Axiomatic Rewriting Theory IV

A stability theorem in Rewriting Theory

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Abstract

One key property of the λ-calculus is that there exists a minimal computation (the head-reduction) \( M \xrightarrow{\text{head}} V \) from a λ-term \( M \) to the set of its head-normal forms. Minimality here means categorical “reflectivity” i.e. that every reduction path \( M \xrightarrow{\text{red}} W \) to a head-normal form \( W \) factors (up to redex permutation) to a path \( M \xrightarrow{\text{head}} V \xrightarrow{\text{head}} W \). This paper establishes a stability à la Berry or poly-reflectivity theorem [D, La, T] which extends the minimality property to Rewriting Systems with critical pairs. The theorem is proved in the setting of Axiomatic Rewriting Systems where sets of head-normal forms are characterized by their frontier property in the spirit of [GK].

1 Axiomatic Rewriting Theory

Rewriting is a versatile model of computation which stretches from Turing Machines and Petri nets to λ-calculus and π-calculus. This versatility has generated in the past a variety of theories which are still poorly interconnected. Axiomatic Rewriting Theory [GLM, M, 1, 3] purports a unified treatment of Rewriting where every particular system is considered as part of a larger picture. General structures are exhibited as axioms and generic properties are established from them. The stability theorem presented here (also called fundamental theorem) is the most significant contribution of this approach. It inherits the conceptual clarity of its domain-theoretic ancestor, Berry stability theorem [Be], and the genericity of its own rewriting-theoretic principles. Informally, the theorem states that there exists from every term \( M \) a cone of minimal computations to the set of head-normal forms.

ARS. The framework of Axiomatic Rewriting Systems (ARSs) encompasses many existing Rewriting Systems like (left-linear) first-order rewriting systems with critical pairs, λ-calculus, call-by-value λ-calculus, λσ-calculus of explicit substitutions, Petri nets, CCS and π-calculus.

Formally, an ARS is a couple \((\mathcal{G}, \triangleright)\) which verifies a series of standardisation axioms formulated in [M, 1]. The couple \((\mathcal{G}, \triangleright)\) is composed by

1. a graph \( \mathcal{G} \), that is a quadruple \((\mathcal{T}, \mathcal{R}, \partial_0, \partial_1)\) consisting of a set \( \mathcal{T} \) of vertices (= terms), a set \( \mathcal{R} \) of arrows (= reduction steps, or redexes), and two functions \( \partial_0, \partial_1 : \mathcal{R} \rightarrow \mathcal{T} \) (= the source and target functions). We write \( u : M \rightarrow N \) when \( \partial_0(u) = M \) and \( \partial_1(u) = N \).

2. a binary relation \( \triangleright \) between paths\(^1\) in \( \mathcal{G} \).

The standardisation axioms try to capture the generic properties of redex permutations in a rewriting system. In particular, one axiom asks that two paths \( f \) and \( g \) are coinitial and cofinal when \( f \triangleright g \) and another axiom that \( f \) is of length 2 and \( g \) of length at least 1. So,

\(^1\)We recall that a path in a graph \( \mathcal{G} \) is a sequence \( f = (M_0, r_1, M_2, \ldots, M_n, r_m, M_{n+1}) \) where \( r_i : M_i \rightarrow M_{i+1} \) for every \( i \in \{1, \ldots, n\} \). We write \( f : M_0 \rightarrow M_n \). The length of \( f \) is \( n+1 \); \( f \) is said to be empty when \( n = 0 \). Two paths \( f : M \rightarrow N \) and \( g : P \rightarrow Q \) are coinitial (resp. cofinal) when \( M = P \) (resp. \( N = Q \)). \( f \triangleright g : M \rightarrow Q \) denotes the concatenation of two paths \( f : M \rightarrow P \) and \( g : P \rightarrow Q \).
every diagram $f \triangleright g$ in an ARS has the form:
\[
\begin{array}{c}
\begin{array}{c}
M \xrightarrow{u} Q \\
u \downarrow \equiv \\
P \xrightarrow{h} N
\end{array}
\end{array}
\begin{array}{c}
f = v; u' \\
g = u; h
\end{array}
\] 

where $u$, $v$ and $u'$ are redexes and $h$ is a path. For lack of space we do not present the axioms here and suggest [GLM, M, I] for a complete description.

