On the subject reduction property for algebraic type systems

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Abstract

Algebraic type systems provide a general framework for the study of the interaction between typed λ-calculi and typed rewriting systems. A major problem in the development of a general theory for algebraic type systems is to prove that typing is preserved under reduction (Subject Reduction lemma). In this paper, we propose a general technique to prove Subject Reduction for a large class of algebraic type systems. The idea is to consider for every (functional) algebraic type system a labelled syntax for which Subject Reduction is easy to prove and then prove the equivalence between the labelled and standard syntaxes when the labelled system is strongly normalising. The equivalence can then be used to recover confluence, strong normalisation and subject reduction for the standard syntax.

1 Introduction

λ-calculus and term-rewriting are two fundamental computational paradigms. When combined, they give rise to the class of algebraic-functional languages ([3, 9, 10, 15]). Recently, H. Geuvers and the first author have proposed a general framework for the classification and study of algebraic-functional languages: *algebraic type systems* ([6]).

Subject Reduction, also known as Type Safeness, states that types are closed under reduction. It is an important property of a type system: for instance it implies that correctness is preserved under evaluation and is needed in most strong normalisation proofs. Unfortunately, it is unknown whether Subject Reduction holds for an arbitrary algebraic type system. Indeed, the reduction relation of algebraic type systems may not be confluent on pseudo-terms (see [9, 16]) and as a result standard techniques to prove Subject Reduction (see [4, 14]) cannot be used.

The problem. Let $\lambda\mathbf{S}$ be an algebraic type system. If $\Gamma \vdash M : A$ and $M \to_{\beta R} N$, then $\Gamma \vdash N : A$.

In this paper, we propose a general technique for proving Subject Reduction for a large class of algebraic type systems (and so provide a partial but useful answer to the problem). The central idea is to consider a labelled syntax for which Subject Reduction is easy to establish and then prove that, under suitable conditions, both syntaxes are equivalent. Our work, complemented with a generic proof of strong normalisation ([6, 19][1]), provides a clear and widely applicable meta-theory of algebraic type systems. A particular application of our work is a proof of strong normalisation of the algebraic λ-cube, see [7] for details. Another, perhaps more important, application is to contribute to a better understanding of the various presentations of type systems. Several presentations are used in the literature, each of which serves a specific purpose. For example, the labelled syntax

[1] These proofs are concerned with a different syntax but may be adapted to that of this paper.

we consider is best suited to give a semantics of type systems[2] (see [2, 19, 22]) while the standard syntax is best suited for proof checking (see [11, 18]). Our work establishes the equivalence between the two presentations for a large class of systems.

Contents of the paper and prerequisites In Section 2, we introduce the standard and labelled syntaxes of algebraic type systems. The Subject Reduction property for the labelled syntax is proved in Section 3 and the equivalence between the labelled and standard syntaxes is proved in Section 4. In Section 5, we consider an application of our results. Finally, we conclude in Section 6.

The paper assumes some basic familiarity with pure type systems (see [4, 14]) and term-rewriting (see [12, 17]).

2 Algebraic type systems

2.1 Preliminaries

Throughout this section, X is an arbitrary set. All relations will be understood as relations over X. Elements of X are called objects.

If $\mathcal{R}$ and $\mathcal{S}$ are binary relations, $\mathcal{R}.\mathcal{S}$ denote their composition. Moreover, for $\mathcal{R}$ an arbitrary relation, we use the following notation (below R stands for Reflexive, S for symmetric and T for transitive, C for closure):

Notion	RC	SC	TC	RTC	RSTC	Inverse	$\mathcal{R}^\omega \cdot (\mathcal{R}^{op})^\omega$
Notation	$\underline{\mathcal{R}}$	$\mathcal{R}^{\leftrightarrow}$	$\mathcal{R}^+$	$\mathcal{R}^\omega$	$=_\mathcal{R}$	$\mathcal{R}^{op}$	$\downarrow_\mathcal{R}$

Some of the relations will written as $\rightarrow_i$, in which case we use an ARS (abstract rewriting system) notation:

Usual notation	$\rightarrow_i^\omega$	$=_{\rightarrow_i}$	$\downarrow_{\rightarrow_i}$
ARS notation	$\twoheadrightarrow_i$	$=_i$	$\downarrow_i$

Definition 1 *A relation $\mathcal{R}$ is*

- locally confluent *if $\mathcal{R}^{op} \cdot \mathcal{R} \subseteq \downarrow_\mathcal{R}$.*
- confluent *if the relations $\downarrow_\mathcal{R}$ and $=_\mathcal{R}$ are equal.*
- Church-Rosser *on an object a if for every b, c such that $b\ (\mathcal{R}^\omega)^{op}\ a\ \mathcal{R}^\omega\ c$ there exists d such that $b\ \mathcal{R}^\omega\ d\ (\mathcal{R}^\omega)^{op}\ c$.*
- strongly normalising *on an object a if there is no infinite sequence*

$$a_0 \mathcal{R} a_1 \mathcal{R} a_2 \mathcal{R} \ldots$$

- canonical *on an object a if it is Church-Rosser and strongly normalising on a.*

Throughout the paper, we will make use of Newman's Lemma.

Lemma 2 (Newman's Lemma) *If $\mathcal{R}$ is locally confluent and strongly normalising on a, then $\mathcal{R}$ is Church-Rosser on a.*

[2] Labels were also used, in a slightly different form, by Salvesen to prove Church-Rosser for extensional pure type systems [21].

2.2 Algebraic type systems

For the sake of clarity, we only consider first-order rewriting.

Definition 3 *A* pre-specification *is a 6-tuple* $\lambda\mathsf{S} = (U, S, F, H, P, D)$ *where*

- U *is a set of* universes*,* S *is a set of* sorts *and* F *is a set of* function symbols*;*
- $H \subseteq (U \cup S) \times U$ *is a set of* axioms *s.t.* $\forall \tau \in S. \exists s \in U.\ (\tau, s) \in H$*.*
- $P \subseteq U \times U \times U$ *is a set of* rules*;*
- $D : F \to S^\star \times S$ *is a* declaration *function.*

For the sake of hygiene, we assume that U, S, F are pairwise disjoint. Throughout the rest of this paper, we let V be a fixed set of variables and let $\sigma, \tau, \ldots$ (resp. $f, g, \ldots$) range over sorts (resp. function symbols). Moreover we define the arity $\mathsf{ar}(f)$ of a function symbol $f \in F$ to be the length of the first component of $D(f)$. K is then defined as the set of function symbols of arity 0.

To complete the specification of an ATS, we introduce algebraic reduction. The approach we follow is inspired from [3, 13] and is equivalent to that of [6].

Definition 4 *Let* $\lambda\mathsf{S} = (U, S, F, H, P, D)$ *be a pre-specification.*

- *The set* L *of algebraic terms is given by the abstract syntax:*

$$L = V | f(L, \ldots, L)$$

where in the last case the number of arguments applied to f *is* $\mathsf{ar}(f)$*.*

- *The set of variables of a term* t *is denoted by* $\mathsf{var}(t)$ *and is defined as usual.*
- *Let* $\xi : V \to S$*. The relation* $:_\xi \subseteq L \times S$ *is defined by the rules*

$$\frac{\xi(x) = \tau}{x :_\xi \tau} \qquad\qquad \frac{t_i :_\xi \tau_i \quad (1 \leq i \leq n)}{f(t_1, \ldots, t_n) :_\xi \sigma} \ \textit{if } D(f) = ((\tau_1 \ \ldots \ \ldots \tau_n), \sigma)$$

- *A* rewrite rule *is a pair* (l, r) *of algebraic terms s.t.* $l \notin V$ *and* $\mathsf{var}(l) \subseteq \mathsf{var}(r)$ *and* $l, r :_\xi \tau$ *for some* $\xi : V \to S$ *and* $\tau \in S$*.*
- *A* rewrite system *is a set of rewrite rules.*

Every rewrite system R may be seen as an unsorted rewrite system and thus induces a relation $\to_{L(R)}$ on L. We can also define a relation $\to_{LL(R)}$ by $a \to_{LL(R)} b$ if $a \to_{L(R)} b$ and $a, b :_\xi \tau$ for some $\xi : V \to S$ and $\tau \in S$.

Definition 5

- *An* ATS specification *is a pair consisting of a pre-specification* $\lambda\mathsf{S} = (U, S, F, H, P, D)$ *and a rewrite system* R*. By abuse of notation, we write* $\lambda\mathsf{S} = (U, S, F, H, P, D, R)$*.*
- *Let* PROPERTY *be a property of relations (e.g. confluent or terminating). A specification* $\lambda\mathsf{S} = (U, S, F, H, P, D, R)$ *is* A-PROPERTY *if* $\to_{L(R)}$ *is* PROPERTY*.*
- *A specification* $\lambda\mathsf{S} = (U, S, F, H, P, D, R)$ *is* functional *if* A *and* P *are partial maps.*

For the remaining of the paper, we assume:

Assumption 6 $\lambda\mathsf{S} = (U, S, F, H, P, D, R)$ *is an* ATS *specification.*

