On a duality between Kruskal and Dershowitz theorems

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Abstract. The article is mainly concerned with the Kruskal tree theorem and the following observation: there is a duality at the level of binary relations between well and noetherian orders. The first step here is to extend Kruskal theorem from orders to binary relations so that the duality applies. Then, we describe the theorem obtained by duality and show that it corresponds to a theorem by Ferreira and Zantema which subsumes Dershowitz's seminal results on recursive path orderings.

1 Introduction

1.1 A duality between well and noetherian relations

This paper investigates a duality between the two well-established concepts of well and noetherian order. An order $\preceq$ on a set $\mathbf{X}$ is *well* when in every sequence $(x_i)_{i \in \mathbb{N}}$ of elements of $\mathbf{X}$:

$$\exists (i, j) \in \mathbb{N}^2, \quad i < j \text{ and } x_i \preceq x_j \tag{1}$$

An order $\prec$ is *noetherian* on $\mathbf{X}$ when it induces no infinite descending chain ... $\prec x_2 \prec x_1$. In other words, letting $\prec^\perp$ denote the complement [1] of $\prec$, an order is noetherian when for every sequence $(x_i)_{i \in \mathbb{N}}$ of elements of $\mathbf{X}$:

$$\exists (i, j) \in \mathbb{N}^2, \quad i < j \text{ and } x_j \prec^\perp x_i \tag{2}$$

The similarity between the definitions (1) and (2) is striking. Letting $\prec^\star$ denote the reverse [2] of $\prec$'s complement $\prec^\perp$, the order $\prec$ is noetherian whenever

$$\prec^\star = (\prec^\perp)^{op} = (\prec^{op})^\perp$$

is well. But we should be very careful here because the relation $\prec^\star$ need not be an order — only a binary relation.

In fact we are forced to consider binary relations instead of orders if we want to enjoy the duality sketched above. We declare that a binary relation $\preceq$ on a set $\mathbf{X}$ is *well* when it verifies an analogue of property (1) that for every sequence $(x_i)_{i \in \mathbb{N}}$ of elements of $\mathbf{X}$,

$$\exists (i, j) \in \mathbb{N}^2, \quad i < j \text{ and } x_i \preceq x_j \tag{3}$$

[1] By complement we mean its complement in $\mathbf{X} \times \mathbf{X}$, that is $\forall (x, y) \in \mathbf{X}^2, x \prec^\perp y \iff \neg(x \prec y)$.

[2] By reverse we mean the relation $\prec^{op}$ such that $\forall (x, y) \in \mathbf{X}^2, x \prec^{op} y \iff y \prec x$.

Similarly, a binary relation $\prec$ on a set $\mathbf{X}$ is declared *noetherian* when an analogue of property (2) holds that for every sequence $(x_i)_{i\in\mathbb{N}}$ of elements of $\mathbf{X}$,

$$\exists(i,j) \in \mathbb{N}^2, \quad i < j \text{ and } x_i \prec^\star x_j \tag{4}$$

The duality on binary relations can be expressed as follows (noting that the operator $(-)^\star$ is idempotent on binary relations):

A binary relation $\prec$ is noetherian if and only if its dual $\prec^\star$ is well.
A binary relation $\preceq$ is well if and only if its dual $\preceq^\star$ is noetherian.

We repeat here that this duality between well and noetherian relations is not visible on orders because the transformation $(\prec\, \mapsto\, \prec^\star)$ does not respect them.

1.2 A duality on Kruskal theorem extended to binary relations

Kruskal theorem states that the tree embedding extension of a well order on labels is also a well order. To justify our shift from orders to binary relations we extend Kruskal theorem to well binary relations — see section 3. Hence, we show that the tree embedding extension of a well binary relation on labels is a well binary relation on trees.

The immediate pay-off of this extension to binary relations is that Kruskal theorem can be *dualised* to another theorem on noetherian binary relations. It turns out that this new theorem — at least its restriction to orders — falls into a class of results developed by Nachum Dershowitz on recursive path orderings, see [2, 3]. Here, we could oversimplify and write:

Kruskal* = Dershowitz

In fact, the duality is already (and secretly) at work in a series of paper [9, 4, 7] where Nash-Williams' proof of Kruskal theorem is *adapted* to establish noetherianity of various path orders.

2 Well Binary Relations (WBRs)

The section adapts some well-known order-theoretic notions and results to binary relations. In particular, lemma 3 establishes that every sequence in a well binary relation contains an increasing sequence.

Definition 1 (sequence, subsequence). *Let $\mathbf{X}$ denote a set. A **sequence** over $\mathbf{X}$ is function from $\mathbb{N}$ to $\mathbf{X}$, where $\mathbb{N}$ denotes the set of natural numbers. We write $(x_i)_{i\in\mathbb{N}}$ the sequence which associates $x_i \in \mathbf{X}$ to $i \in \mathbb{N}$. A **subsequence** $g = (y_j)_{j\in\mathbb{N}}$ of $f = (x_i)_{i\in\mathbb{N}}$ is any sequence such that $g = f \circ \phi$ for ϕ a strictly monotone function from $\mathbb{N}$ to $\mathbb{N}$. We say that f **contains** g and write $(y_j)_{j\in\mathbb{N}} = (x_{\phi(i)})_{i\in\mathbb{N}}$.*

