_0 is the only application from the empty set to the singleton and s^1_0 permute the two elements of $[2]$)

$$\langle 0, i_0\rangle : [0] \to [1] \qquad\qquad \langle 0, s^1_0\rangle : [2] \to [2]$$

The morphism $alloc$ is given by the pair $\langle 1, id_{[1]}\rangle : [1] \to [0]$. Then we can compute that we have the following 2-cells :

$$\begin{tikzcd}
 & {[1]} \arrow[dr, bend left, "{\langle 1,id_{[1]}\rangle}"] \arrow[d, phantom, "\Downarrow i_0"] & \\
{[0]} \arrow[ur, bend left, "{\langle 0,i_0\rangle}"] \arrow[rr, bend right, "{\langle 0,id_{[0]}\rangle}"'] & {} & {[0]}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & {[2]} \arrow[rr, "{\langle 1,id_{[2]}\rangle}"] & {} \arrow[dd, phantom, "\Downarrow s^1_0"] & {[1]} \arrow[dr, "{\langle 1,id_{[1]}\rangle}"] & \\
{[2]} \arrow[ur, "{\langle 0,s^1_0\rangle}"] \arrow[drr, "{\langle 1,id_{[2]}\rangle}"'] & & & & {[0]} \\
 & & {[1]} \arrow[urr, bend right, "{\langle 1,id_{[1]}\rangle}"'] & &
\end{tikzcd}$$

o) A 2-functor into monads.

We will turn now to the definition of the 2-functor

$$\mathscr{T} \quad : \quad \mathcal{P}Inj^{op(1,2)} \quad \longrightarrow \quad \mathcal{M}nd$$

We naturally set $\mathcal{T}([n])$ to be the pair $\langle Set, T_n\rangle$ corresponding to the state monad on n registers $T_n : Set \to Set$. Then given a partial injection

$$\langle p, f\rangle \quad : \quad [m] \quad \longrightarrow \quad [n]$$

we have to describe a morphism $T_m \to T_n$, that is a pair of an endofunctor F on Set and a natural transformation $\lambda^{\langle p,f\rangle} : T_m F \to F T_n$. Since we observed earlier that the functor $(V \Rightarrow -)$ allowed us to move from T_n to T_{n+1}, we set $F = (V^p \Rightarrow -)$ and construct the following natural transformation

$$\lambda^{\langle p,f\rangle}_A : V^m \Rightarrow (V^p \Rightarrow A) \times V^m \longrightarrow V^p \Rightarrow V^n \Rightarrow A \times V^n$$

Let A be a set, $h \in V^m \Rightarrow (V^p \Rightarrow A) \times V^m$ and $v^p \in V^p$, $v^n \in V^n$ be vectors of values. We must construct an object

$$\lambda^{\langle p,f\rangle}_A(f)(v^p)(v^n) \quad \in \quad A \times V^n$$

First note that the concatenation of v^p and v^n gives us a vector $(v^n \cdot v^p) \in V^{n+p}$. Pulling back that vector along f gives us an element in V^m on which we can evaluate h yielding a pair :

$$(h_0, v^m) = h \circ f^*(v^n \cdot v^p) \in (S^p \Rightarrow A) \times V^m$$

Now, evaluating the first component on v^p gives us an element $h_0(v^p) \in A$. The last thing we need is a vector of values in v^n. It is obtained by first rewriting $(v^n \cdot v^p)$ by pushing forward v^m along f and then projecting on the n first components of the vector via $\pi^{n+p}_n : V^{n+p} \to V^n$. More formally, for $v \in V^m$ and $w \in V^{n+p}$, we introduce an operation $f^w_* : V^m \to V^{n+p}$

$$(f^w_*(v))(i) = \begin{cases} v_j & \text{if } \{j\} = f^{-1}(i) \\ w_i & \text{otherwise} \end{cases}$$

and we finally get as result for $\lambda^{\langle p,f\rangle}_A(f)(v^p)(v^n)$

$$(h_0(v^p), \pi^{n+p}_p \circ f^{(v^n \cdot v^p)}_*(v^m)) \in A \times V^n$$

Putting everything together gives us the following definition for $\lambda^{\langle p,f\rangle}$:

$$h \mapsto \lambda v^p.\lambda v^n.(eval_{v^p} \times \pi^{n+q}_n \circ f^{(v^n \cdot v^p)}_*) \circ h \circ f^*(v^n \cdot v^p) \tag{9}$$