Axiom	$\frac{}{\vdash_{\mathcal{R}} c : s}$	if $(c, s) \in H$
Function	$\frac{\Gamma \vdash_{\mathcal{R}} t_1 : \sigma_1 \quad \dots \quad \Gamma \vdash_{\mathcal{R}} t_n : \sigma_n}{\Gamma \vdash_{\mathcal{R}} f(t_1, \dots, t_n) : \tau}$	if $Df = ((\sigma_1 \dots \sigma_n), \tau)$
Start	$\frac{\Gamma \vdash_{\mathcal{R}} A : s}{\Gamma, x : A \vdash_{\mathcal{R}} x : A}$	if $x \notin \Gamma$, $x \in V$
Weakening	$\frac{\Gamma \vdash_{\mathcal{R}} t : A \quad \Gamma \vdash_{\mathcal{R}} B : s}{\Gamma, x : B \vdash_{\mathcal{R}} t : A}$	if $x \notin \Gamma$ and $t \in S \cup U \cup V \cup K$
Product	$\frac{\Gamma \vdash_{\mathcal{R}} A : s_1 \quad \Gamma, x : A \vdash_{\mathcal{R}} B : s_2}{\Gamma \vdash_{\mathcal{R}} \Pi x : A.B : s_3}$	if $(s_1, s_2, s_3) \in P$
Application	$\frac{\Gamma \vdash_{\mathcal{R}} t : \Pi x : A.B \quad \Gamma \vdash_{\mathcal{R}} u : A}{\Gamma \vdash_{\mathcal{R}} t\ u : B[u/x]}$	
Abstraction	$\frac{\Gamma, x : A \vdash_{\mathcal{R}} t : B \quad \Gamma \vdash_{\mathcal{R}} (\Pi x : A.B) : s}{\Gamma \vdash_{\mathcal{R}} \lambda x : A.t : \Pi x : A.B}$	
Conversion	$\frac{\Gamma \vdash_{\mathcal{R}} u : A \quad \Gamma \vdash_{\mathcal{R}} B : s}{\Gamma \vdash_{\mathcal{R}} u : B}$	if $A \mathcal{R} B$

Table 1: $\mathcal{R}$-DEDUCTIVE SYSTEM FOR THE STANDARD SYNTAX

2.3 Standard syntax

The set T of *pseudo-terms* is defined by the abstract syntax:

$$T = V|U|S|TT|\Pi V : T.T|\lambda V : T.T|f(T, \dots, T)$$

where in the last case, the number of arguments applied to f is $\mathsf{ar}(f)$[3].

In order to provide a uniform framework for the systems used in the literature, the rules for derivation, in Table 1, are parametrised by a binary relation $\mathcal{R}$ on pseudo-terms, see [20] for a similar idea. For lack of space, only one deductive system $\vdash$ is considered here. The definition below makes use of contexts, substitutions and β-reduction. These are defined as usual.

Definition 7 - $M \to_R N$ *if there exists a context* $C[.]$, *a rule* (l, r) *and a substitution* θ *s.t.* $M \equiv C[\theta l]$ *and* $N \equiv C[\theta r]$.

- $\to_{mix} \;=\; \to_\beta \cup \to_R$.

- $\vdash \;=\; \vdash_{\downarrow_{mix}}$.

2.4 Labelled syntax

The labelled syntax differs from the standard one by having labelled abstractions and labelled applications. The set T_e of labelled pseudo-terms is defined by the abstract syntax:

$$T_e = V|U|S|\mathsf{app}^{\Pi \mathsf{var}:T_e.T_e}(T_e, T_e)|\Pi V : T_e.T_e|\lambda^{\Pi \mathsf{var}:T_e.T_e} V.T_e|f(T_e, \dots, T_e)$$

where in the last case, the number of arguments applied to f is $\mathsf{ar}(f)$.

As for the standard syntax, we consider a class of deductive systems indexed by a binary relation $\mathcal{R}$ on (labelled) pseudo-terms. The rules for derivation are given in Table 2. Two specific deductive systems will be considered.

Definition 8 ([2]) - Algebraic *reduction* $\to_R$ *is defined in the same way as for the standard syntax;*

[3] In other words, we only consider fully applied algebraic terms. Such a restriction is crucial when η-reduction is considered.

Axiom	$\frac{}{\vdash^e_{\mathcal{R}} c : s}$	if $(c, s) \in H$
Function	$\frac{\Gamma \vdash^e_{\mathcal{R}} t_1 : \sigma_1 \quad \ldots \quad \Gamma \vdash^e_{\mathcal{R}} t_n : \sigma_n}{\Gamma \vdash^e_{\mathcal{R}} f(t_1, \ldots, t_n) : \tau}$	if $Df = ((\sigma_1 \ldots \sigma_n), \tau)$
Start	$\frac{\Gamma \vdash^e_{\mathcal{R}} A : s}{\Gamma, x : A \vdash^e_{\mathcal{R}} x : A}$	if $x \notin \Gamma$, $x \in V$
Weakening	$\frac{\Gamma \vdash^e_{\mathcal{R}} t : A \quad \Gamma \vdash^e_{\mathcal{R}} B : s}{\Gamma, x : B \vdash^e_{\mathcal{R}} t : A}$	if $x \notin \Gamma$ and $t \in S \cup U \cup V \cup K$
Product	$\frac{\Gamma \vdash^e_{\mathcal{R}} A : s_1 \quad \Gamma, x : A \vdash^e_{\mathcal{R}} B : s_2}{\Gamma \vdash^e_{\mathcal{R}} \Pi x : A.B : s_3}$	if $(s_1, s_2, s_3) \in P$
Application	$\frac{\Gamma \vdash^e_{\mathcal{R}} t : \Pi x : A.B \quad \Gamma \vdash^e_{\mathcal{R}} u : A}{\Gamma \vdash^e_{\mathcal{R}} \mathsf{app}^{\Pi x:A.B}(t, u) : B[u/x]}$	
Abstraction	$\frac{\Gamma, x : A \vdash^e_{\mathcal{R}} t : B \quad \Gamma \vdash^e_{\mathcal{R}} (\Pi x : A.B) : s}{\Gamma \vdash^e_{\mathcal{R}} \lambda^{\Pi x:A.B} x.t : \Pi x : A.B}$	
Conversion	$\frac{\Gamma \vdash^e_{\mathcal{R}} u : A \quad \Gamma \vdash^e_{\mathcal{R}} B : s}{\Gamma \vdash^e_{\mathcal{R}} u : B}$	if $A \mathcal{R} B$

Table 2: $\mathcal{R}$-Deductive system for the labelled syntax

- Tight β-*reduction* $\rightarrow_{\beta_t}$ *is defined as the compatible closure of*

$$\mathsf{app}^{\Pi x:A.B}(\lambda^{\Pi x:A.B} x.M, N) \rightarrow M[N/x]$$

- Loose β-*reduction* $\rightarrow_{\beta_l}$ *is defined as the compatible closure of*

$$\mathsf{app}^{\Pi x:A'.B'}(\lambda^{\Pi x:A.B} x.M, N) \rightarrow M[N/x]$$

- $\rightarrow_{mixt} \quad = \quad \rightarrow_R \cup \rightarrow_{\beta_t}$ *and* $\rightarrow_{mixl} \quad = \quad \rightarrow_R \cup \rightarrow_{\beta_l}$.

- $\vdash^e_t \quad = \quad \vdash^e_{\downarrow_{mixt}}$ *and* $\vdash^e_l \quad = \quad \vdash^e_{\downarrow_{mixl}}$.

Tight β-reduction requires the abstraction and application labels to match. In contrast loose β-reduction which does not impose any condition on labels.

Lemma 9 *1.* $\rightarrow_{\beta_t}$ *is locally confluent.*

2. If $\lambda\mathbf{S}$ *is A-confluent, then* $\rightarrow_{mixt}$ *is locally confluent.*

Proof by induction on the structure of the terms. ■

It is unclear whether tight β-reduction, which is not left-linear, is confluent.

Throughout the paper, we will use the following standard terminology: a labelled pseudo-term M is *legal* w.r.t. $\vdash^e_{\mathcal{R}}$ if there is a context Γ and a pseudo-term A such that $\Gamma \vdash^e_{\mathcal{R}} M : A$. A labelled pseudo-context Γ is *legal* w.r.t. $\vdash^e_{\mathcal{R}}$ if there two pseudo-terms M and A such that $\Gamma \vdash^e_{\mathcal{R}} M : A$.

2.5 Subject Reduction for the standard syntax

Before embarking on technicalities, let us analyze where the standard proof of subject reduction breaks down. The problem arises when trying to prove subject reduction for β-reduction: as noticed in [3, 13], one cannot prove subject reduction by induction on the length of the derivations. Indeed, the induction step

$$\frac{\Gamma \vdash \lambda x : A'.b : \Pi x : A.B \quad \Gamma \vdash a : A}{\Gamma \vdash (\lambda x : A'.b)\ a : B[a/x]}$$

fails if one wants to prove $\Gamma \vdash b[a/x] : B[a/x]$. If we follow the proof of subject reduction for pure type systems (see [4, 14]), the induction step should be completed in four steps:

1. deduce from the generation lemma that $\Gamma, x : A' \vdash b : B'$ for some B' such that $\Pi x : A.B \downarrow_{mixt} C_1 \downarrow_{mixt} \ldots \downarrow_{mixt} C_n \downarrow_{mixt} \Pi x : A'.B'$ (where the C_i's are legal);
2. use confluence to derive $A \downarrow_{mixt} A'$ and $B \downarrow_{mixt} B'$;
3. apply the conversion rule and substitution to get $\Gamma \vdash b[a/x] : B'[a/x]$;
4. apply the conversion rule once more to get $\Gamma \vdash b[a/x] : B[a/x]$.

However the induction step cannot be completed (at step 2) because confluence may fail in presence of algebraic rewriting.

To circumvent this problem we propose a different strategy to develop the meta-theory of $\vdash$ for functional, A-confluent ATSs. The strategy is an adaptation of a technique applied originally on Pure Type Systems, (see [2, 19] for details). We proceed in three steps:

1. prove subject reduction of $\rightarrow_{mixt}$ for a class of deductive systems $\vdash^{e}_{\mathcal{R}}$;
2. prove strong normalisation of the labelled syntax using subject reduction if necessary;
3. deduce from functionality and strong normalisation (a) the equivalence between labelled and unlabelled syntaxes (b) confluence, strong normalisation and subject reduction for the standard syntax.