A CONCRETE EXAMPLE. As an illustration we explicit the construction of the ARS $(G_{\lambda}, \triangleright_{\lambda})$ corresponding to the $\lambda$-calculus. $G_{\lambda}$ is the transition graph of the calculus; its vertices are the $\lambda$-terms up to $\alpha$-conversion, its arrows $u : M \rightarrow P$ are the $\beta$-redexes from $M$ to $P$. The relation $\triangleright_{\lambda}$ relates two paths $f \triangleright_{\lambda} g$ precisely when:

1. $f = v; u'$ and $g = u; h$ where $u$, $v$ and $u'$ are $\beta$-redexes and $h$ is a path,
2. the $\beta$-redex $u$ is not nested under $v$ wrt. the tree ordering,
3. $u'$ is the (unique) residual of $u$ after $v$ and $h$ develops the residuals of $v$ after $u$.

Two paradigmatic examples of such redex permutations $f \triangleright_{\lambda} g$ are:

\[
\begin{array}{c}
PQ \xrightarrow{u} P'Q \\
u \downarrow \equiv_1 \downarrow u' \\
PQ' \xrightarrow{v'} P'Q'
\end{array}
\begin{array}{c}
\Delta P \xrightarrow{v} \Delta P' \\
u \downarrow \equiv_2 \downarrow u' \\
PP \xrightarrow{v_1; v_2} P'P'
\end{array}
\]

where $P \rightarrow P'$ and $Q \rightarrow Q'$ are two $\beta$-redexes and $\Delta = (\lambda x.xx)$.

The standardisation axioms are easily verified on $(G_{\lambda}, \triangleright_{\lambda})$ which therefore is an ARS.

2 The fundamental theorem

DISJOINT VS NESTED. One distinctive feature of ARSs compared to asynchronous transition systems [WN] or concurrent transition systems [S] is that every permutation $f \triangleright g$ is "oriented". This accounts for the so-called standardisation procedure in Rewriting Theory; see [Ba, HL, Bo, GLM, CK] for details. In particular, a permutation $f \triangleright g$ is declared disjoint when $g \triangleright f$ and standardising when $\neg (g \triangleright f)$.

The concepts of nested and disjoint redexes can be derived as follows: two redexes $u$ and $v$ are declared disjoint, $u \not| v$, when there exists a disjoint permutation $uv; f \triangleright g$; a redex $u$ nests (or contains) another redex $v$, $u \prec v$, when there exists a standardising permutation $v; g \triangleright u; f$. The axiomatics of ARSs insures that $u \not\prec v$, $u \not| v$ and $v \not\prec u$ are exclusive.

Observe that the definitions of nested and disjoint redex correspond to the usual notions in the case of $(G_{\lambda}, \triangleright_{\lambda})$. Check in diagram (1) that the first permutation $uv; u' \triangleright u; v'$ is disjoint and that the second one $v; u' \triangleright u; v_1; v_2$ is standardising.

COMPATIBILITY. By definition, two coinitial redexes $u$ and $v$ can be permuted when $v \prec u$, $uv; f$ or $u \prec v$. We say in any of the three cases that $u$ and $v$ are compatible, otherwise that they are incompatible or conflicting. In $(G_{\lambda}, \triangleright_{\lambda})$ in particular, every two coinitial redexes are compatible.

LÉVY PERMUTATION EQUIVALENCE. The Lévy permutation equivalence $\equiv$ on the paths of $(G, \triangleright)$ is defined as the least equivalence relation (reflexive, symmetric, transitive) such that $d_1; f; d_2 \equiv d_1; g; d_2$ whenever there are paths $d_1 : M \rightarrow P$, $d_2 : Q \rightarrow N$ and $f, g : P \rightarrow Q$ such that $f \triangleright g$.