Finally, given a 2-cell

$$\alpha \quad : \quad \langle p, f\rangle \longrightarrow \langle q, g\rangle \quad : \quad [m] \longrightarrow [n]$$

that is an injection $\alpha : [q] \hookrightarrow [p]$ such that $f = (id_{[n]} + \alpha) \circ g$, we must describe a natural transformation

$$\theta \quad : \quad (V^q \Rightarrow -) \quad \longrightarrow \quad (V^p \Rightarrow -)$$

satisfying (7). The operation V^α of pulling back along α sends an element of V^p to V^q and precomposing by this map gives the desired natural transformation $\theta = (- \circ V^\alpha)$. The required equation (7) comes from the fact that pulling back along f is the same as pulling back along $(id_{[n]} + \alpha) \circ g$, that is first pulling back along α and then along g.

p) The fibration over $\mathcal{P}Inj$

We described in section [III] a way to construct a fibration over a base $\mathcal{B}$ given a 2-functor $\mathcal{B} \to \mathcal{M}nd$. Let us unroll the definition of the category $\int \mathscr{T}$ built from the 2-functor $\mathscr{T}$ introduced in the previous section.

Its objects are pairs of an natural number n and an algebras of the global state monad on n registers. A morphism between a T_n-algebra (A, α) and a T_m-algebra (B, β) is a pair of a partial injection $(p, f) : [n] \to [m]$ and a T_n-algebra morphism $A \to S^p \Rightarrow B$. The T_n-algebra structure on $S^p \Rightarrow B$ is obtained by pulling back the T_m-algebra structure of B along f :

$$T_n(S^p \Rightarrow B) \xrightarrow{\lambda_B^{(p,f)}} S^p \Rightarrow T_m(B) \xrightarrow{S^p \Rightarrow \beta} S^p \Rightarrow B$$

Moreover, if $A \to S^p \Rightarrow B$ and $A \to S^q \Rightarrow C$ are T_n-algebra morphisms respectively over (p, f) and (q, g), the 2-cells between them are given by the 2-cells between (p, f) and (q, g) in $\mathcal{P}Inj$, that is injections $\alpha : [q] \hookrightarrow [p]$ such that $f = (id_{[n]} + \alpha) \circ g$.

From this description, the fibration $\pi_{\mathscr{T}} : \int \mathscr{T} \to \mathcal{P}Inj$ sends a T_n-algebra to the cardinal $[n]$, a morphism in $\int \mathscr{T}$ to the partial injection present as its first component and a 2-morphism to itself.

q) Algebras as sections

Now, we can consider the category of sections $\Gamma(\pi_{\mathscr{T}})$ of the fibration $\pi_{\mathscr{T}}$, that is the full subcategory of 2-functors F from $\mathcal{P}Inj$ to $\int \mathscr{T}$ such that $\pi_{\mathscr{T}} \circ F = id_{\mathcal{P}Inj}$. An object $\mathscr{S} \in \Gamma(\pi_{\mathscr{T}})$ consist of the following data :

- a family $(A_n)_{n \in \mathbb{N}}$ of sets where $\mathscr{S}(n) = A_n$ is endowed with a T_n-algebra structure
- for each natural number p and injection $f : [m] \hookrightarrow [n+p]$, a T_m-algebra morphisms
$$\varphi_{\langle p,f \rangle} \quad : \quad A_m \quad \longrightarrow \quad S^p \Rightarrow A_n$$
- such that any 2-cell, that is any equation $f = (id_n + \alpha) \circ g$ induce a commutative diagram

$$\begin{tikzcd} & A_m \arrow[dl, "\varphi_{\langle q,g \rangle}"'] \arrow[dr, "\varphi_{\langle p,f \rangle}"] & \\ V^q \Rightarrow A_n \arrow[rr, "- \circ V^\alpha"] & & V^p \Rightarrow A_n \end{tikzcd}$$