We treat Steps 1 and 3 thoroughly. Step 2 is treated in [2, 19] for Pure Type Systems and by the first author in a companion paper [7] for Algebraic Type Systems.

3 The subject reduction property for the labelled syntax

Definition 10 *$\mathcal{S}$ has the Subject Reduction Property w.r.t $\mathcal{Q}$ ($\mathcal{Q}$-SR) if*

$$\Gamma \vdash^{e}_{\mathcal{S}} t : A \text{ and } t\mathcal{Q}u \Rightarrow \Gamma \vdash^{e}_{\mathcal{S}} u : A$$

In this section, we prove the Subject Reduction property w.r.t. $\rightarrow_{mixt}$ for a large class of $\mathcal{R}$-deductive systems.

The standard proof of subject reduction (in [4, 14]) uses a *frontier* property: at each derivation step,

$$\frac{\Gamma_1 \vdash t_1 : A_1 \quad \cdots \quad \Gamma_k \vdash t_k : A_k}{\Delta \vdash u : B}$$

Δ and u can be constructed from the Γ_i's and the t_i's. Labelled systems do not fulfill this property because of the Application rule (where B appears in $\lambda^{\Pi x:A.B} x.t$). To recover this frontier property, we consider a variant $\Vdash_{\mathcal{R}}$ of the labelled syntax, where the Application rule is replaced by:

$$\text{Application+} \quad \frac{\Gamma \Vdash_{\mathcal{R}} t : \Pi x : A.B \qquad \Gamma \Vdash_{\mathcal{R}} u : A \qquad \Gamma \Vdash_{\mathcal{R}} \Pi x : A.B : s}{\Gamma \Vdash_{\mathcal{R}} \mathsf{app}^{\Pi x:A.B}(t, u) : B[u/x]}$$

Proposition 15 will show that this modification has in general no consequence on the set of derivable judgements. For now, we prove Subject Reduction for $\Vdash_{\mathcal{R}}$. Some preliminary closure results are needed.

Lemma 11 (Generation lemma) ($\mathbf{G}_c$) *if $\Gamma \Vdash_{\mathcal{R}} c : E$ and $c \in U \cup S$, there exists $s \in S$ such that $(c, s) \in H$ and $s\ \mathcal{R}^{\omega}\ E$;*

($\mathbf{G}_f$) *if $\Gamma \Vdash_{\mathcal{R}} f(t_1, \ldots, t_n) : E$ with $\mathsf{D}(f) = ((\sigma_1, \ldots, \sigma_n), \tau)$, then $\Gamma \Vdash_{\mathcal{R}} t_i : \sigma_i$ for $i = 1, \ldots, n$ and $\tau\ \mathcal{R}^{\omega}\ E$;*

($\mathbf{G}_x$) *if* $\Gamma \Vdash_{\mathcal{R}} x : E$, *then there exists* B *such that* $(x : B) \in \Gamma$ *and* $B\ \mathcal{R}^{\omega}\ E$;

($\mathbf{G}_{\mathsf{app}}$) *any derivation of* $\Gamma \Vdash_{\mathcal{R}} \mathsf{app}^{\Pi x:A.B}(M, N) : E$ *contains a derivation of* $\Gamma \Vdash_{\mathcal{R}} M : \Pi x : A.B$ *and* $\Gamma \Vdash_{\mathcal{R}} N : A$ *and* $\Gamma \Vdash_{\mathcal{R}} \Pi x : A.B : s$ *for some universe* s. *Moreover* $B[N/x]\mathcal{R}^{\omega}E$.

($\mathbf{G}_{\Pi}$) *any derivation of the judgement* $\Gamma \Vdash_{\mathcal{R}} (\Pi x : A.B) : E$ *contains derivations of* $\Gamma \Vdash_{\mathcal{R}} A : s_1$ *and* $\Gamma, x : A \Vdash_{\mathcal{R}} B : s_2$ *for some universes* s_1, s_2. *Moreover there exists* $s_3 \in U$ *such that* $(s_1, s_2, s_3) \in P$ *and* $s_3\mathcal{R}^{\omega}E$.

($\mathbf{G}_{\lambda}$) *any derivation of* $\Gamma \Vdash_{\mathcal{R}} \lambda^{\Pi x:A.B}x.b : E$ *contains a derivation of* $\Gamma, x : A \Vdash_{\mathcal{R}} b : B$ *and* $\Gamma \Vdash_{\mathcal{R}} \Pi x : A.B : s$ *for some universe* s. *Moreover* $(\Pi x : A.B)\mathcal{R}^{\omega}E$.

($\mathbf{G}_{\Gamma}$) *any derivation of* $\Gamma, x : A \Vdash_{\mathcal{R}} M : B$ *contains a derivation of* $\Gamma \Vdash_{\mathcal{R}} A : s$ *for some universe* s.

Lemma 12 (Substitution lemma) *Let* $\Gamma_1, x : A, \Gamma_2$ *be a context, let* a, b, B *be pseudo-terms. If* $\mathcal{R}$ *is closed under substitution then*

$$\left.\begin{array}{c}\Gamma_1, x : A, \Gamma_2 \Vdash_{\mathcal{R}} b : B \\ \Gamma_1 \Vdash_{\mathcal{R}} a : A\end{array}\right\} \Rightarrow \quad \Gamma_1, \Gamma_2[a/x] \Vdash_{\mathcal{R}} b[a/x] : B[a/x]$$

Lemma 13 (Correctness of Types) *(***C***) Suppose that* $\mathcal{R}$ *is closed under substitution. If* $\Gamma \Vdash_{\mathcal{R}} a : A$ *and* $A \notin U$, *then* $\Gamma \Vdash_{\mathcal{R}} A : s$ *for some universe* s.

Proof by induction on the structure of the derivation of $\Gamma \Vdash_{\mathcal{R}} a : A$. ■

The next result gives three general conditions for Subject Reduction to hold. $\mathbf{H}_1$ is needed to apply the above closure lemmas while $\mathbf{H}_2$ and $\mathbf{H}_3$ are needed to apply the induction hypothesis via a back-and-forth reasoning.

Theorem 14 (Subject Reduction Theorem) *Let* $\mathcal{R}$ *be a relation such that*

$\mathbf{H}_1$ $\mathcal{R}$ *is closed under substitution,*

$\mathbf{H}_2$ *if* $Q_1 \rightarrow_{mixt} Q_2$ *then* $Q_1\ \mathcal{R}\ Q_2$

$\mathbf{H}_3$ *if* $Q_1 \rightarrow_{mixt} Q_2$ *then* $P[Q_2/x]\ \mathcal{R}\ P[Q_1/x]$ *where* P *is any labelled pseudo-term.*

Assume $\Gamma \Vdash_{\mathcal{R}} M : A$ *and* $M \rightarrow_{mixt} M'$. *Then* $\Gamma \Vdash_{\mathcal{R}} M' : A$.

Proof see Appendix. ■

Proposition 15 *If* $\mathcal{R}$ *is closed under substitution then for every judgement* (Γ, M, A):

$$\Gamma \vdash^{e}_{\mathcal{R}} M : A \quad \Leftrightarrow \quad \Gamma \Vdash_{\mathcal{R}} M : A$$

Proof both implications are proved by induction on the structure of derivations. The implication ($\Rightarrow$) is proved using Correctness of Types. ■

Corollary 16 *If* $\mathcal{R}$ *verifies the hypotheses* $\mathbf{H}_1$, $\mathbf{H}_2$ *and* $\mathbf{H}_3$, *then it has the* $\rightarrow_{mixt}$*-SR property. In particular,* $\vdash^{e}_{l}$ *and* $\vdash^{e}_{t}$ *have the* $\rightarrow_{mixt}$*-SR property.*

4 Equivalence results

In this section, we establish under certain conditions an equivalence between (a) labelled deductive systems (b) $\vdash^{e}_{t}$ and $\vdash$. Only the most important equivalence results are stated here. There are further, more general, results which we omit for the lack of space.

4.1 A general equivalence result for labelled deductive systems

Throughout this subsection, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$ denote binary relations on labelled pseudo-terms.

Definition 17

- $\mathcal{R} \sqsubseteq \mathcal{S}$ *if for all judgements* (Γ, M, A),

$$\Gamma \vdash^{e}_{\mathcal{R}} M : A \Rightarrow \Gamma \vdash^{e}_{\mathcal{S}} M : A$$

- $\mathcal{R} \simeq \mathcal{S}$ *if* $\mathcal{R} \sqsubseteq \mathcal{S} \sqsubseteq \mathcal{R}$.

- $\mathcal{R} < \mathcal{S}$ *if for all judgements* (Γ, M, A),

$$(\Gamma \vdash^{e}_{\mathcal{S}} M : A \quad \text{and} \quad \Gamma \vdash^{e}_{\mathcal{S}} B : s \quad \text{and} \quad A\mathcal{R}B) \quad \Rightarrow \quad \Gamma \vdash^{e}_{\mathcal{S}} M : B$$

- $\mathcal{R} \lesseqgtr \mathcal{S}$ *if* $\mathcal{R} < \mathcal{S} < \mathcal{R}$.

Remark that $<$ is *not* transitive. Working at an abstract level, we show that all the labelled deductive systems satisfying certain properties are equivalent.

Proposition 18 $\mathcal{R} < \mathcal{S} \Rightarrow \mathcal{R} \sqsubseteq \mathcal{S}$ *and* $\mathcal{R} \lesseqgtr \mathcal{S} \Leftrightarrow \mathcal{R} \simeq \mathcal{S}$.