Definition 2 (good, bad, perfect). *Let $(\mathbf{X}, \preceq)$ be a set equipped with a binary relation $\preceq$. A sequence f over $\mathbf{X}$ is **good** w.r.t $\preceq$ when there exists two natural numbers i and j such that $i < j$ and $x_i \preceq x_j$. Otherwise it is **bad**. The sequence f is **perfect** when every subsequence of it is good. An **infinite increasing chain** over $(\mathbf{X}, \preceq)$ is a sequence $(x_i)_{i\in\mathbb{N}}$ such that $\forall(i,j) \in \mathbb{N}^2, i < j \Rightarrow x_i \preceq x_j$.*

Definition 3. *The relation $\preceq$ is a **well binary relation** (WBR) on $\mathbf{X}$ when every sequence over $\mathbf{X}$ is good w.r.t $\preceq$.*

Lemma 1. *Every sequence is perfect in a well binary relation.*

Lemma 2. *Every subsequence of a perfect sequence is perfect.*

Lemma 3. *Every perfect sequence over $(\mathbf{X}, \preceq)$ contains an infinite increasing chain.*

Proof We will show that any infinite sequence over $(\mathbf{X}, \preceq)$ contains an infinite subsequence which is either bad or infinitely increasing. Let the sequence be $f = (x_i)_{i \in Nn}$.

We consider the graph whose vertices are the natural numbers and edges $\{(i,j) \mid i < j\}$. An edge (i,j) is coloured red when $x_i \preceq x_j$ and blue otherwise.

From the infinite version of Ramsey theorem, see for instance pp.392 theorem 6.1 in [1], if there is a red-colored monochromatic countably infinite complete subgraph, this means the existence of an infinite increasing subsequence. If there is a blue-colored one, this means the existence of a bad subsequence. □

Lemma 4. *Suppose that $\preceq_1$ and $\preceq_2$ are two binary relations on $\mathbf{X}$. Every sequence perfect w.r.t $\preceq_1$ and $\preceq_2$ is also perfect w.r.t $(\preceq_1 \cap \preceq_2)$.*

Proof Let $f = (x_i)_{i \in \mathbb{N}}$ be perfect w.r.t $\preceq_1$ and $\preceq_2$. We show that every subsequence $g = (y_i)_{i \in \mathbb{N}}$ of f is good w.r.t $\preceq_1 \cap \preceq_2$. By lemma 1, g is perfect w.r.t $\preceq_1$. By lemma 3, g contains an increasing sequence w.r.t $\preceq_1$, viz. there is a strictly monotone function $\phi : \mathbb{N} \to \mathbb{N}$ such that

$$\forall (i,j) \in \mathbb{N}^2, i < j \Rightarrow y_{\phi(i)} \preceq_1 y_{\phi(j)}$$

As a subsequence of f, this sequence $(y_{\phi(i)})_{i \in \mathbb{N}}$ is good w.r.t $\preceq_2$. So, there is a couple of indexes $i < j$ such that $y_{\phi(i)} \preceq_2 y_{\phi(j)}$. We conclude with the observation that the two indices $I = \phi(i)$ and $J = \phi(j)$ verify $I < J$ and $y_I(\preceq_1 \cap \preceq_2)y_J$. Therefore, $g = (y_i)_{i \in \mathbb{N}}$ is good w.r.t $\preceq_1 \cap \preceq_2$. □

Lemma 5. *Suppose that $\preceq_1$ and $\preceq_2$ are WBRs on $\mathbf{X}$. Then $\preceq_1 \cap \preceq_2$ is a WBR on $\mathbf{X}$.*

Proof Let $f = (x_i)_{i \in \mathbb{N}}$ be any sequence over $\mathbf{X}$. We have to show that f is good w.r.t $\preceq_1 \cap \preceq_2$. By lemma 1, f is perfect w.r.t $\preceq_1$ and $\preceq_2$ and by lemma 4 it is perfect w.r.t $\preceq_1 \cap \preceq_2$ and a fortiori good. We conclude. □

3 Kruskal Theorem

This section is devoted to Kruskal theorem. Its proof is “abstract” in the sense that it does not proceed on the syntactical structure of the terms but on an abstract structure which mirrors them.

3.1 The abstract system

An abstract decomposition system is a 8-tuple $(\mathcal{T}, \mathcal{L}, \mathcal{V}, \preceq, \preceq_{\mathcal{L}}, \preceq_{\mathcal{V}}, \dot{\longrightarrow}, \vdash)$ where

- $\mathcal{T}$ is a set of *terms* noted $t, u, ...$, equipped with a binary relation $\preceq$,
- $\mathcal{L}$ is a set of *labels*, noted $f, g, ...$, equipped with a binary relation $\preceq_{\mathcal{L}}$,
- $\mathcal{V}$ is a set of *vectors*, noted $T, U, ...$, equipped with a binary relation $\preceq_{\mathcal{V}}$,
- $\dot{\longrightarrow}$ is a relation between terms, labels and vectors, for instance $t \xrightarrow{f} T$,
- $\vdash$ is a relation between vectors and terms, for instance $t \vdash T$.

Example.— We briefly present the concrete structures we have in mind in the case of the (multiset) Kruskal theorem. In that case, $\mathcal{T}$ is the set of ground trees built on a signature $\mathcal{F}$, $\mathcal{L}$ is just $\mathcal{F}$, and $\mathcal{V}$ is the set of finite multisets of elements of $\mathcal{T}$. $\preceq_{\mathcal{L}}$ is a well order on $\mathcal{L} = \mathcal{F}$, $\preceq$ is the tree embedding of the order $\preceq_{\mathcal{L}}$, and $\preceq_{\mathcal{V}}$ is the multiset embedding of $\preceq$. The operator $\dot{\longrightarrow}$ is the tree deconstructor: if $t = f(s_1, ..., s_k)$, then $t \xrightarrow{f} \{\{s_1, ..., s_k\}\}$. The operator $\vdash$ is the multi-set deconstructor: if $T = \{\{s_1, ..., s_k\}\}$, then $T \vdash s_i$ for every $i \in [1...k]$. A more detailed presentation is given in sections 3.4 and 3.5.