OPEN-STABLE SETS. We follow the spirit of [GK] and declare that a set $V$ of vertices is open-stable in $(G, \triangleright)$ when the three properties below are verified by every permutation diagram $v; u' \triangleright u; f$:

\[
\begin{array}{c}
\begin{array}{c}
M \xrightarrow{u} Q \\
u \downarrow \equiv \\
P \xrightarrow{f} N
\end{array}
\end{array}
\begin{array}{c}
v; u' \triangleright u; f
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
M \xrightarrow{u} Q \\
u \downarrow \equiv \\
P \xrightarrow{f} N
\end{array}
\end{array}
\begin{array}{c}
v; u' \triangleright u; f
\end{array}
\]
[head] if \( u \not\rightarrow v \) and \( Q \in \mathcal{V} \), then \( M \in \mathcal{V} \),

[stable] if \( u \not\rightarrow v \) and \( P, Q \in \mathcal{V} \), then \( M \in \mathcal{V} \),

[closed] if \( u \not\rightarrow v \) and \( Q \in \mathcal{V} \) then \( M \in \mathcal{V} \) or \( N \in \mathcal{V} \).

Observe that the condition [closed] is automatically verified when the set \( \mathcal{V} \) is reduction-closed because \( Q \in \mathcal{V} \) and \( Q \rightarrow N \) implies \( N \in \mathcal{V} \).

**The Fundamental Theorem.** Let \( \mathcal{V} \) be an openable set in the ARS \( (\mathcal{G}, \rightarrow) \). The following stability theorem is established in section 4:

For every term \( M \), there exists a cone \( (e_i : M \rightarrow V_i)_{i \in I} \) of paths with each \( V_i \in \mathcal{V} \) such that for every path \( f : M \rightarrow W \) with \( W \in \mathcal{V} \), there exists one and only one \( i \in I \) such that \( f \) factors as \( f \equiv e_i ; h \) for some path \( h \).

\[
\begin{array}{c}
M \\
\downarrow f \quad \downarrow h \\
V_i \\
W
\end{array}
\]

Moreover, the path \( h \) is unique up to \( \equiv \).

The theorem formalises the idea that there exists from every term \( M \) a cone of minimal computations to the set of head-normal forms \( \mathcal{V} \).

A series of examples follows.

**Example 1.** The set \( \mathcal{V}_\lambda \) of head-normal forms is openable in the \( \lambda \)-calculus ARS \( (\mathcal{G}_\lambda, \not\rightarrow) \). The [head] and [stable] conditions derive from the fact that a \( \beta \)-redex \( u : M \rightarrow V \) from \( M \not\in \mathcal{V}_\lambda \) to \( V \in \mathcal{V}_\lambda \) is necessarily \( M \)'s head \( \beta \)-redex. The [closed] condition is a consequence of reduction-closedness of \( \mathcal{V}_\lambda \). So, the fundamental theorem applies on \( \mathcal{V}_\lambda \) and repeats Levy's interpretation of Wadsworth's theorem, see [Ba] Theorem 8.3.11. Observe that the cone \( (e_i : M \rightarrow V_i)_{i \in I} \) is empty or singleton because the \( \lambda \)-calculus admits no conflict (orthogonality). This path is precisely the head-reduction described syntactically by Wadsworth.

**Example 2.** Another instance of the theorem was described by Plotkin in the \( \lambda \)-calculus [P]. Recall that a value is a variable or a \( \lambda \)-term of the form \( \lambda x.M \) and that the \( \beta \)-reduction \( (\lambda x.M)[V/x] \rightarrow M[V/x] \) is the \( \beta \)-rule restricted to value arguments \( V \). Since the \( \lambda \)-calculus verifies the axioms in [P] and the set of values is openable (w.r.t. the \( \lambda \)-calculus), the fundamental theorem implies that there is a head-reduction in the \( \lambda \)-calculus. In fact, this non-trivial reduction was described by Plotkin in [P] as the strategy which implements Landin's SECD machine. See also [FH].

**Example 3.** The fundamental theorem has been used by the author to prove that the weak evaluation strategy defined by Abadi and Cardelli on their object-\( \psi \)-calculus is complete, see [AC] pages 60-65. More generally, the fundamental theorem bridges the rewriting (one-step) and operational (big-step) semantics of programming languages.