We will now show that this category $\Gamma(\pi_{\mathscr{T}})$ of sections is equivalent to the category of model of the indexed Lawvere theory $\mathcal{T}$ equipped with a block-algebra structure as introduced by Power. This will suffice to prove our claim that the algebras of the local state monad can be considered as the aforementioned sections.

Let's begin by noting that there is an obvious inclusion functor

$$\mathcal{J} \quad : \quad Inj \quad \longrightarrow \quad \mathcal{P}Inj$$

sending each natural number to itself and an injection f to the pair $\langle 0, f \rangle$. Also recall that for any natural number n, the catgeory of n-mnemoids coincides with both the category $\mathcal{T}_n$-Mod of models of the Lawvere theory $\mathcal{T}_n$ of state on n registers and the category T_n-Alg of algebra of the state monad T_n on n registers.

Given a section $\mathscr{S} \in \Gamma(\pi_{\mathscr{T}})$, we can precompose it with $\mathcal{J}$ and consider it as taking its value in $\mathcal{T}_n$-Mod instead of T_n-alg, thus yielding a model $\mathcal{M}_s$ of the indexed Lawvere theory of local state $\mathcal{T}$. Moreover, we can equip $\mathcal{M}_{\mathscr{S}}$ with a block-algebra structure by setting for each natural number $n \in \mathbb{N}$

$$\texttt{block}_n = \mathscr{S}\langle 1, id_{n+1} \rangle \quad : \quad A_{n+1} \quad \longrightarrow \quad V \Rightarrow A_n$$

The two required commutating diagrams (4) and (5) for the block-algebra structure are respectively the image by $\mathscr{S}$ of the following 2-cells

$$\begin{tikzcd} & {[n+1]} \arrow[dr, "\langle 1, id_{[n+1]} \rangle"] \arrow[d, phantom, "\Downarrow i_n"] & \\ {[n]} \arrow[ur, "\langle 0, i_n \rangle"] \arrow[rr, bend right, "\langle 0, id_{[n]} \rangle"'] & {} & {[n]} \end{tikzcd}$$

$$\begin{tikzcd} & {[n+2]} \arrow[r, "\langle 1, id_{[n+2]} \rangle"] & {[n+1]} \arrow[dr, "\langle 1, id_{[n+1]} \rangle"] \arrow[d, phantom, "\Downarrow s_n^{n+1}"] & \\ {[n+2]} \arrow[ur, "\langle 0, s_n^{n+1} \rangle"] \arrow[rr, "\langle 1, id_{[n+2]} \rangle"'] & & {[n+1]} \arrow[r, "\langle 1, id_{[n+1]} \rangle"'] & {[n]} \end{tikzcd}$$

Reciprocally, we can build a section $\mathscr{S}_{\mathcal{M}}$ from a model $\mathcal{M}$ of the indexed Lawvere theory for local state $\mathcal{T}$ equipped with a block-algebra structure $(\texttt{block}_n)_{n \in \mathbb{N}}$. For an object $[n] \in \mathcal{P}Inj$, we set $\mathscr{S}_{\mathcal{M}}(n)$ to be the T_n-algebra isomorphic to $\mathcal{M}_n$. Then for a partial injection

$$\langle p, f \rangle \quad : \quad [m] \quad \longrightarrow \quad [n]$$

we set $\mathscr{S}_{\mathcal{M}}\langle p, f \rangle$ to be the following composite