Proof see Appendix. ■

Proposition 19 *Assume that $\mathcal{S}$ has the $\mathcal{Q}$-SR property and is closed under substitutions.*

$$\begin{array}{rcl} \mathcal{Q} < \mathcal{S} \text{ and } \mathcal{R} < \mathcal{S} & \Rightarrow & \mathcal{Q} \cdot \mathcal{R} < \mathcal{S} \\ \mathcal{R} < \mathcal{S} \text{ and } \mathcal{Q}^{op} < \mathcal{S} & \Rightarrow & \mathcal{R} \cdot \mathcal{Q}^{op} < \mathcal{S} \\ \mathcal{Q}^{\leftrightarrow} < \mathcal{S} & \Rightarrow & \mathcal{S} \simeq \mathcal{Q}^{\omega} \cdot \mathcal{S} \cdot (\mathcal{Q}^{op})^{\omega} \end{array}$$

Proof see Appendix. ■

Corollary 20 (Equivalence Lemma) *Assume that $\mathcal{S}$ has the $\mathcal{Q}$-SR property and is closed under substitutions.*

$$\begin{array}{rcl} \mathcal{Q}^{\leftrightarrow} < \mathcal{S} & \Rightarrow & \downarrow_{\mathcal{Q}} \sqsubseteq \mathcal{S} \\ \mathcal{Q}^{\leftrightarrow} < \mathcal{S} < \downarrow_{\mathcal{Q}} & \Rightarrow & \downarrow_{\mathcal{Q}} \simeq \mathcal{S} \end{array}$$

Proof see Appendix. ■

Theorem 21 (Equivalence theorem) *Let $\mathcal{R}$ verify the hypotheses* $\mathbf{H}_1$, $\mathbf{H}_2$ *and* $\mathbf{H}_3$. *Then* $\downarrow_{mixt} \sqsubseteq \mathcal{R}$. *Moreover* $\mathcal{R} < \downarrow_{mixt} \quad \Rightarrow \quad \downarrow_{mixt} \simeq \mathcal{R}$.

Proof see Appendix. ■

4.2 More labelled equivalences

In this subsection, we prove two further equivalence results for the labelled syntax. Both results will be used to prove the equivalence between the labelled and unlabelled syntaxes.

The first result is concerned with showing that under suitable conditions, $\vdash^{e}_{\downarrow_{\mathcal{R}}}$ is equivalent to another, easier to use, deductive system.

Definition 22 *Let $\mathcal{R}$ be a binary relation. The relation* $\mathbf{T}(\mathcal{R})$ *is defined by*

$$t \ \mathbf{T}(\mathcal{R}) \ u \quad \Longleftrightarrow \quad (t \ \mathcal{R} \ u \text{ and } t \text{ and } u \text{ are legal w.r.t. } \vdash^{e}_{\downarrow_{\mathcal{R}}})$$

We have:

Lemma 23 *If $\mathcal{R}$ is closed under substitution and $\vdash^{e}_{\downarrow_{\mathcal{R}}}$ has $\mathcal{R}$-SR then $\downarrow_{\mathcal{R}}\simeq\downarrow_{\mathbf{T}(\mathcal{R})}$.*

Proof see Appendix. ■

We write $\rightarrow_{\mathbf{T}(mixt)}$ for $\mathbf{T}(\rightarrow_{mixt})$.

Corollary 24 $\downarrow_{\mathbf{T}(mixt)}\simeq\downarrow_{mixt}$.

An interesting point about $\rightarrow_{\mathbf{T}(mixt)}$ is that it is confluent when $\rightarrow_{mixt}$ is canonical on legal terms of $\vdash^{e}_{t}$.

The second result is concerned with an equivalence between $\vdash^{e}_{t}$ and $\vdash^{e}_{l}$. We start with a preliminary result:

Lemma 25 *Assume $\rightarrow_{\mathbf{T}(mixt)}$ is confluent. For every judgement $\Gamma \vdash^{e}_{t} M : A$ and labelled pseudo-term M',*

$$M \rightarrow_{mixl} M' \quad \Rightarrow \quad M \twoheadrightarrow^{+}_{mixt} M'$$

Proof see Appendix. ■

Proposition 26 *Assume $\rightarrow_{\mathbf{T}(mixt)}$ is confluent. Then $\downarrow_{mixt}\simeq\downarrow_{mixl}$.*

Proof the direct inclusion $\downarrow_{mixt}\sqsubseteq\downarrow_{mixl}$ follows from the inclusion $\downarrow_{mixt} \subseteq \downarrow_{mixl}$. The reverse inclusion $\downarrow_{mixl}\sqsubseteq\downarrow_{mixt}$ follows from Equivalence Lemma: Remark that we need Lemma 25 to show that $\vdash_{t}$ has the $\rightarrow_{mixl}$-SR property and that $(\downarrow_{mixl})^{\leftrightarrow} <\downarrow_{mixt}$. ■

4.3 Equivalence between labelled and unlabelled syntaxes

In this section, we establish the main equivalence result between labelled and unlabelled syntaxes. For the lack of space, we only consider the equivalence between $\vdash$ and $\vdash^{e}_{t}$.

There is an obvious translation from T_e to T which erases labels:

Definition 27 (The translation) *The map $||\cdot|| : T_e \rightarrow T$ is defined inductively as follows:*

- $||x|| = x$
- $||s|| = s$,
- $||\tau|| = \tau$,
- $||\Pi x : A.B|| = \Pi x : ||A||.||B||$,
- $||f(t_1, \ldots, t_n)|| = f(||t_1||, \ldots, ||t_n||)$,
- $||\lambda^{\Pi x:A.B} x.M|| = \lambda x : ||A||.||M||$,
- $||\mathsf{app}^{\Pi x:A.B}(M, N)|| = ||M||\ ||N||$.

Erasure preserves typing.

Lemma 28 $\Gamma \vdash^{e}_{t} M : A \quad \Rightarrow \quad ||\Gamma|| \vdash ||M|| : ||A||$

Proof by an easy structural induction on the derivation of $\Gamma \vdash^{e}_{t} M : A$. ■

The fundamental fact about labels is that, under suitable conditions, every derivable judgement can be labelled without losing derivability. Throughout this subsection, we assume:

Assumption 29 *$\lambda\mathbf{S}$ is a functional algebraic type system. Moreover $\rightarrow_{mixt}$ is canonical on legal terms of $\vdash^{e}_{t}$.*

We start with some preliminary results.

Proposition 30 (Unicity of types) *Assume $\Gamma \vdash_t^e M : A$ and $\Gamma \vdash_t^e M : B$. Then $A =_{T(mixt)} B$.*

Proof by induction on the structure of the derivation of $\Gamma \vdash_t^e M : A$. ■

Corollary 31 *Assume that $\Gamma \vdash_t^e M : A$ and $\Delta \vdash_t^e N : B$. If $\Gamma =_{\mathbf{T}(mixt)} \Delta$ and $M =_{\mathbf{T}(mixt)} N$, then $A =_{\mathbf{T}(mixt)} B$.*

Proof by confluence of $\to_{T(mixt)}$, there exists Ξ and P such that $\Gamma, \Delta \twoheadrightarrow_{T(mixt)} \Xi$ and $M, N \twoheadrightarrow_{T(mixt)} P$. By Subject Reduction, $\Xi \vdash_t^e P : A$ and $\Xi \vdash_t^e P : B$. By Unicity of Types, $A =_{\mathbf{T}(mixt)} B$. ■

Next we define for each legal term its canonical form.

Definition 32 (canonical forms) *Let M be legal. We define M^{can} as:*

- $x^{\mathsf{can}} = x$
- $s^{\mathsf{can}} = s$
- $\tau^{\mathsf{can}} = \tau$
- $(\Pi x : A.B)^{\mathsf{can}} = \Pi x : A^{\mathsf{can}}.B^{\mathsf{can}}$
- $(f(t_1, \ldots, t_n))^{\mathsf{can}} = f(t_1^{\mathsf{can}}, \ldots, t_n^{\mathsf{can}})$
- $(\lambda^{\Pi x:A.B} x.M)^{\mathsf{can}} = \lambda^{\Pi x:A^{\mathsf{can}}.B^{\mathsf{nf}}} x.M^{\mathsf{can}}$
- $(\mathsf{app}^{\Pi x:A.B}(M, N))^{\mathsf{can}} = \mathsf{app}^{\Pi x:A^{\mathsf{nf}}.B^{\mathsf{nf}}}(M^{\mathsf{can}}, N^{\mathsf{can}})$

where A^{nf} denotes the normal form of A w.r.t $\to_{mixt}$.

We remark that $||M|| = ||M^{\mathsf{can}}||$. One important property of $.^{\mathsf{can}}$ is that it identifies terms which have equal erasures.

Lemma 33 (Unicity of the canonical translation)

1. *for every legal contexts Γ and Δ, $||\Delta|| \equiv ||\Gamma|| \Rightarrow \Delta^{\mathsf{can}} \equiv \Gamma^{\mathsf{can}}$.*
2. *for every derivations $\Gamma \vdash_t^e M : C$ and $\Delta \vdash_t^e N : D$,*

$$(||\Gamma|| \equiv ||\Delta||) \wedge (||M|| \equiv ||N||) \quad \Rightarrow \quad M^{\mathsf{can}} \equiv N^{\mathsf{can}}$$

Proof See Appendix. ■

In order to be able to prove the equivalence between $\vdash$ and $\vdash_t^e$, it is necessary to show that standard reductions may be lifted to labelled ones. The following result is also useful to deduce subject reduction and strong normalisation from labelled subject reduction and strong normalisation.