Definition 4. *The binary relation $\rhd$ on $\mathcal{T}$ is defined as*

$$t \rhd u \iff \exists (f, T) \in \mathcal{L} \times \mathcal{V}, t \xrightarrow{f} T \vdash u$$

An elementary term t *is a minimal term w.r.t $\rhd$, that is* $\forall u \in \mathcal{T}, \neg(t \rhd u)$.

3.2 The axiomatics

Six axioms will be needed to prove our Kruskal theorem. We present them informally first, then formally.

- Axiom I asks that a term cannot be deconstructed for ever,
- Axiom II asks that every sequence $(t_i)_{i\in\mathbb{N}}$ of elementary terms contains a "comparable pair" $t_i \preceq t_j$ with $i < j$,
- Axioms III and IV are abstract adaptation of the syntactical requirements of the *subterm property*:
$$t \preceq u_i \Rightarrow t \preceq f(u_1, ..., u_m)$$
and of the *monotonicity* of the syntactical contexts $C[-]$:
$$\forall C[-], \quad t \preceq u \Rightarrow C[t] \preceq C[u]$$
- Axiom V is less traditional. We trace it back to [6, 7]'s notion of lifting — here in its dual presentation. The axiom asks that $\preceq_{\mathcal{V}}$ is well on any subset $\mathcal{W}$ of $\mathcal{V}$ when $\preceq$ is well on the set $\mathcal{W}_{\vdash} \subset \mathcal{T}$ composed of the elements of the vectors of $\mathcal{W}$,
- Axiom VI asks that a term t has only a finite number of principal subterms.

Axiom I [well-foundedness] There is no infinite chain $t_1 \rhd t_2 \rhd ...$
Axiom II [elementary terms] The relation $\preceq$ is a WBR on the set of elementary terms.
Axiom III [subterm property] $\forall (t, u, u') \in \mathcal{T}^3, \forall (f, U) \in \mathcal{L} \times \mathcal{V}$,

$$\text{if } t \preceq u' \text{ and } u \xrightarrow{f} U \vdash u', \text{ then } t \preceq u.$$

Axiom IV [monotonicity] $\forall (t, u) \in \mathcal{T}^2$, $\forall (f, g) \in \mathcal{L}^2$ and $\forall (T, U) \in \mathcal{V}^2$,
Suppose that $t \xrightarrow{f} T$ and $u \xrightarrow{g} U$ and $f \preceq_{\mathcal{L}} g$ and $T \preceq_{\mathcal{V}} U$.

$$\text{If } \forall t' \in \mathcal{T}, T \vdash t' \Rightarrow (t' \preceq u \text{ and } t' \neq u), \text{ then } t \preceq u.$$

Axiom V [lifting] Take any subset $\mathcal{W}$ of $\mathcal{V}$. Letting

$$\mathcal{W}_{\vdash} = \{t \in \mathcal{T} \mid \exists T \in \mathcal{W}, T \vdash t\}$$

we ask that $\preceq_{\mathcal{V}}$ is a WBR on $\mathcal{W}$ whenever $\preceq$ is a WBR on $\mathcal{W}_{\vdash}$.
Axiom VI [finitely branching] Take any vector T. We ask that the number of terms t such that $T \vdash t$ is finite.

3.3 The axiomatic Kruskal theorem

Theorem 1 (axiomatic Kruskal). *Suppose that $(\mathcal{T}, \mathcal{L}, \mathcal{V}, \preceq, \preceq_{\mathcal{L}}, \preceq_{\mathcal{V}}, \dot{\longrightarrow}, \vdash)$ verifies the axioms I — VI. If $\preceq_{\mathcal{L}}$ is a WBR on $\mathcal{L}$, then $\preceq$ is a WBR on $\mathcal{T}$.*

Proof The proof in six steps uses Nash-Williams' minimal bad sequence argument and therefore is not constructive.
[Step 1]. Suppose that $\preceq$ is not a WBR on $\mathcal{T}$. By axiom I, there is a "minimal" bad sequence $t_0, ..., t_i, ...$ such that every sequence $u_0, ..., u_i, ...$ is good as soon as $\exists k \in \mathbb{N}$ such that $[t_0, ..., t_k] = [u_0, ..., u_k]$ and $t_{k+1} \rhd u_{k+1}$. By axiom II, there exists an index $M \in \mathbb{N}$ such that every term t_{M+k} decomposes into a vector T_k and a term u_k for $k \in \mathbb{N}$:

$$t_{M+k} \xrightarrow{f_k} T_k \vdash u_k$$

[Step 2]. Let ϕ be any function from $\mathbb{N}$ to $\mathbb{N}$ such that

$$\forall i \in \mathbb{N}, \quad \phi(i) \geq \phi(0) \tag{5}$$

Let $v_0, ..., v_i, ...$ any sequence of terms such that

$$\forall i \in \mathbb{N}, \quad t_{M+\phi(i)} \xrightarrow{f_{\phi(i)}} T_{\phi(i)} \vdash v_i \tag{6}$$