**Example 4.** Examples 1, 2 and 3 consider non-conflicting systems where the theorem characterises at most one reduction path which, for correctness reasons, is generally chosen by implementers. But the distinctive feature of the fundamental theorem (and its conceptual novelty) is that it also applies to Rewriting Systems with conflicts (critical pairs). In that case the cone \( (e_i : M \rightarrow V_i)_{i \in I} \) may contain several paths, even be infinite. A simple example is the first order Rewriting System \( A \rightarrow A \) and \( A \rightarrow B \) where \( \mathcal{V} = \{B\} \). The set \( \{B\} \) is openable because \( B \) is a normal form. The minimal cone \( (e_i : A \rightarrow B)_{i \in I} \) contains every path \( A \rightarrow \cdots \rightarrow A \rightarrow B \) and is therefore infinite.

**Example 5.** A well-known calculus with conflict is the \( \lambda \sigma \)-calculus, a \( \lambda \)-calculus with explicit substitutions, see [ACCL] for details. Its eleven rules and eleven critical pairs combine to describe rich and subtle substitution mechanisms. Many traditional techniques were adapted to understand its rewriting dynamics but the whole picture was still unclear until the fundamental theorem. In fact, the theorem offers the first picture of the various reduction paths from a \( \lambda \sigma \)-term to its values (the \( \lambda \sigma \)-head-normal forms).

To see why the fundamental theorem applies let us say that the \( \lambda \sigma \)-calculus is a left-linear first-order rewriting system, therefore an ARS, and that the set \( \mathcal{V}_{\lambda \sigma} \) of \( \lambda \sigma \)-head-normal forms is openable for the same reasons as \( \mathcal{V}_\lambda \) is openable in example 1. Consequently, there exists from every \( \lambda \sigma \)-term \( M \) a canonical cone \( (e_i : M \rightarrow V_i)_{i \in I} \) of head-reductions to \( \mathcal{V}_{\lambda \sigma} \).

As an aside, it should be noted that this cone is finite. This result, proved syntactically in [M] and axiomatically in [2], has the very important consequence
that every needed strategy\textsuperscript{3} normalises — a property
which would be very difficult (and depressing) to es-
stablish with traditional techniques.

**Example 6.** Petri nets form another interesting class
of Rewriting Systems with conflict. A Petri net is a quadru-
ple \( N = (P, T, \text{pre}, \text{post}) \) where \( P \) is the set of
places and \( T \) is the set of transitions of the net. A state
of the net \( N \) is a multiset of elements of \( P \). To every
transition \( t \) the functions \( \text{pre, post} : T \rightarrow S \) associate
a pre-condition \( \text{pre}(t) \) and a post-condition \( \text{post}(t) \)
in the set of states \( S \). The multiset \( \text{pre}(t) \) is supposed
here to be non empty. We follow the spirit of [S] and
construct the ARS \((G, \triangleright)\) associated to the net \( N \).
Its transition graph \( \mathcal{G} \) is defined as follows:

1. its vertices are the states of \( N \),
2. its arrows \( p \rightarrow q \) are the triples \((p, t, q)\) such that
   \( p = p_0 \uplus \text{pre}(t) \) and \( q = p_0 \uplus \text{post}(t) \) for some state
   \( p_0 \), where \( \uplus \) is the multiset addition.

Two paths \( u; u' \triangleright u; u' \) are related precisely when:

1. \( u, v, w, w' \) and \( w' \) are redexes \( u = (p_1, t_1, p_2), v =
   (p_3, t_2, p_4) \) and \( w = (p_5, t_3, p_6) \),
2. both \( \text{pre}(t_1) \cap \text{pre}(t_2) \) and \( \text{post}(t_1) \cap \text{post}(t_2) \) are
   empty multisets (this implies that \( p_2 = p_1 \)).

The ARS \((G, \triangleright)\) is "linear" in the sense that every two
compatible redexes are disjoint. Let us fix a place \( p \)
and define \( \mathcal{V}_p \) as the set of states containing at least
one occurrence of \( p \). \( \mathcal{V}_p \) then is open-closed: it veri-
fies the [head] condition by linearity, the [stable] con-
dition because the post-conditions of disjoint redexes
are disjoint, and the [closed] condition because the pre-
conditions of \( u \) and \( u' \) are equal whenever \( u; u' \triangleright u; u' \).
So, the fundamental theorem applies here although \( \mathcal{V}_p \)
is not necessarily reduction-closed.