Lemma 34 *Assume $\Gamma \vdash_t^e M : A$.*

1. *If $||M|| \to_{mix} N$ then there exists N' such that $M \twoheadrightarrow_{mixl}^+ N'$ and $||N'|| \equiv N$.*
2. *If $||M|| \twoheadrightarrow_{mix}^+ N$ then there exists N' such that $M \twoheadrightarrow_{mixt}^+ N'$ and $||N'|| \equiv N$.*
3. *$\to_{mix}$ is confluent and strongly normalising on $||M||$.*
4. *If $\Gamma \vdash_t^e N : B$ with $||M|| \downarrow_{mix} ||N||$, then $M \downarrow_{mixt} N$.*

Proof see Appendix. ■

Collecting the previous results, we get:

Proposition 35 *If $\Gamma \vdash M : A$ is derivable then there exists a derivable judgement $\Gamma_\bullet \vdash_t^e M_\bullet : A_\bullet$ such that $\Gamma_\bullet, M_\bullet$ and $A_\bullet$ are canonical and*

$$||\Gamma_\bullet|| \equiv \Gamma \text{ and } ||M_\bullet|| \equiv M \text{ and } ||A_\bullet|| \equiv A$$

Proof see Appendix ■

4.4 Aside: deductive systems with one-step conversion rule

Algebraic type systems are often defined with a deductive system using one-step conversion ([3, 6, 13]). Unfortunately, it seems unclear how to prove Subject Reduction or an equivalence result for those systems. Yet there is a slightly bigger deductive system for which Subject Reduction holds.

Definition 36 *$\rightarrow_{mix1}$ is the smallest reduction relation on labelled pseudo-terms such that for every $M, N, N' \in T_e$ and $x \in \mathsf{FV}(M)$, $N \rightarrow_{mixt} N' \quad \Rightarrow \quad M[N/x] \rightarrow_{mix1} M[N'/x]$. The relation* $\mathsf{1step}$ *is defined as the symmetric closure of $mix1$.*

We have

Corollary 37 $\mathsf{1step}$ *has the $\rightarrow_{mixt}$-SR property and* $\mathsf{1step} \simeq \downarrow_{mixt}$*.*

Proof the first part follows from Theorem 14; the second part from the first and Theorem 21. ■

5 Application

Under suitable conditions, subject reduction for $\vdash$ can be deduced from subject reduction of $\vdash^e_t$.

Proposition 38 *Let $\lambda\mathsf{S}$ be a functional* ATS*. Assume $\rightarrow_{mixt}$ is canonical on legal terms of $\vdash^e_t$. Then $\vdash$ has the Subject Reduction property w.r.t. $\rightarrow_{mix}$. Moreover $\rightarrow_{mix}$ is canonical on legal terms of $\vdash$.*

Proof Assume $\Gamma \vdash M : A$. By Proposition 35, there exists a derivation $\Gamma_\bullet \vdash^e_t M_\bullet : A_\bullet$ with the expected translation property. By Lemma 34, there exists $N_\bullet$ such that $M_\bullet \twoheadrightarrow_{mixt} N_\bullet$ with $||N_\bullet|| \equiv N$. By Subject Reduction, $\Gamma_\bullet \vdash^e_t N_\bullet : A_\bullet$ and by translation $\Gamma \vdash N : A$. The second part of the proposition follows from Lemma 34. ■

As a corollary, we get strong normalisation of the algebraic λ-cube.

Corollary 39 *A-canonical systems of the algebraic λ-cube are strongly normalising.*

Proof by Proposition 38, it is enough to prove strong normalisation of $\rightarrow_{mixt}$ on legal terms of $\vdash^e_t$. See [7] for such a proof. ■

6 Conclusion

Proving the equivalence between the various formulations of algebraic or pure type systems is a vital exercise. It contributes to a better understanding of type systems and allows to derive results from one formalism to another. The main technical contribution of this paper is a proof of subject reduction for functional, A-confluent algebraic type systems which are strongly normalising for the labelled syntax. Although we have been unable to prove Subject Reduction for an arbitrary algebraic type system, our result is interesting because it is based on a simple technique and applies to an important class of algebraic type systems. Moreover, the technique in itself is interesting as it is very general and may be used in other type-theoretic frameworks where the reduction relation is not confluent on pseudo-terms. These include:

- pure and algebraic type systems with η-reduction,
- pure type systems with congruence types ([5])
- classical pure type systems ([8]).

As such, it constitutes the first general technique to prove subject reduction for (unlabelled) type systems with a non-confluent reduction relation.

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Appendix: proofs

Proof of Theorem 14 the proof proceeds along the same lines as in [4]. The following two facts are proved by simultaneous induction on the structure of derivations:

1. If $\Gamma \Vdash_{\mathcal{R}} M : C$ and $M \to_{mixt} M'$, then $\Gamma \Vdash_{\mathcal{R}} M' : C$.
2. If $\Gamma \Vdash_{\mathcal{R}} M : C$ and $\Gamma \to_{mixt} \Gamma'$, then $\Gamma' \Vdash_{\mathcal{R}} M : C$.

We treat the cases where the last rule is an abstraction, an application or a function rule:

- *abstraction rule:* assume $M \equiv \lambda^{\Pi x:A.B} x.t$ and $C \equiv \Pi x : A.B$. 2 follows from the induction hypothesis. As for 1, the interesting case is when the reduction occurs in A or in B, i.e. $M' \equiv \lambda^{\Pi x:A'.B} x.t$ or $M' \equiv \lambda^{\Pi x:A.B'} x.t$.

  Subcase 1: $M' \equiv \lambda^{\Pi x:A'.B} x.t$. In this case, we use the induction hypothesis to conclude $\Gamma, x : A' \Vdash_{\mathcal{R}} t : B$ and $\Gamma \Vdash_{\mathcal{R}} \Pi x : A'.B : s$. We may then apply the abstraction rule to get $\Gamma \Vdash_{\mathcal{R}} \lambda^{\Pi x:A'.B} x.t : \Pi x : A'.B$. By $\mathbf{H}_3$, we may apply the conversion rule to get $\Gamma \Vdash_{\mathcal{R}} \lambda^{\Pi x:A'.B} x.t : \Pi x : A.B$.

  Subcase 2: $M' \equiv \lambda^{\Pi x:A.B'} x.t$. By induction hypothesis, $\Gamma \Vdash_{\mathcal{R}} \Pi x : A.B' : s$. By $\mathbf{G}_\Pi$, $\Gamma, x : A \Vdash_{\mathcal{R}} B' : s'$ for a universe s'. We apply the conversion rule thanks to $\mathbf{H}_2$ to deduce $\Gamma, x : A \Vdash_{\mathcal{R}} M : B'$. By abstraction, $\Gamma \Vdash_{\mathcal{R}} \lambda^{\Pi x:A.B'} x.M : (\Pi x : A.B')$. We next apply the conversion rule thanks to $\mathbf{H}_3$ to deduce $\Gamma \Vdash_{\mathcal{R}} \lambda^{\Pi x:A.B'} x.M : (\Pi x : A.B)$.

- *application rule:* assume $M \equiv \mathsf{app}^{\Pi x:A.B}\ t\ u$ and $C \equiv B[u/x]$. It is easy to prove 2. As for 1, we treat four subcases:

  Subcase 1: top reduction: $M \equiv \mathsf{app}^{\Pi x:A.B}(\lambda^{\Pi x:A.B} x.b, u)$ and $M' \equiv b[u/x]$. By $\mathbf{G}_\lambda$: $\Gamma, x : A \Vdash_{\mathcal{R}} b : B$. By the Substitution Lemma, $\Gamma \Vdash_{\mathcal{R}} b[u/x] : B[u/x]$.

  Subcase 2: inside u: $M \equiv \mathsf{app}^{\Pi x:A.B}(t, u)$ and $M' \equiv \mathsf{app}^{\Pi x:A.B}(t, u')$ with $u \to_{mixt} u'$. By induction hypothesis, $\Gamma \Vdash_{\mathcal{R}} u' : A$. By construction, $\Gamma \Vdash_{\mathcal{R}} (\Pi x : A.B) : s$. By application+, $\Gamma \Vdash_{\mathcal{R}} \mathsf{app}^{\Pi x:A.B}(t, u') : B[u'/x]$. By $\mathbf{G}_\Pi$: $\Gamma, x : A \Vdash_{\mathcal{R}} B : s'$ for some universe s'. By substitution lemma, $\Gamma \Vdash_{\mathcal{R}} B[u/x] : s'$ We apply a conversion rule thanks to $\mathbf{H}_3$ to deduce $\Gamma \Vdash_{\mathcal{R}} \mathsf{app}^{\Pi x:A.B}(t, u') : B[u/x]$.

  Subcase 3: inside B: just like Subcase 2 of the abstraction rule: if $M \equiv \mathsf{app}^{\Pi x:A.B}(t, u)$ and $M' \equiv \mathsf{app}^{\Pi x:A.B'}(t, u)$ with $B \to_{mixt} B'$ then by induction hypothesis on the premise $\Gamma \Vdash_{\mathcal{R}} (\Pi x : A.B) : s$ we deduce $\Gamma \Vdash_{\mathcal{R}} (\Pi x : A.B') : s$. By conversion thanks to $\mathbf{H}_2$ we deduce $\Gamma \Vdash_{\mathcal{R}} t : (\Pi x : A.B')$. By application+, $\Gamma \Vdash_{\mathcal{R}} \mathsf{app}^{\Pi x:A.B'}(t, u) : B'[u/x]$. By $\mathbf{G}_\Pi$, $\Gamma, x : A \Vdash_{\mathcal{R}} B : s'$ for some universe s'. By substitution lemma, $\Gamma \Vdash_{\mathcal{R}} B[u/x] : s'$. By conversion and $\mathbf{H}_3$, $\Gamma \Vdash_{\mathcal{R}} \mathsf{app}^{\Pi x:A.B'}(t, u) : B[u/x]$.

Subcase 4: inside A: simpler than the preceding case because A does not appear in the type of $\mathsf{app}^{\Pi x:A.B}(t,u)$.