We claim that the sequence $(v_i)_{i \in \mathbb{N}}$ is good w.r.t $\preceq$. Indeed, by minimality of $(t_i)_{i \in \mathbb{N}}$ and $t_{M+\phi(0)} \rhd v_0$, the sequence

$$t_0, ..., t_{M+\phi(0)-1}, v_0, v_1, ..., v_i, ...$$

is good. This means that there is a "comparable pair" in that sequence. We show by case analysis that this pair occurs in $(v_i)_{i \in \mathbb{N}}$ and conclude:

1. Suppose that $t_i \preceq t_j$ for $0 \leq i < j < M + \phi(0)$. Then $(t_i)_{i \in \mathbb{N}}$ is good, which contradicts its construction.
2. Suppose that $t_i \preceq v_j$ for $i < M + \phi(0)$ and $j \in \mathbb{N}$. Then, $t_i \preceq t_{M+\phi(j)}$ by axiom III. Because $i < M + \phi(0) \leq M + \phi(j)$ by hypothesis (5), the relation $t_i \preceq t_{M+\phi(j)}$ contradicts the construction of $(t_i)_{i \in \mathbb{N}}$ as a bad sequence.

3. The cases 1. and 2. are impossible. The only remaining case is $v_i \preceq v_j$ for some $0 \leq i < j$ — hence $(v_i)_{i\in\mathbb{N}}$ is good.

[Step 3]. Let $\mathcal{W}$ be the set consisting of all the T_i's. We claim that $\preceq$ is well on the set

$$\mathcal{W}_{\vdash} = \{t \in \mathcal{T} \mid \exists T \in \mathcal{W}, T \vdash t\}$$

Take any sequence $w_0, ..., w_i, ...$ of terms in $\mathcal{W}_{\vdash}$. For every $i \in \mathbb{N}$, there is a minimum index $P(i) \in \mathbb{N}$ such that $T_{P(i)} \vdash w_i$. Let P be the minimum index among these $P(i)$'s, and $i_0 \in \mathbb{N}$ such that $P = P(i_0)$. We consider the subsequence $(v_i)_{i\in\mathbb{N}} = (w_{i_0+i})_{i\in\mathbb{N}}$ of $(w_i)_{i\in\mathbb{N}}$ and show that it is good. Indeed, the function $\phi : \mathbb{N} \to \mathbb{N}$ defined by $i \mapsto P(i_0 + i)$ verifies the two conditions 5 and 6 of [step 2]:

$$\forall i \in \mathbb{N}, \quad \phi(i) \geq \phi(0) \quad \wedge \quad t_{M+\phi(i)} \xrightarrow{f_{\phi(i)}} T_{\phi(i)} \vdash v_i$$

We deduce that $(v_i)_{i\in\mathbb{N}}$ is good, and a fortiori $(w_i)_{i\in\mathbb{N}}$ is good. We conclude that $\preceq$ is well on $\mathcal{W}_{\vdash}$.

[Step 4]. From this fact and axiom V, $\preceq_{\mathcal{V}}$ is well on $\mathcal{W}$. In particular, the sequence $(T_i)_{i\in\mathbb{N}}$ is perfect w.r.t $\preceq_{\mathcal{V}}$. The sequence $(f_0, T_0), ..., (f_i, T_i), ...$ being perfect w.r.t $\preceq_{\mathcal{L}} \times \mathcal{T}^2$ and $\mathcal{L}^2\times \preceq_{\mathcal{V}}$ is also perfect w.r.t $(\preceq_{\mathcal{L}} \times \mathcal{T}^2) \cap (L^2 \times \preceq_{\mathcal{V}}) = \preceq_{\mathcal{L}} \times \preceq_{\mathcal{V}}$ by lemma 4.

From lemma 3, there exists a strictly monotone function $\psi : \mathbb{N} \to \mathbb{N}$ such that

$$(f_{\psi(0)}, T_{\psi(0)}), ..., (f_{\psi(i)}, T_{\psi(i)}), ...$$

is an increasing sequence w.r.t $\preceq_{\mathcal{L}} \times \preceq_{\mathcal{V}}$.

[Step 5]. The proof that there exists an index $j \in \mathbb{N}$ such that

$$\forall t' \in \mathcal{T}, T_{\psi(0)} \vdash t' \Rightarrow (t' \preceq t_{M+\psi(j)} \text{ and } t' \neq t_{M+\psi(j)})$$

is left as exercise. Note that it uses axiom VI.

[Step 6]. We conclude with axiom IV. Let $I = \psi(0)$ and $J = \psi(j)$. Here, the vector $T_{\psi(0)}$ can be rewritten as T_I and the index $M + \psi(j)$ as $M + J$. Step 1 shew that

$$t_{M+I} \xrightarrow{f_I} T_I \text{ and } t_{M+J} \xrightarrow{f_J} T_J$$

Step 4 established that

$$f_I \preceq_{\mathcal{L}} f_J \text{ and } T_I \preceq_{\mathcal{V}} T_J$$

Step 5 established that

$$\forall t' \in \mathcal{T}, T_I \vdash t' \Rightarrow (t' \preceq t_{M+J} \text{ and } t' \neq t_{M+J})$$

We apply axiom IV and derive $t_{M+I} \preceq t_{M+J}$, hence contradicting the construction of $(t_i)_{i\in\mathbb{N}}$ as a bad sequence. We conclude that $\preceq$ is a WBR on $\mathcal{T}$. □

Remark. — Axioms IV and VI appear as luxury from the point of view of Kruskal theorem. Indeed, they can be replaced by a simpler requirement:

Axiom IV[bis] $\forall (t,u) \in \mathcal{T}^2$, $\forall (f,g) \in \mathcal{L}^2$ and $\forall (T,U) \in \mathcal{V}^2$,
If $t \xrightarrow{f} T$ and $u \xrightarrow{g} U$ and $f \preceq_{\mathcal{L}} g$ and $T \preceq_{\mathcal{V}} U$, then $t \preceq u$.