**Example 7.** One interesting and maybe unex-
pected point about our stability theorem is that it ap-
pies also to non sequential Rewriting Systems. Take
for instance the parallel or operator defined on the
signature \( \{\bot, \text{true}\} \) by the rules \( \text{par(true, x)} \rightarrow \text{true},
\text{par(x, true)} \rightarrow \text{true}, \bot \rightarrow \text{true} \). The set \( \mathcal{V} =
\{\text{true}\} \) is open-stable and the minimal cone
\((\text{e}_1 : \text{par}((\bot, \bot)) \rightarrow \text{true})_\epsilon \)\textsuperscript{4}
takes the two following computations:

\[ e_1 : \text{par}(\bot, \bot) \rightarrow \text{par}(\text{true}, \bot) \rightarrow \text{true} \]
\[ e_2 : \text{par}(\bot, \bot) \rightarrow \text{par}(\bot, \text{true}) \rightarrow \text{true} \]

The traditional classification of \( \text{par}(-, -) \) as a non
stable operator can be interpreted as the effect of
projecting the two incompatible computations \( e_1 \) and
\( e_2 \) to compatible morphisms \((\bot, \bot) \rightarrow (\text{true}, \bot)
\) and \((\bot, \bot) \rightarrow (\bot, \text{true}) \) in the denotational model.

**Example 8.** Many symbolic constraint problems can
be expressed and solved in the framework of Conditional
Rewriting Systems [BK], see [CDJK]. The funda-
mental theorem should help to design lazy evalua-
tion techniques for them.

3 An axiomatic toolbox

This section recalls the various tools developed in
[GLM, M, 1, 2, 3].

**Standardisation theorem.** In the following, we
write \( d \Rightarrow e \) when there is a sequence of oriented
permutations operating from \( d \) to \( e \), and \( d \approx e \)
when all the permutations are disjoint in the sequence\textsuperscript{4}. In
particular, \( d \approx e \) implies \( d \Rightarrow e \) and \( e \Rightarrow d \). Notice
that the Lévy permutation equivalence \( \equiv \) is simply
the symmetric transitive closure of \( \Rightarrow \).

An example is \( v_1; v'_2 \Rightarrow u; v_2; v'_3 \) in the ARS \((G; \triangleright)\):

\[ \Delta(Ia) \]

\[ \Delta(a) \]

\[ \Delta = \lambda x. xx \]

\[ I = \lambda x. x \]

\[ v_1; v'_2 \Rightarrow u; v_1; v'_2 \]

\[ v_1; v'_3 \Rightarrow v_2; v'_3 \]

A path \( d \) is standard when no standardising permuta-
3See [M, 2] for a definition of needed strategies in Rewriting

\textsuperscript{4}In fact [1, 2, 3] interprets every ARS \((G, \triangleright)\) as a 2-category

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\( C \) whose cells \( a : d \Rightarrow e \) between paths \( d : M \rightarrow P \) and \( e : M \rightarrow P \)

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\( C \) whose cells \( a : d \Rightarrow e \) between paths \( d : M \rightarrow P \) and \( e : M \rightarrow P \)
tion can operate on \( d \) after a sequence of disjoint permutations. In particular, \( d 
Rightarrow e \) implies \( d \simeq e \).

The **axiom of standardisation** theorem established in [GLM, M, 1] states that there exists a standard path in every Lévy permutation class and that this path is unique up to disjoint permutations: If we associate to each path \( d \) a standard path \( \|d\) such that \( d \equiv \|d \), the uniqueness property can be expressed as (2):

\[
\text{For every paths } d \text{ and } e, \quad d \equiv e \Rightarrow \|d \simeq \|e \quad (2)
\]

**EXTERNAL PATH.** A path \( e : M \rightarrow P \) is external, see [M, 2], when for every standard path \( f : P \rightarrow Q \), the path \( e; f \) is standard. Externally of \( e \) can also be expressed this way:

\[
\forall Q \in T, \forall f : P \rightarrow Q, \quad e; f \simeq \|e; f
\]

For example, any head-redex \( (\lambda x.M)N \rightarrow M[N/x] \) is external in \( (\mathcal{G}, \beta \Delta) \) whereas the \( \beta \)-redex \( \Delta(Ia) \rightarrow \Delta a \) is not external. Observe also that any normalising and standard path is external by definition.