$$\mathscr{S}_{\mathcal{M}}(m) \quad \overset{\mathscr{S}_{\mathcal{M},f}}{\longrightarrow} \quad \mathscr{S}_{\mathcal{M}}(n+p) \quad \overset{\texttt{block}_n^p}{\longrightarrow} \quad V^p \Rightarrow \mathscr{S}_{\mathcal{M}}(n)$$

where the left arrow $\mathscr{S}_{\mathcal{M},f}$ is given up to isomorphism by $\mathcal{M}_f$ and the right arrow $\texttt{block}_n^p$ is the following composite

$$(V^{p-1} \Rightarrow \texttt{block}_{\langle n \rangle}) \circ \cdots \circ (V \Rightarrow \texttt{block}_{\langle n+p-1 \rangle}) \circ \texttt{block}_{\langle n+p \rangle}$$

Finally, we must show that any 2-cell in the base

$$\alpha \ : \ \langle p, f \rangle \ \longrightarrow \ \langle q, g \rangle \ : \ [m] \longrightarrow [n]$$

induces a commuting diagram in the total space :

$$\begin{tikzcd} & \mathscr{S}_{\mathcal{M}}(n+p) \arrow[r, "\texttt{block}_n^p"] & V^p \Rightarrow \mathscr{S}_{\mathcal{M}}(n) \\ \mathscr{S}_{\mathcal{M}}(m) \arrow[ur, "\mathscr{S}_{\mathcal{M},f}"] \arrow[dr, "\mathscr{S}_{\mathcal{M},g}"'] & & \\ & \mathscr{S}_{\mathcal{M}}(n+q) \arrow[r, "\texttt{block}_n^q"] \arrow[uu, dotted] & V^q \Rightarrow \mathscr{S}_{\mathcal{M}}(n) \arrow[uu, "V^\alpha"'] \end{tikzcd}$$

It is proved by first instanciating the dotted arrow by the morphism obtained from $\mathcal{M}_{id_{[n]}+\alpha}$ trough the isomorphism and then proving separately that the left triangle and right square commute.

For the left triangle, the definition of the 2-cell α give us

$$f = (id_{[n]} + \alpha) \circ g$$

and by functoriality of $\mathcal{M}$ we get

$$\mathcal{M}_f = \mathcal{M}_{id_{[n]}+\alpha} \circ \mathcal{M}_g$$

and we can conclude by applying the isomorphism for m-mnemoids. For the right square, first recall that the category Inj is generated by the operations $i_0 : [0] \to [1]$ and $s_0^1 : [2] \to [2]$. Given a presentation of α in terms of these two operations, we can easily prove by induction on the presentation that the square commutes, using the diagrams (4) and (5) in the base case.

It is not difficult to see that the two transformations that we just described are inverse of each other and that they extends straightforwadly to morphisms. Hence we obtain the following :

Theorem 1. *The category $\Gamma(\pi_{\mathscr{T}})$ of sections of the fibration $\pi_{\mathscr{T}}$ is equivalent to the category of models of the indexed Lawvere theory $\mathcal{T}$ equipped with a block algebra structure.*

Thus an algebra of the local state monad can be seen as well as a section of $\pi_{\mathscr{T}}$, since the former is proved by Power to be equivalent to a model of $\mathcal{T}$ equipped with a block-algebra structure. However, the proof of equivalence relies on Beck's monadicity theorem which prevent us from having a conceptual overview of the variations between the different presentations of the local state monad.

V. Connection with the algebraic presentation

We exhibit in this section a direct proof of isomorphism between our category of sections $\Gamma(\pi_{\mathscr{T}})$ and the category of algebra of the local state monad. Due to tha complicated nature of the local state monad T on $[Inj, Set]$, we chose to introduce here the presentation given by Melliès in [12] of the local state monad T_{Res} on $[Res, Set]$ where Res is a category obtained from Inj by adding some morphisms as explained thereafter.