- *function rule:* if $M \equiv f(t_1,\ldots,t_n)$ with $\mathsf{D}(f) = ((\sigma_1,\ldots,\sigma_n),\tau)$, then $A \equiv \tau$. The only interesting case here is when M is a redex, i.e. when $M \equiv f(t_1,\ldots,t_n)$ is matched to a rewrite rule $l \rightarrow r$ (of sort τ) by some substitution θ. So let $M \equiv \theta l$ and $M' \equiv \theta r$.

  Fact 40 *Assume M is an algebraic term of sort τ. Assume $\mathsf{FV}(M) = \{x_1,\ldots,x_n\}$ with $x_i \in V_{\sigma_i}$ for $i = 1,\ldots,n$. Then $\Gamma \equiv x_1 : \sigma_1,\ldots,x_n : \sigma_n \Vdash_{\mathcal{R}} M : \tau$.*

  So we know $\Delta \Vdash_{\mathcal{R}} l : \tau$ and $\Delta \Vdash_{\mathcal{R}} r : \tau$ for the canonical context Δ associated to l. Moreover, $\Gamma \Vdash_{\mathcal{R}} \theta x_i : \tau_i$ for every $(x_i : \tau_i) \in \Delta$. By substitution, $\Gamma \Vdash_{\mathcal{R}} M' : \tau$.

■

Proof of Proposition 18 the first statement is proved by structural induction on the derivations of $\vdash^e_{\mathcal{R}}$. The direct implication of the second statement follows immediately. As for the reverse implication of the second statement, suppose that $\mathcal{R} \simeq \mathcal{S}$; we show $\mathcal{R} < \mathcal{S}$. Assume

$$\Gamma \vdash^e_{\mathcal{S}} M : A \quad \Gamma \vdash^e_{\mathcal{S}} B : s \quad A\mathcal{R}B$$

implies thanks to $\mathcal{S} \sqsubseteq \mathcal{R}$ that

$$\Gamma \vdash^e_{\mathcal{R}} M : A \quad \Gamma \vdash^e_{\mathcal{R}} B : s \quad A\mathcal{R}B$$

By $\mathcal{R}$-conversion, $\Gamma \vdash^e_{\mathcal{R}} M : B$. By $\mathcal{R} \simeq \mathcal{S}$: $\Gamma \vdash^e_{\mathcal{S}} M : B$. Henceforth $\mathcal{R} < \mathcal{S}$ and symmetrically $\mathcal{R} \lesseqgtr \mathcal{S}$. Remark that $\mathcal{R} \lesseqgtr \mathcal{S}$ implies $\mathcal{R} \simeq \mathcal{S}$ with the first statement. So we are done. ■

Proof of Proposition 19 We prove the first statement. Suppose that

$$\Gamma \vdash^e_{\mathcal{S}} M : A \quad \Gamma \vdash^e_{\mathcal{S}} B : s \quad A(\mathcal{Q} \cdot \mathcal{R})B$$

There exists a pseudo-term C such that $A\mathcal{Q}C\mathcal{R}B$. We use that $\mathcal{S}$ is closed by substitution: $\Gamma \vdash^e_{\mathcal{S}} A : s'$ for some universe s' follows by Correctness of Types. By $\mathcal{Q}$-SR: $\Gamma \vdash^e_{\mathcal{S}} C : s'$. By $\mathcal{Q} < \mathcal{S}$: $\Gamma \vdash^e_{\mathcal{S}} M : C$. By $\mathcal{R} < \mathcal{S}$: $\Gamma \vdash^e_{\mathcal{S}} M : B$.

The proof of the second statement is (nearly) dual. Suppose that:

$$\Gamma \vdash^e_{\mathcal{S}} M : A \quad \Gamma \vdash^e_{\mathcal{S}} B : s \quad A(\mathcal{R} \cdot \mathcal{Q}^{op})B$$

There is a pseudo-term C such that $A\mathcal{R}C\mathcal{Q}^{op}B$. By $\mathcal{Q}$-SR: $\Gamma \vdash^e_{\mathcal{S}} C : s$. By $\mathcal{R} < \mathcal{S}$: $\Gamma \vdash^e_{\mathcal{S}} M : C$. By $\mathcal{Q}^{op} < \mathcal{S}$: $\Gamma \vdash^e_{\mathcal{S}} M : B$. We are done.

The proof of the third statement: Note that $\mathcal{S} < \mathcal{S}$. Hence, we may apply the first and second statement as many times as wished. By continuity of $<$:

$$\mathcal{Q}^{\omega} \cdot \mathcal{S} \cdot (\mathcal{Q}^{op})^{\omega} < \mathcal{S}$$

We deduce from $\mathcal{S} \subseteq \mathcal{Q}^{\omega} \cdot \mathcal{S} \cdot (\mathcal{Q}^{op})^{\omega}$ that $\mathcal{S} \sqsubseteq \mathcal{Q}^{\omega} \cdot \mathcal{S} \cdot (\mathcal{Q}^{op})^{\omega}$. We are done with the first statement of proposition 18. ■

Proof of Corollary 20 the last statement is easy to prove with proposition 18. As for the first statement, we prove the following sequence of inequalities

$$\downarrow_{\mathcal{Q}} \equiv \mathcal{Q}^{\omega} \cdot (\mathcal{Q}^{op})^{\omega} \sqsubseteq \mathcal{Q}^{\omega} \cdot \underline{\mathcal{S}} \cdot (\mathcal{Q}^{op})^{\omega} \simeq \underline{\mathcal{S}} \simeq \mathcal{S}$$

We proceed in reverse order. $\underline{\mathcal{S}} \simeq \mathcal{S}$ is easy. It follows that $\underline{\mathcal{S}}$ has the $\mathcal{Q}$-SR property and that $\mathcal{Q} < \underline{\mathcal{S}}$. We apply Proposition 19 to get $\underline{\mathcal{S}} \simeq \mathcal{Q}^{\omega} \cdot \underline{\mathcal{S}} \cdot (\mathcal{Q}^{op})^{\omega}$. The last inequality follows $\mathcal{Q}^{\omega} \cdot (\mathcal{Q}^{op})^{\omega} \subseteq \mathcal{Q}^{\omega} \cdot \underline{\mathcal{S}} \cdot (\mathcal{Q}^{op})^{\omega}$. ■

Proof of Theorem 21 we only prove the first part as the second part is easy. Let $\mathcal{Q}$ be $\rightarrow_{mixt}$. It follows from $\mathbf{H}_2$ and $\mathbf{H}_3$ that $\mathcal{Q}^{\leftrightarrow} \subseteq \mathcal{R}$ therefore $\mathcal{Q}^{\leftrightarrow} < \mathcal{R}$. On the other hand Theorem 14 shows that $\mathcal{R}$ has the $\mathcal{Q}$-SR property. Hence we can apply corollary 20 to get $\downarrow_{mixt} \sqsubseteq \mathcal{R}$. ■

Proof of Lemma 23 the direction $\downarrow_{\mathbf{T}(\mathcal{R})} \sqsubseteq \downarrow_{\mathcal{R}}$ is the consequence of $\mathbf{T}(\mathcal{R}) \subset \mathcal{R}$. The reverse direction is a nice application of lemma 18. To prove that $\mathcal{R} < \downarrow_{\mathbf{T}(\mathcal{R})}$ suppose that $\Gamma \vdash^{e}_{\downarrow_{\mathbf{T}(\mathcal{R})}} M : A$ and $\Gamma \vdash^{e}_{\downarrow_{\mathbf{T}(\mathcal{R})}} B : s$ and $A \downarrow_{\mathcal{R}} B$. The following properties induce $A \downarrow_{\mathbf{T}(\mathcal{R})} B$:

1. the relation $\downarrow_{\mathcal{R}}$ is closed under substitution, so A is legal w.r.t. $\vdash^{e}_{\downarrow_{\mathcal{R}}}$ by Correctness of Types,
2. B is legal, $A \downarrow_{\mathcal{R}} B$ and $\vdash^{e}_{\downarrow_{\mathcal{R}}}$ has $\mathcal{R}$-SR.

Hence, the Conversion rule can be applied in $\vdash^{e}_{\downarrow_{\mathbf{T}(\mathcal{R})}}$ in order to get $\Gamma \vdash^{e}_{\downarrow_{\mathbf{T}(\mathcal{R})}} M : B$. We conclude that $\downarrow_{\mathcal{R}} < \downarrow_{\mathbf{T}(\mathcal{R})}$ and so $\downarrow_{\mathcal{R}} \sqsubseteq \downarrow_{\mathbf{T}(\mathcal{R})}$. ■

Proof of Lemma 25 by induction on the length of the derivation. Note that we only have to prove the result for $M \rightarrow_{\beta_l} M'$ as $\rightarrow_{R} \subseteq \rightarrow_{mixt}$. The only interesting case is when the last rule is an application rule and the subject of the judgement is a redex w.r.t. $\rightarrow_{\beta_l}$. So assume the last rule is

$$\frac{\Gamma \vdash^{e}_{t} \lambda^{\Pi x:A.B} x.t : (\Pi x : A'.B') \qquad \Gamma \vdash^{e}_{t} u : A'}{\Gamma \vdash^{e}_{t} \mathsf{app}^{\Pi x:A'.B'}(\lambda^{\Pi x:A.B} x.t, u) : B'[u/x]}$$

with $M \equiv \mathsf{app}^{\Pi x:A'.B'}(\lambda^{\Pi x:A.B} x.t, u)$ and $M' \equiv t[u/x]$. To show that $M \twoheadrightarrow_{mixt} M'$. We use the fact that $\downarrow_{\mathbf{T}(mixt)} \simeq \downarrow_{mixt}$. By generation on $\vdash^{e}_{\downarrow_{\mathbf{T}(mixt)}}$ (which is equivalent to $\vdash^{e}_{t}$), $\Pi x : A.B \;\; =_{\mathbf{T}(mixt)} \;\; \Pi x : A'.B'$. By confluence of $\rightarrow_{\mathbf{T}(mixt)}$, there exists A'' and B'' such that $A, A' \twoheadrightarrow_{mixt} A''$ and $B, B' \twoheadrightarrow_{mixt} B''$. Therefore

$$M \twoheadrightarrow_{mixt} \mathsf{app}^{\Pi x:A''.B''}(\lambda^{\Pi x:A''.B''} x.t, u) \rightarrow_{\beta_t} t[u/x]$$

and we are done. ■

Proof of Lemma 33 by induction on the derivation of $\Gamma \vdash^{e}_{t} M : C$. We treat the case where the last rule is an application, an abstraction or a weakening:

- *application:* assume the last step is

$$\frac{\Gamma \vdash^{e}_{t} t : \Pi x : A.B \qquad \Gamma \vdash^{e}_{t} u : A}{\Gamma \vdash^{e}_{t} \mathsf{app}^{\Pi x:A.B}(t, u) : B[u/x]}$$

  with $M \equiv \mathsf{app}^{\Pi x:A.B}(t, u)$. 1 is easy to prove. As for 2, assume $N \equiv \mathsf{app}^{\Pi x:A'.B'}(t', u')$, $||\Gamma|| \equiv ||\Delta||$ and $||M|| \equiv ||N||$. We show $M^{\mathsf{can}} \equiv N^{\mathsf{can}}$, i.e.

$$\mathsf{app}^{\Pi x:A^{\mathsf{nf}}.B^{\mathsf{nf}}}(t^{\mathsf{can}}, u^{\mathsf{can}}) \equiv \mathsf{app}^{\Pi x:A'^{\mathsf{nf}}.B'^{\mathsf{nf}}}(t'^{\mathsf{can}}, u'^{\mathsf{can}})$$

  By $\mathbf{G}_{\mathsf{app}}$, $\Delta \vdash^{e}_{t} t' : \Pi x : A'.B'$ and $\Delta \vdash^{e}_{t} u' : A'$. Note that $||M|| \equiv ||N|| \Rightarrow (||t|| \equiv ||t'||$ and $||u|| \equiv ||u'||)$. We can use the induction hypothesis on the premises $\Gamma \vdash^{e}_{t} t : \Pi x : A.B$ and $\Gamma \vdash^{e}_{t} u : A$ and deduce that $\Gamma^{\mathsf{can}} \equiv \Delta^{\mathsf{can}}$, $t^{\mathsf{can}} \equiv t'^{\mathsf{can}}$ and $u^{\mathsf{can}} \equiv u'^{\mathsf{can}}$. By $\rightarrow_{mixt}$-SR, $\Gamma \downarrow_{\mathbf{T}(mixt)} \Delta$ and $t \downarrow_{\mathbf{T}(mixt)} t'$. By corollary 31 applied on

$$\Gamma \vdash^{e}_{t} t : (\Pi x : A.B) \text{ and } \Delta \vdash^{e}_{t} t' : (\Pi x : A'.B')$$

  we deduce $(\Pi x : A.B) =_{\mathbf{T}(mixt)} (\Pi x : A'.B')$. By confluence, $(\Pi x : A.B) \downarrow_{\mathbf{T}(mixt)} (\Pi x : A'.B')$. Hence $A \downarrow_{mixt} A'$ and $B \downarrow_{mixt} B'$. By Correctness of Types, $\Gamma \vdash^{e}_{t} \Pi x : A.B : s$ and $\Delta \vdash^{e}_{t} \Pi x : A'.B' : s'$ and hence $\Pi x : A.B$ and $\Pi x : A'.B'$ are strongly normalising. So $A^{\mathsf{nf}} \equiv A'^{\mathsf{nf}}$ and $B^{\mathsf{nf}} \equiv B'^{\mathsf{nf}}$, and we are done.

- *abstraction:* assume the last step is

$$\frac{\Gamma, x : A \vdash^{e}_{t} t : B \qquad \Gamma \vdash^{e}_{t} \Pi x : A.B : s}{\Gamma \vdash^{e}_{t} \lambda^{\Pi x:A.B} x.t : \Pi x : A.B}$$

with $M \equiv \lambda^{\Pi x:A.B} x.t$. 1 is easy to prove. As for 2, assume $N \equiv \lambda^{\Pi x:A'.B'} x.t'$, $||\Gamma|| \equiv ||\Delta||$ and $||M|| \equiv ||N||$. To show $M^{\mathsf{can}} \equiv N^{\mathsf{can}}$, i.e.

$$\lambda^{\Pi x:A^{\mathsf{can}}.B^{\mathsf{nf}}} x.t^{\mathsf{can}} \equiv \lambda^{\Pi x:A'^{\mathsf{can}}.B'^{\mathsf{nf}}} x.t'^{\mathsf{can}}$$

By $\mathbf{G}_\lambda$, $\Delta, x : A' \vdash_t^e t' : B'$. Note that $||M|| \equiv ||N||$ implies that $||A|| \equiv ||A'||$, hence $||\Gamma, x : A|| \equiv ||\Delta, x : A'||$. We can use the induction hypothesis on $\Gamma, x : A \vdash_t^e t : B$ and deduce $t^{\mathsf{can}} \equiv t'^{\mathsf{can}}$ and $A^{\mathsf{can}} \equiv A'^{\mathsf{can}}$.

We are left to show $B^{\mathsf{nf}} \equiv B'^{\mathsf{nf}}$. By corollary 31, $B =_{\mathbf{T}(mixt)} B'$. By confluence, $B \downarrow_{\mathbf{T}(mixt)} B'$. By Correctness of Types, either B is a sort or $\Gamma, x : A \vdash_t^e B : s_0$. Similarly, either B' is a sort or $\Delta, x : A' \vdash_t^e B' : s_1$. In all cases both B and B' are strongly normalising. Hence $B^{\mathsf{nf}} \equiv B'^{\mathsf{nf}}$ and we are done.

- *weakening:* assume the last step is

$$\frac{\Gamma \vdash_t^e M : B \qquad \Gamma \vdash_t^e A : s}{\Gamma, x : A \vdash_t^e M : B}$$

with M a variable or a sort or an universe. Assume $\Delta, x : A'$ is a legal context with $||\Gamma, x : A|| \equiv ||\Delta, x : A'||$. Then $||\Gamma|| \equiv ||\Delta||$ and $||A|| \equiv ||A'||$. Necessarily, $\Delta \vdash_t^e A' : s'$ so we may apply the induction hypothesis on $\Gamma \vdash_t^e A : s$ to conclude $\Delta^{\mathsf{can}} \equiv \Gamma^{\mathsf{can}}$ and $A^{\mathsf{can}} \equiv A'^{\mathsf{can}}$. This proves 1. As for 2, assume $||N|| \equiv ||M||$. Then $N \equiv M$ because M a variable or a sort or an universe. So we are done.

■

Proof of Lemma 34

- first note that it is not true for an arbitrary M because algebraic rewrite rules might not be left-linear. Indeed, consider the rewrite rule

$$f(x, x) \to x$$

If A and A' have no common reduct, then the term

$$f(\mathsf{app}^{\Pi x:A.B}(x, y), \mathsf{app}^{\Pi x:A'.B}(x, y))$$

is in normal form while we have the reduction

$$||f(\mathsf{app}^{\Pi x:A.B}(x, y), \mathsf{app}^{\Pi x:A'.B}(x, y))|| \equiv f(x\ y, x\ y) \to x\ y$$

The lemma is proved by structural induction on the derivation of $\Gamma \vdash_t^e M : A$. We treat the cases where the last rule of the derivation is (function) or (application).

- *function:* then $M \equiv f(t_1, \ldots, t_n)$. The only interesting case is when $||M||$ itself is a redex, i.e. when there exists a rule $l \to r$ and a substitution θ with domain $\mathsf{FV}(l)$ such that $\theta l \equiv f(||t_1||, \ldots, ||t_n||)$ and $\theta r \equiv N$. Take l_0 linear with $\mathsf{FV}(l) \cap \mathsf{FV}(l_0) = \emptyset$ and ρ a renaming with domain $\mathsf{FV}(l_0)$ (it may rename two distinct variables with the same name) such that $\rho l_0 \equiv l$. There exists a labelled substitution θ' with domain $\mathsf{FV}(l_0)$ such that $\theta' l_0 \equiv M$. We know that for every $x \in \mathsf{FV}(l_0)$, we have $||\theta' x|| \equiv \theta \circ \rho(x)$. Hence for every $x, y \in \mathsf{FV}(l_0)$, $\rho x \equiv \rho y \Rightarrow ||\theta' x|| \equiv ||\theta' y||$. By Lemma 33, it follows $(\theta' x)^{\mathsf{can}} \equiv (\theta' y)^{\mathsf{can}}$. Define a labelled substitution θ'' with domain $\mathsf{FV}(l_0)$ by $\theta'' x \equiv (\theta' x)^{\mathsf{can}}$. There exists a substitution θ_0 with domain $\mathsf{FV}(l)$ such that $\theta''(x) \equiv \theta_0 \circ \rho(x)$ for every $x \in \mathsf{FV}(l_0)$. Define $N' \equiv \theta_0 r$. Then $M \twoheadrightarrow_{mixl} N'$.

  To show $||N'|| \equiv N$. Let $x \in \mathsf{FV}(l)$. There exists y such that $\rho y \equiv x$. We have

$$||\theta_0 x|| \equiv ||\theta_0(\rho y)|| \equiv ||\theta'' y|| \equiv ||(\theta' y)^{\mathsf{can}}|| \equiv ||\theta' y|| \equiv \theta(\rho y) \equiv \theta x$$

  Hence $||\theta_0 x|| \equiv \theta x$ for every $x \in \mathsf{FV}(l)$ and we are done.