In axiom IV[bis], the condition

$$\forall t' \in \mathcal{T}, T \vdash t' \Rightarrow (t' \preceq u \text{ and } t' \neq u) \tag{7}$$

disappears and [step 5] is not necessary in the proof of theorem 1. But duality is our reason to keep condition 7. In fact, the dual of condition (7) stands among the fundamentals of recursive path orderings — see section 4.1 for further discussion.

3.4 Application I: Higman theorem, in [8]

Let $\mathcal{L}$ be a set of labels equipped with a binary relation $\preceq_{\mathcal{L}}$. The set $\mathcal{T}$ is constructed as the set of finite strings on $\mathcal{L}$ equipped with the following binary relation $\preceq$.

$$(a_1, ..., a_m) \preceq (b_1, ..., b_n) \iff \begin{array}{l} \exists \text{ an embedding } \phi : [1...m] \to [1...n] \\ \text{such that } \forall i \in [1...m], a_i \preceq_{\mathcal{L}} b_{\phi(i)} \end{array}$$

where an (Higman) embedding $[1...m] \to [1...n]$ is a strictly monotone function.

Higman proved in [8] that $\preceq$ is a well order whenever $\preceq_{\mathcal{L}}$ is a well order.

Theorem 2 (Higman). *If $\preceq_{\mathcal{L}}$ is well on $\mathcal{L}$, then $\preceq$ is well on $\mathcal{T}$.*

We show that theorem 2 is an instance of our theorem 1. To do this, we identify the sets $\mathcal{V}$ and $\mathcal{T}$ and define $\vdash$ as the identity relation on $\mathcal{T}$. The relation $\xrightarrow{f}$ is defined as follows:

$$(a_1, ..., a_m) \xrightarrow{f} (b_1, ..., b_n) \iff (a_1, ..., a_m) = (f, b_1, ..., b_n)$$

Let us check that the decomposition system $(\mathcal{T}, \mathcal{L}, \mathcal{V}, \preceq, \preceq_{\mathcal{L}}, \preceq_{\mathcal{V}}, \xrightarrow{\cdot}, \vdash)$ verifies the axioms I — VI of section 3.2.

1. $t \rhd u$ implies that u's length is strictly smaller than t's length. In particular, there is no infinite sequence $t_1 \rhd t_2 \rhd ....$
2. the only elementary term is the empty string ϵ, and $\epsilon \preceq \epsilon$,
3. Letting $u' = (a_1, ..., a_n)$, $u \xrightarrow{f} u'$ means that $u = (f, a_1, ..., a_n)$. If $\phi : [1...m] \to [1...n]$ is an embedding corresponding to $t \preceq u'$, then $i \mapsto \phi(i)+1$ is an embedding $[1...m] \to [1...n+1]$ corresponding to $t \preceq u$. Therefore, $t \preceq u'$ implies $t \preceq u$.
4. Letting $T = (a_1, ..., a_m)$ and $U = (b_1, ..., b_n)$, $T \preceq_{\mathcal{V}} U$ means here that $T \preceq U$, hence that there is an embedding $\phi : [1...m] \to [1...n]$ such that $\forall i \in [1...m], a_i \preceq_{\mathcal{L}} b_{\phi(i)}$. Letting $\psi : [1...m+1] \to [1...n+1]$ be the function $1 \mapsto 1$ and $\forall i \in [1...m], 1+i \mapsto 1+\phi(i)$, ψ embeds $t = (f, a_1, ..., a_m)$ into $u = (g, b_1, ..., b_n)$ so that $t \preceq u$ — because $f \preceq_{\mathcal{L}} g$.

5-6. immediate.

This shows in six easy steps that Higman theorem is an instance of theorem 1. Simultaneously, it extends Higman theorem to binary relations.

3.5 Application II: Kruskal theorem, in [10]

We show here that the original Kruskal theorem is an instance of theorem 1.

A varyadic signature $(\mathcal{F}, \texttt{arity})$ is a set $\mathcal{F}$ equipped with a function $\texttt{arity} : \mathcal{F} \rightarrow \wp\mathbb{N}$. Intuitively, the set $\texttt{arity}(f)$ contains the possible arities of f. Occurrences are finite strings of natural numbers which can be ordered by $\leq_{\mathcal{O}}$, the precedence order defined as follows: $o \leq_{\mathcal{O}} o'$ when the string o' extends the strings o.

Let $\mathcal{T}$ be the set of trees constructed from a varyadic signature $\mathcal{L}$. Every tree t in $\mathcal{T}$ is characterised by a function $o \mapsto t_o$ from its set $\mathcal{O}_t$ of occurrences to the set $\mathcal{L}$ of labels. Also, every tree t defines a function $o \mapsto [o]_t$ from $\mathcal{O}_t$ to $\mathbb{N}$ characterised by:

$$\forall i \in \mathbb{N}, \quad oi \in \mathcal{O}_t \iff i \in [1...[o]_t]$$

So, $[o]_t = 2$ means that $o1$ and $o2$ are occurrences of t, but not $o3$. And $[o]_t = 0$ means that o is a leaf occurrence of t.