Indeed, given a covariant presheaf on Res

$$A \quad : \quad Res \quad \longrightarrow \quad Set$$

the local state monad T_{Res} takes the following particularly nice expression on objects :

$$T_{Res}A \quad : \quad [n] \quad \longmapsto \quad T_n A_n = V^n \Rightarrow A_n \times V^n$$

That is the local state monad T_{Res} coincides pointwise with the state monad T_n on n registers.

r) The category Res ***of resources***

The category Res of resources has natural number as objects and *resource morphisms* as 1-cells. A resource morphism f between $[m]$ and $[n]$ is defined as a function

$$f \quad : \quad [m] \quad \longrightarrow \quad [n] \cup V$$

where every element in $[n]$ has *at most* one antecedent. As such, any injection $f : [m] \to [n]$ can be seen as a resource morphism and thus there is an induced embedding

$$\iota \quad : \quad Inj \quad \longrightarrow \quad Res$$

sending each object to itself and coercing transparently injections to resource morphisms.

There is another description of resource morphisms as equivalence class which may seem less natural but will simplify greatly the presentation of the connection to $\Gamma(\pi_{\mathscr{T}})$. The idea is that a resource morphism can be seen as a pair of an injection and a vector of values up to some equivalence relation relating pairs sending the same inputs to the same outputs. This can be expressed formally with the following coend formula :

$$Res(m, n) \cong \int^{p \in Inj} Inj(m, n+p) \times V^p$$

That is a resource morphism from $[m]$ to $[n]$ is a pair$[f, v]_p$ composed of an injection $f : [m] \to [n+p]$ and a vector of values $v \in v^p$. Moreover two such pairs $[f, v]_{p_1}$ and $[g, w]_{p_2}$ are identified when there exists an injection $\alpha : [p_1] \to [p_2]$ such that :

$$(id_n + \alpha) \circ f = g \qquad\qquad v = V^\alpha(w)$$

This reformulation of resource morphisms enable us to give a concise description of pull back and push forward operations for resources morphisms in terms of the very same operations for injections. Recall that we introduced and used the operations of pulling back a state along an injection and updating a previous state by pushing forward a newer but smaller state along an injection in (9). Given a resource morphism $\mathbf{f} : [m] \to [n]$ and a piece of state $v^n \in V^n$, we hence have the two following operations :

$$\begin{array}{cccc} \mathbf{f}^* & : & V^n & \longrightarrow & V^m \\ \mathbf{f}_*^{v_n} & : & V^m & \longrightarrow & V^n \end{array}$$

Moreover, if $\mathbf{f}$ is represented by the pair $[f, v^p]_p$ of an injection $f : [m] \to [n+p]$ and a vector of values $v^p \in V^p$, then we have the following definitions :

$$[f, v_p]^*(v_n) = f^*(v_n \cdot v_p) \tag{10}$$

$$[f, v_p]_*^{v_n}(v_m) = f_*^{(v_n \cdot v_p)}(v_m) \tag{11}$$

It is not difficult to calculate that these definitions does not depends on the choice of the representant taken for $\mathbf{f}$.

s) A local state monad on $[Res, Set]$

The construction of the local state monad T_{Res} on $[Res, Set]$ start from the observation that the local state monad T on $[Inj, Set]$ can be decomposed as the composition of a *change of base* monad $\mathscr{B}$ and a *fiber* monad $\mathscr{F}$ regulated by a distributivity law

$$\lambda \quad : \quad \mathscr{B} \circ \mathscr{F} \quad \longrightarrow \quad \mathscr{F} \circ \mathscr{B}$$

where the change of base monad $\mathscr{B}$ is obtained from the following adjunction

$$\begin{tikzcd}
{[Inj, Set]} \arrow[rr, bend left, "\exists_\iota"] \arrow[rr, phantom, "\perp"] & & {[Res, Set]} \arrow[ll, bend left, "\iota^*"]
\end{tikzcd}$$

From the general theory of distributivity laws [1], it then follows that the fiber monad $\mathscr{F}$ can be lifted to a monad T_{Res} on $[Res, Set]$ whose algebras coincides with the algebras of the local state monad on $[Inj, Set]$, hence the name of local state monad on $[Res, Set]$ for T_{Res}.