- *application:* let $M \equiv \mathsf{app}^{\Pi x:C.D}(t,u)$ and $||M|| \equiv ||t||||u|| \to_{mix} N'$. We use the induction hypothesis if the reduction occurs in $||t||$ or $||u||$. When $||M||$ itself is a β-redex then $t \equiv \lambda^{\Pi x:C'.D'} x.t'$ and $N \equiv ||t'||[||u||/x]$. The loose head reduction of M leads to $N' \equiv t'[u/x]$. We are done with the following equality:

$$||N'|| \equiv \ ||t'[u/x]|| \equiv ||t'||[||u||/x] \ \equiv N$$

- it is proved by induction on the length of the reduction sequence $||M|| \twoheadrightarrow^{+}_{mix} N$. Assume $||M|| \to_{mix} P$. By 1. there exists P' such that $M \twoheadrightarrow^{+}_{mixl} P'$ and $||P'|| \equiv P$. By Lemma 25, $M \twoheadrightarrow^{+}_{mixt} P'$. By Subject Reduction, P' is legal. So we can apply the induction hypothesis on P'.

- the strong normalisation part is proved by induction on the length of the longest $\to_{mixt}$-reduction sequence starting from M. Assume that $||M|| \to_{mix} N$. By 2 there exists N' such that $M \twoheadrightarrow^{+}_{mixt} N'$ and $||N'|| \equiv N$. By Subject Reduction, N' is legal so we can apply the induction hypothesis. Hence $\to_{mix}$ is strongly normalising on $N \equiv ||N'||$. The property is true for any N such that $||M|| \to_{mix} N$. Henceforth $\to_{mix}$ is strongly normalising on $||M||$. As for the Church-Rosser property, assume $||M|| \twoheadrightarrow_{mix} N_1$ and $||M|| \twoheadrightarrow_{mix} N_2$. Then there exist N_1' and N_2' such that $M \twoheadrightarrow_{mixt} N_1'$ and $M \twoheadrightarrow_{mixt} N_2'$ with $||N_1'|| \equiv N_1$ and $||N_2'|| \equiv N_2$. By confluence of $\to_{mixt}$, there exists a labelled pseudo-term P with $N_1' \twoheadrightarrow_{mixt} P$ and $N_2' \twoheadrightarrow_{mixt} P$. We can translate $N_i' \twoheadrightarrow_{mixt} P$ into $||N_i'|| \twoheadrightarrow_{mix} ||P||$. It follows that $N_1 \twoheadrightarrow_{mix} ||P||$ and $N_2 \twoheadrightarrow_{mix} ||P||$.

- assume $||M|| \downarrow_{mix} ||N||$. Hence there exists Q such that $||M||, ||N|| \twoheadrightarrow_{mix} Q$. By 2, there exists Q_1 and Q_2 such that $M \twoheadrightarrow_{mixt} Q_1$ and $M \twoheadrightarrow_{mixt} Q_2$. Moreover $||Q_1|| \equiv ||Q_2|| \equiv Q$. Hence $Q_1^{\mathsf{can}} \equiv Q_2^{\mathsf{can}} \equiv Q'$. Thus we have $M, N \twoheadrightarrow_{mixt} Q'$ i.e. $M \downarrow_{mixt} N$.

■

Proof of Proposition 35 the proof proceeds by structural induction on the derivation of $\Gamma \vdash M : A$.

- *conversion:* suppose that $\Gamma \vdash M : B$ is derived from $\Gamma \vdash M : A$ and $\Gamma \vdash B : s$ with $A \downarrow_{mix} B$. It follows by induction that $\Gamma_\bullet \vdash^e_t M_\bullet : A_\bullet$ and $\Gamma_\circ \vdash^e_t B_\circ : s$ with

$$||\Gamma_\bullet|| \equiv \Gamma \equiv ||\Gamma_\circ|| \quad ||M_\bullet|| \equiv M \quad ||A_\bullet|| \equiv A \quad ||B_\circ|| \equiv B$$

By lemma 33, $\Gamma_\bullet \equiv \Gamma_\circ$. By Correctness of Types, $\Gamma_\bullet \vdash^e_t A_\bullet : s'$ for a given universe s'. The last statement of Lemma 34 deduces $A \downarrow_{mixt} B$ from $\Gamma_\bullet \equiv \Gamma_\circ$ and $||A|| \downarrow_{mix} ||B||$. By conversion

$$\Gamma_\bullet \vdash M_\bullet : B_\circ$$

with the required translation (and canonicity) features:

$$||\Gamma_\bullet|| \equiv \Gamma \text{ and } ||M_\bullet|| \equiv M \text{ and } ||B_\circ|| \equiv N$$

- *application:* suppose that $\Gamma \vdash tu : B[u/x]$ is derived from $\Gamma \vdash t : (\Pi x : A.B)$ and $\Gamma \vdash u : A$ with application rule. By induction there exists two derivable judgements

$$\Gamma_\bullet \vdash^e_t t_\bullet : (\Pi x : A.B)_\bullet \text{ and } \Gamma_\circ \vdash^e_t u_\circ : A_\circ \quad (1)$$

with the good translation properties. Define $A_\bullet, B_\bullet$ as $(\Pi x : A.B)_\bullet \equiv \Pi x : A_\bullet.B_\bullet$; by $\mathbf{G}_\Pi$ and Correctness of Types, $\Gamma_\bullet \vdash^e_t A_\bullet : s'$ and $\Gamma_\circ \vdash^e_t A_\circ : s''$ for some universes s', s''. By lemma 33

$$\Gamma_\bullet \equiv \Gamma_\circ \text{ and } A_\bullet \equiv A_\circ$$

because $A_\circ$ is canonical. By (1) and an application rule it follows that:

$$\Gamma_\bullet \vdash^e_t \mathsf{app}^{\Pi x:A_\bullet.B_\bullet}(t_\bullet, u_\circ) : B_\bullet[u_\circ/x]$$

By the Subject Reduction Property and conversion:

$$\Gamma_\bullet \vdash^e_t \mathsf{app}^{\Pi x:A_\bullet^{\mathsf{nf}}.B_\bullet^{\mathsf{nf}}}(t_\bullet, u_\circ) : (B_\bullet[u_\circ/x])^{\mathsf{can}}$$

We easily check the three equalities

$$||\Gamma_\bullet|| \equiv \Gamma \text{ and } ||\mathsf{app}^{\Pi x:A_\bullet^{\mathsf{nf}}.B_\bullet^{\mathsf{nf}}}(t_\bullet, u_\circ)|| \equiv tu \text{ and } ||(B_\bullet[u_\circ/x])^{\mathsf{can}}|| \equiv B[u/x]$$

and the canonicity of $\Gamma_\bullet$, $\mathsf{app}^{\Pi x:A_\bullet^{\mathsf{nf}}.B_\bullet^{\mathsf{nf}}}(t_\bullet, u_\circ)$ and $(B_\bullet[u_\circ/x])^{\mathsf{can}}$.

- *abstraction:* assume the last step is

$$\frac{\Gamma, x : A \vdash t : B \qquad \Gamma \vdash \Pi x : A.B : s}{\Gamma \vdash \lambda x : A.t : \Pi x : A.B}$$

and $M \equiv \lambda x : A.t$. By induction hypothesis:

$$\Gamma_\bullet, x : A_\bullet \vdash^e_t t_\bullet : B_\bullet \qquad \Gamma_\circ \vdash^e_t (\Pi x : A.B)_\circ : s \qquad (2)$$

for some canonical contexts $\Gamma_\bullet, \Gamma_\circ$. By Correctness of Types, $\Gamma_\bullet, x : A_\bullet \vdash^e_t B_\bullet : s'''$ for some universe s'. Let $A_\circ$ and $B_\circ$ be defined as $(\Pi x : A.B)_\circ \equiv \Pi x : A_\circ.B_\circ$. By $\mathbf{G}_\Pi$:

$$\Gamma_\circ \vdash^e_t A_\circ : s_1 \qquad \Gamma_\circ, x : A_\circ \vdash^e_t B_\circ : s_2$$

for some universes s_1, s_2. By lemma 33

$$\Gamma_\bullet \equiv \Gamma_\circ \quad A_\bullet \equiv A_\circ \quad B_\bullet \equiv B_\circ$$

By abstraction and (2):

$$\Gamma_\bullet \vdash^e_t \lambda^{\Pi x:A_\bullet.B_\bullet} x.t_\bullet : \Pi x : A_\bullet.B_\bullet$$

By the Subject Reduction Property and a conversion rule:

$$\Gamma_\bullet \vdash^e_t \lambda^{\Pi x:A_\bullet^{\mathsf{can}}.B_\bullet^{\mathsf{nf}}} x.t_\bullet : (\Pi x : A_\bullet.B_\bullet)^{\mathsf{can}}$$

We easily check the three equalities

$$||\Gamma_\bullet|| \equiv \Gamma \quad ||\lambda^{\Pi x:A_\bullet^{\mathsf{can}}.B_\bullet^{\mathsf{nf}}} x.t_\bullet|| \equiv \lambda x : A.t \quad ||(\Pi x : A_\bullet.B_\bullet)^{\mathsf{can}}|| \equiv \Pi x : A.B$$

and the respective canonicity properties.

- *weakening:* assume the last step is:

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash B : s}{\Gamma, x : B \vdash t : A} \text{ if } x \notin \Gamma \text{ and } t \in S \cup U \cup V \cup K$$

By induction there exists $\Gamma_\bullet \vdash^e_t t_\bullet : A_\bullet$ and $\Gamma_\circ \vdash^e_t B_\circ : s$. By lemma 33 $\Gamma_\circ \equiv \Gamma_\bullet$ and henceforth $\Gamma_\bullet, x : B_\circ \vdash^e_t t_\bullet : A_\bullet$ since $x \notin \Gamma$ and $t \in S \cup U \cup V \cup K$.

■