An (Kruskal) embedding ϕ from $t \in \mathcal{T}$ to $u \in \mathcal{T}$ is a function from $\mathcal{O}_t$ to $\mathcal{O}_u$ such that:

1. $\forall (o, o') \in \mathcal{O}_t^2, \;\; o \leq_{\mathcal{O}} o' \Rightarrow \phi(o) \leq_{\mathcal{O}} \phi(o')$,
2. $\forall o \in \mathcal{O}_t$, there is a strictly monotone function $\psi_o : [1...[o]_t] \rightarrow [1...[\phi(o)]_u]$ such that $\forall i \in [1...[o]_t]$, $\phi(o)\psi_o(i) \leq_{\mathcal{O}} \phi(oi)$.

Let $\mathcal{L}$ be equipped with a binary relation $\preceq_{\mathcal{L}}$. The binary relation $\preceq$ on $\mathcal{T}$ is constructed as follows:

$$t \preceq u \iff \text{there exists an embedding } \phi : \mathcal{O}_t \rightarrow \mathcal{O}_u \text{ such that } \forall o \in \mathcal{O}_t, t_o \preceq_{\mathcal{L}} u_{\phi(o)}$$

Theorem 3 (Kruskal). *If $\preceq_{\mathcal{L}}$ is well on $\mathcal{L}$, then $\preceq$ is well on $\mathcal{T}$.*

We show that theorem 3 is an instance of our theorem 1. To do this, we define the set $\mathcal{V}$ as the set of finite strings on $\mathcal{T}$ equipped with the Higman binary relation $\preceq_{\mathcal{V}}$ defined in section 3.4:

$$(t_1, ..., t_m) \preceq_{\mathcal{V}} (u_1, ..., u_n) \iff \exists \text{ a strictly monotone } \phi : [1...m] \rightarrow [1...n] \text{ such that } \forall i \in [1...m], t_i \preceq u_{\phi(i)}$$

The relation $\xrightarrow{f}$ for $f \in \mathcal{L}$ is the tree deconstructor:

$$f(t_1, ..., t_m) \xrightarrow{f} (u_1, ..., u_n) \iff (t_1, ..., t_m) = (u_1, ..., u_n)$$

The relation $\vdash$ is the string deconstructor:

$$\forall i \in [1...m], \quad (t_1, ..., t_m) \vdash t_i$$

The axiomatics of section 3.2 is checked on $(\mathcal{T}, \mathcal{L}, \mathcal{V}, \preceq, \preceq_{\mathcal{L}}, \preceq_{\mathcal{V}}, \dot{\longrightarrow}, \vdash)$ as it is in section 3.4. Note that axiom V is theorem 2, and that axiom VI means that a finite string can only decompose to a finite number of elements.

So, Kruskal theorem is an instance of theorem 1 and as such extends to binary relations.

4 Dershowitz RPO theorems

In this section, we dualise the theorem 1 of section 3 and obtain a theorem on noetherian binary relations. To simplify the presentation, we write $\succ$ instead of $\prec^{op}$.

4.1 Axioms I—VI and theorem 1 dualised

We explain how to dualise the axioms and the theorem. Take a decomposition system $\Sigma = (\mathcal{T}, \mathcal{L}, \mathcal{V}, \prec, \prec_{\mathcal{L}}, \prec_{\mathcal{V}}, \dot{\longrightarrow}, \vdash)$. Then, theorem 1 applied to the decomposition system $\Sigma' = (\mathcal{T}, \mathcal{L}, \mathcal{V}, \prec^\star, \prec^\star_{\mathcal{L}}, \prec^\star_{\mathcal{V}}, \dot{\longrightarrow}, \vdash)$ reads as follows:

> If the decomposition system Σ' verifies the axioms I—VI and $\prec^\star_{\mathcal{L}}$ is a WBR on $\mathcal{L}$, then $\prec^\star$ is a WBR on $\mathcal{T}$.

By duality, we transform this assertion into:

> If the decomposition system Σ' verifies the axioms I—VI and $\prec_{\mathcal{L}}$ is noetherian on $\mathcal{L}$, then $\prec$ is noetherian on $\mathcal{T}$.

Now, we must express the conditions on Σ such that Σ' verifies the axioms I—VI. For instance, Σ' verifies axiom III when

$$\forall (t, u, u') \in \mathcal{T}^3, \forall (f, U) \in \mathcal{L} \times \mathcal{V}, \text{if } t \prec^\star u' \text{ and } u \xrightarrow{f} U \vdash u', \text{then } t \prec^\star u.$$

This is rephrased by contraposition as:

$$\forall (t, u, u') \in \mathcal{T}^3, \forall (f, U) \in \mathcal{L} \times \mathcal{V}, \text{if } t \succ u \text{ and } u \xrightarrow{f} U \vdash u', \text{then } t \succ u'.$$

Hence, Σ' verifies axiom III if and only if Σ verifies the previous assertion, called hereafter axiom III*. We list the "dualised" axioms we obtain:

Axiom I* [well-foundedness] There is no infinite chain $t_1 \rhd t_2 \rhd \ldots$

Axiom II* [elementary terms] There is no infinite chain $t_1 \succ t_2 \succ \ldots$ of elementary terms.

Axiom III* [subterm property] $\forall (t, u, u') \in \mathcal{T}^3, \forall (f, T) \in \mathcal{L} \times \mathcal{V}$,

$$\text{if } t \succ u \text{ and } u \xrightarrow{f} U \vdash u', \text{then } t \succ u'.$$

Axiom IV* [decomposability] $\forall (t, u) \in \mathcal{T}^2, \forall (f, g) \in \mathcal{L}^2$ and $\forall (T, U) \in \mathcal{V}^2$,
If $t \xrightarrow{f} T$ and $u \xrightarrow{g} U$ and $t \succ u$, then either:

1. $f \succ_{\mathcal{L}} g$,
2. or $T \succ_{\mathcal{V}} U$,
3. or $\exists t' \in \mathcal{T}$ such that $T \vdash t'$ and $t' \succ u$,
4. or $T \vdash u$.