As we announced at the beginning of this section, the monad T_{Res} act as follows on a covariant presheaf $F \in [Res, Set]$

$$T_{Res}F \quad : \quad [n] \quad \longmapsto \quad T_n(F(n))$$

Then, if $\mathbf{f} : [m] \to [n]$ is a resource morphism, we have the following definition of $T_{Res}F$ on morphisms

$$T_{Res}F(\mathbf{f}) \ : \begin{cases} V^m \Rightarrow F(m) \times V^m \ \longrightarrow \ V^n \Rightarrow F(n) \times V^n \\ \qquad h \qquad \longmapsto \ \lambda v.(F(\mathbf{f}) \times \mathbf{f}^v_*) \circ h \circ \mathbf{f}^*(v) \end{cases}$$

The action of the monad T_{Res} on a natural transformation $\theta = (\theta_n)_{n\in\mathbb{N}}$ is simply to apply the relevant state monad T_n on n register, that is $(T_{Res}\theta)_n = T_n\theta_n$. The unit and multiplication are also given pointwise by those of T_n for each natural number n.

t) A second proof of equivalence

Now we turn to the proof of the following theorem :

Theorem 2. *The category T_{res}-Alg of the algebras of the local state monad T_{Res} on $[Res, Set]$ is equivalent to the category $\Gamma(\pi_{\mathscr{F}})$ of sections of the fibration $\pi_{\mathscr{F}}$.*

Given an object in T_{Res}-alg, that is a pair of a covariant presheaf F and an algebra map α

$$F \ : \ Res \ \longrightarrow \ Set \qquad\qquad \alpha \ : \ T_{Res}F \ \longrightarrow \ F$$

we construct a section $\mathscr{S}$ as follows. On an object of $\mathcal{P}Inj$, that is a finite cardinal $[n]$, we set $\mathscr{S}(n)$ to be the T_n-algebra $\langle F(n), \alpha_n\rangle$. Then we need to define $\mathscr{S}$ on a partial injection

$$\langle p, f\rangle \quad : \quad [m] \quad \longrightarrow \quad [n]$$

In order to do so, first note that we have an operation

$$F([f, -]_p) \quad : \quad V^p \times F(m) \quad \longrightarrow \quad F(n)$$

Hence, we give the following definition for $\mathscr{S}(f)$

$$\begin{array}{ccc} \mathscr{S}(m) & \longrightarrow & V^p \Rightarrow \mathscr{S}(n) \\ h & \longmapsto & \lambda v^p.F([f, v_p])(h) \end{array}$$

We ought to check that this function is indeed a T_m-algebra morphism. This can be derived from the following commuting diagram :

$$\begin{tikzcd}
T_m\mathscr{S}(m) \arrow[r, "T_m\mathscr{S}(f)"] \arrow[dr, "\psi"'] \arrow[dd, "\alpha_m"'] & T_m(V^p \Rightarrow \mathscr{S}(n)) \arrow[d, "\lambda^{\langle p,f\rangle}_{\mathscr{S}()}"] \\
& V^p \Rightarrow T_n\mathscr{S}(n) \arrow[d, "V^p\Rightarrow\alpha_n"] \\
\mathscr{S}(m) \arrow[r, "\mathscr{S}(f)"] & V^p \Rightarrow \mathscr{S}(n)
\end{tikzcd}$$

where $\psi = h \mapsto \lambda v^p.T_{Res}F([f, v^p])(h)$. The lower rectangle commutes because F is a T_{Res}-algebra and the commutation of the upper triangle follows from a little calculation.