Axiom V* [lifting] Take any subset $\mathcal{W}$ of $\mathcal{V}$. Letting $\mathcal{W}_\vdash = \{t \in \mathcal{T} \mid \exists T \in \mathcal{W}, T \vdash t\}$, we ask that $\succ_{\mathcal{V}}$ is noetherian on $\mathcal{W}$ whenever $\succ$ is noetherian on $\mathcal{W}_\vdash$.

Axiom VI* [finitely branching] Take any vector T. We ask that the number of terms t such that $T \vdash t$ is finite.

We obtain the following dual theorem:

Theorem 4 (Dual Kruskal). *Let $\Sigma = (\mathcal{T}, \mathcal{L}, \mathcal{V}, \prec, \prec_{\mathcal{L}}, \prec_{\mathcal{V}}, \dot{\longrightarrow}, \vdash)$ be a decomposition system which verifies the axioms I* — VI*. If $\succ_{\mathcal{L}}$ is noetherian on $\mathcal{L}$, then $\succ$ is noetherian on $\mathcal{T}$.*

Proof The previous discussion shows that Σ verifies the axioms I* — VI* if and only if Σ' verifies the axioms I — VI. We conclude with theorem 1. □

It is interesting that the dual of axiom IV is not a monotonicity requirement, but a decomposability one, as one could guess from the result of [7] and the last remark of section 1.2.

Monotonicity* = Decomposability

If we stop a moment and think of axiom IV[bis] at the end of section 3.3, we realise that its dual is just axiom IV* where disjunctions 3. and 4. disappear. This absence would have the disastrous effect to banish most recursive path orderings from the axiomatics!

4.2 Application III: Ferreira and Zantema's theorem, in [7]

We show that Ferreira and Zantema's theorem 16 in [7] is an instance of theorem 4.

Definition 5 (Ferreira-Zantema). *Let $\mathcal{F}$ be a signature and $\mathcal{X}$ be a set of variables. Let the set $\mathcal{T}(\mathcal{F}, \mathcal{X})$ of trees on $\mathcal{F}$ and $\mathcal{X}$ be equipped with a partial order $\succ$. We define a* term lifting *to be a partial order $\succ^{\Lambda}$ such that the following holds: for every $A \subset \mathcal{T}(\mathcal{F}, \mathcal{X})$, if $\succ$ restricted to A is noetherian, then $\succ^{\Lambda}$ restricted to $\bar{A}$ is also noetherian, where $\bar{A} = \{f(t_1, ..., t_m) \mid f \in \mathcal{F}, n \in arity(f), \text{ and } \forall i \in [1...m], t_i \in A\}$*

Ferreira and Zantema's theorem is then expressed as follows:

Theorem 5 (Ferreira-Zantema). *Let $\succ$ be a partial order on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and let $\succ^{\Lambda}$ be a term lifting of $\succ$. Suppose that $\succ$ has the subterm property:*

$$\forall f(t_1, ..., t_m) \in \mathcal{T}(\mathcal{F}, \mathcal{X}), \forall i \in [1...m], \quad f(t_1, ..., t_m) \succ t_i$$

And suppose that $\succ$ satisfies that for every term $f(t_1, ..., t_m)$ and $g(u_1, ..., u_n)$ in $\mathcal{T}(\mathcal{F}, \mathcal{X})$ with $m, n \in \mathbb{N}$:

If $s = f(s_1, ..., s_m) \succ g(t_1, ..., t_n) = t$, then either
- *$\exists i \in [1...m], s_i \succ t$ or $s_i = t$,*
- *or $s \succ^{\Lambda} t$.*

Then $\succ$ is noetherian on $\mathcal{T}(\mathcal{F}, \mathcal{X})$.

Ferreira and Zantema show in [7] that the noetherianity of the semantic path order [9] and of the general[3] path order [4] are direct consequences of their theorem 5.

We will qualify theorem 5 as an instance of theorem 4 by exhibiting the correct decomposition system Σ. Our solution is trivial but in a sense quite unexpected. The two sets $\mathcal{T}$ and $\mathcal{V}$ are identified as $\mathcal{T}(\mathcal{F}, \mathcal{X})$, and $\mathcal{L}$ is simply an arbitrary singleton $\{\clubsuit\}$. $\prec_{\mathcal{L}}$ is the empty relation (hence noetherian) on $\mathcal{L}$, $\prec_{\mathcal{V}} = \prec^{\Lambda}$ and our relation $\prec$ is Ferreira-Zantema's partial order $\prec$. The relation $\xrightarrow{\clubsuit}$ is just the identity relation on $\mathcal{T} = \mathcal{V}$:

[3] Therefore of the original recursive path order defined in [2], see [4] for a detailed discussion.

$$t \xrightarrow{\clubsuit} T \iff t = T$$

The relation $\vdash$ is defined as the tree deconstructor:

$$T \vdash t \iff T = f(t_1, ..., t_m) \text{ and } \exists i \in [1...m], t = t_i$$

We check that the decomposition system Σ verifies axioms I*—VI* when the three hypothesis in theorem 5 are fulfilled:

1. $t \xrightarrow{\clubsuit} T \vdash u$ implies that u is a principal subterm of t. Hence axiom I*,
2. The constant and variable terms are the only elementary terms in $\mathcal{T}(\mathcal{F}, \mathcal{X})$. Note that the second hypothesis in theorem 4 implies that $(t \succ u \Rightarrow t \succ^{\Lambda} u)$ whenever t and u are elementary. We claim that $\succ^{\Lambda}$ is noetherian on the set of elementary terms and conclude. Indeed $\succ$ is trivially noetherian on the empty set $\emptyset$ whose completion $\bar{\emptyset}$ is the set of elementary terms. The term lifting $\succ^{\Lambda}$ is therefore noetherian on the elementary terms. We conclude that axiom II* holds.
3. Axiom III* is a consequence of transitivity of $\succ$ and subterm property hypothesis in theorem 5,
4. The second hypothesis of theorem 5 can be reformulated as:
   If $s = f(s_1, ..., s_m) \succ g(t_1, ..., t_n) = t$, $s \xrightarrow{\clubsuit} S$ and $t \xrightarrow{\clubsuit} T$, then either
   - $\exists s' \in \mathcal{T}$ such that $S \vdash s'$ and $s' \succ t$,
   - or $S \vdash t$,
   - or $S \succ_{\mathcal{V}} T$.

   Axiom IV* follows.
5. Axiom V* is a consequence of the first hypothesis of theorem 5 that $\succ^{\Lambda}$ is a term lifting,
6. Axiom VI* is immediate.

5 Conclusion

The axiomatic proof of Kruskal theorem originates from an attempt to extend the scope of the recursive path ordering (RPO) techniques to higher-order calculi like the λ-calculus. Two directions are suggested here to tackle the problem.

In [12], a proof of strong normalisation of the simply-typed λ-calculus is presented very close in spirit to Nash-Williams' minimal bad sequence argument: A "zoom-in" strategy is shown to be perpetual and terminating on every simply-typed λ-term, qed. Unfortunately, this proof does not transfer yet to a general RPO technique for higher-order calculi.

Also, the discovery of a duality between Dershowitz theorems on RPOs and Kruskal theorem on well-quasi-orderings conveys the hope for a similar dualisation of Robertson-Seymour "graph minor" theorem (see [16] for a survey) into a theorem on RPOs applicable to graphs. Since λ-terms are graphs, this is certainly another approach to a theory of RPOs for higher-order calculi,

I would like to conclude the article with two open problems. Is there a constructive proof like [15] of Kruskal theorem on binary relations? Is it possible to axiomatize Kruskal-Dershowitz theorems à la [5, 14]?

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References

1. *Handbook of Mathematical Logic*, edited by J. Barwise, Studies in Logic and Foundations of Computer Science, North-Holland, 1977.
2. N. Dershowitz, “Orderings for term rewriting systems”. Theoretical Computer Science 17, 3, pp.279-301, 1982.
3. N. Dershowitz, “Termination of rewriting”. Journal of Symbolic Computation 3, 1 and 2, pp.69-116, 1987.
4. N. Dershowitz and C. Hoot, “Topics in termination”. in Proceedings of the 5th conference on Rewriting Techniques and Applications, C. Kirchner ed., LNCS 690, Springer, pp.198-212, 1983.
5. A. Ehrenfeucht, D. Hausslet and G. Rozenberg, “On regularity of context-free languages”, Theoretical Computer Science 27 (3), pp.311-332, 1983.
6. M. C. F. Ferreira and H. Zantema, “Syntactical Analysis of Total Termination”, proceedings of the 4th International Conference on Algebraic and Logic Programming, editors G. Levi and M. Rodrigues-Artalejo, Springer Lecture Notes in Computer Science, volume 850, pages 204 - 222, 1994.
7. M. C. F. Ferreira and H. Zantema, “Well-foundedness of Term Orderings”, in Conditional Term Rewriting Systems, proceedings fourth international workshop CTRS-94, editor N. Dershowitz, Springer Lecture Notes in Computer Science, volume 968, pages 106 - 123, 1995.
8. G. Higman, “Ordering by divisibility in Abstract Algebra”, Proc. London Math. Soc, 3(2), 326-336, 1952.
9. S. Kamin and J. J. Lévy, “Two generalizations of the recursive path ordering”, University of Illinois, 1980.
10. J. B. Kruskal, “Well-quasi-ordering, the tree theorem, and Vázsonyi’s conjecture”, Trans. Amer. Math. Soc. 95, pp.210-225, 1960.
11. P. Lescanne, “Well rewrite orderings”, In Proceedings, Fifth Annual IEEE Symposium on Logic in Computer Science, pages 249-256, Philadelphia, Pennsylvania, 4-7 June 1990. IEEE Computer Society Press.
12. P.-A. Melliès, “Description abstraite des systèmes de réécriture”, Thèse de l’Université Paris VII, Décembre 1996.
13. C. S. L. A. Nash-Williams, “On well-quasi ordering finite trees”, Proceedings Cambridge Phil. Soc. 59, 833—382, 1963.
14. L. Puel, “Using unavoidable sets of trees to generalize Kruskal’s theorem”, Journal of Symbolic Computation 8 , 335-382, 1989.
15. M. Rathjen, A. Weiermann, “Proof-theoretic investigations on Kruskal’s theorem”, Annals of Pure and Applied Logic, 60, 1993.
16. C. Thomassen, “Embeddings and Minors”, in *Handbook of Combinatorics*, Volume 1, R. L. Graham, M. Grötschel, L. Lovász editors, North-Holland, 1995.