Finally, given a 2-cell in $\mathcal{P}Inj$

$$\alpha \ : \ \langle p, f\rangle \ \longrightarrow \ \langle q, g\rangle \ : \ [m] \longrightarrow [n]$$

we must check that $\mathscr{S}(f) = (- \circ V^\alpha) \circ \mathscr{S}(g)$, which comes from the next calculation where $h \in \mathscr{S}(m)$:

$$\begin{aligned} (- \circ V^\alpha) \circ \mathscr{S}(g)(h) &= (- \circ V^\alpha)(\lambda v^q.F([g, v^q]_q)(h)) \\ &= \lambda v^p.F([g, V^\alpha(v^p)]_q)(h) \\ &= \lambda v^p.F([(id_{[n]} + \alpha) \circ g, v^p]_p)(h) \\ &= \lambda v^p.F([f, v^p]_p)(h) = \mathscr{S}(f)(h) \end{aligned}$$

Reciprocally, starting from a section $\mathscr{S} \in \Gamma(\pi_{\mathscr{F}})$, we construct a covariant presheaf $F \in [Res, Set]$ and equip it with a T_{Res}-algebra structure. On a object $[n] \in Res$, we set $F(n) = A_n$ to be the set underlying the T_n-algebra $\mathscr{S}(n) = \langle A_n, \alpha_n\rangle$. Then given a resource morphism $\mathbf{f} \in Res([m], [n])$ represented by a pair $[f, v]_p$

$$f \quad : \quad [m] \ \longrightarrow \ [n + p] \qquad\qquad v^p \in V^p$$

we can take

$$\begin{array}{cccc} F(\mathbf{f}) & : & F(m) & \longrightarrow & F(n) \\ & & h & \longmapsto & \mathscr{S}\langle p, f\rangle(h)(v^p) \end{array}$$

This definition does not depends on the choice of the representant, because any other representant $\mathbf{f} = [g, v^q]_q$ will be related to the first by an injection α which can be seen as a 2-cell in $\mathcal{P}Inj$

$$\alpha \quad : \quad \langle p, f\rangle \longrightarrow \langle q, g\rangle \quad : \quad [m] \longrightarrow [n]$$

from which follows with a little calculation that for any $h \in F(m)$

$$\mathscr{S}\langle p, f\rangle(h)(v^p) = \mathscr{S}\langle q, g\rangle(h)(v_q)$$

The T_{Res}-algebra map on F is then taken as the family $(\alpha_n)_{n\in\mathbb{N}}$ of maps, where $\alpha_n : T_nA_n \to A_n$ is a T_n-algebra map.

VI. Conclusion

In this paper, we develop one step further the careful algebraic analysis of the local state monad initiated fifteen years ago with the seminal work by Plotkin and Power. To that purpose, we benefit from the deep connection (which we also clarify in the process) between the notion of indexed Lawvere theories formulated by Power and the algebraic presentations of the local state monad performed by Staton and Melliès. This unified point of view leads us to a purely conceptual description of the local state monad T as the result of "gluing together" a family of global state monads $(T_n)_{n\in\mathbb{N}}$. The glueing itself is performed by applying a Grothendieck construction on an adapted notion of $\mathcal{C}$-indexed monad, simply defined as a 2-functor from $\mathcal{C}^{op}$ to Street's 2-category Mnd of monads. In this 2-categorical way, we are able to see for the first time an algebra A of the local state monad T as a section $(A_n)_{n\in\mathbb{N}}$ of a specific notion of fibration, whose fibers are the categories of T_n-algebras.

One main purpose of future research will be to combine this fibrational description of the local state monad with the recent work on the proof-relevant semantics of local effects designed by by Benton, Hofmann and Nigam [2]. To that purpose, one needs to understand how the shift from discrete fibrations to general 2-fibrations performed here interacts (by the appropriate sheaf or descent conditions) with the setoidal notion of fibration formulated in [2]. This would enable one to reformulate their monadic description as a local and purely equational account, which we believe is a useful step towards formalization in proof assistants.

Another research direction will be to describe other "local effects". We are specifically interested in a notion of heap monad reflecting the behaviour of registers with self reference and pointers. Our fibrational description of the "local heap monad" should bridge the gap with more direct accounts of heaps like separation logic. This investigation is interestingly related to the fibrational account of refinement types recently developed by Melliès and Zeilberger [14].

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