We say that a path \( m : M \rightarrow Q \) is **internal** when there is no internal path \( e : M \rightarrow P \) (except the identity) such that \( m \equiv e; f \).

**A FACTORISATION THEOREM.** In the following, \( \mathcal{C} \) denotes the free category on \( \mathcal{G} \) (its morphisms are the paths \( d : M \rightarrow P \)) and \( [\mathcal{C}] \) the category \( \mathcal{C} \) quotiented by the permutation equivalence \( \equiv \) (its morphisms are the equivalence classes of paths wrt. \( \equiv \)). We write \( [-] : \mathcal{C} \rightarrow [\mathcal{C}] \) the functor which associates to every path \( d \) its equivalence class \( \|d \).

Write \( \mathcal{E} \) and \( \mathcal{M} \) for the classes of external and internal paths in \( \mathcal{C} \) and \( [\mathcal{E}] \) and \( [\mathcal{M}] \) for their image in \( [\mathcal{C}] \). A factorisation theorem established in [3] states that \( ([\mathcal{E}]), [\mathcal{M}] \) is a factorisation system of \( [\mathcal{C}] \) in the sense of Freyd and Kelly [FK]. This means that the three following properties are verified in every *Axiomatic Rewriting System*:

1. \( [\mathcal{E}] \) and \( [\mathcal{M}] \) are categories,
2. every morphism \( f \) can be factored as \( f = \epsilon; m \) with \( \epsilon \in [\mathcal{E}] \) and \( m \in [\mathcal{M}] \),
3. the factorisation is functorial: if \((\epsilon; m); v = u; (f; n)\) where \( \epsilon, f \in [\mathcal{E}] \) and \( m, n \in [\mathcal{M}] \), there is a unique \( u \) rendering commutative the diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{\epsilon} & N \\
\downarrow v & \ & \downarrow u \\
M' & \leftarrow f & N'
\end{array}
\]

Conceptually, factorisation means that the efficient (external) part of a computation can always be separated from its junk. Technically, it permits to focus one’s attention to the space of external paths.

**4 Proof of the fundamental theorem**

**METHODOLOGY.** We prove the fundamental theorem in three steps:

1. we prove that every internal path \( m \) is Lévy equivalent to a sequence of internal redexes,
2. we prove that every internal path \( m : M \rightarrow P \) is useless in the sense that \( M \notin \mathcal{V} \) implies \( P \notin \mathcal{V} \),
3. we compare the various external paths from \( M \) to \( \mathcal{V} \) and establish that one of them is **minimal** in every computational direction. This exhibits a cone \((e_i : M \rightarrow V_i)_{i \in I}\) of minimal external paths; we then conclude with factorisation and step 2.

Each step \( i \) is tackled in section 4.i.

We will not introduce the standardisation axioms here, see [GLM, M, 1] for details. Instead we present in Figure 1 the seven properties of every ARS \((\mathcal{G}, \triangleright)\) that we use in the course of the proof.

**4.1 The segmentation theorem**

By factorisation, \( \mathcal{M} \) is a subcategory of \( \mathcal{C} \). In particular, every path \( \triangleright_1 \cdots \triangleright_n \) consisting of internal redexes \( \triangleright_i \)'s is internal. We establish the converse property here that every internal path \( m : P \rightarrow Q \) is Lévy equivalent to a sequence \( \triangleright_1 \cdots \triangleright_n \) of internal redexes. The discovery that this **segmentation** property holds in every ARS was a big, big surprise, and in fact the starting point of this paper.

The situation differs notably in the case of external paths. A reduction path like

\[
((\lambda a) b) s[t] \xrightarrow{\text{Beta}} a b \cdot \text{id} s[t] \xrightarrow{\text{Clis}} a((b \cdot \text{id}) \circ s)[t]
\]
4.2 Internal paths are "junk"

We prove that an internal redex is useless by first characterising standard paths as the paths which do not contain any anti-standard path. Remark: this characterisation was our original definition of standard path in [GLM].

ANTI-STANDARD PATH. Anti-standard paths\(^5\) are defined by induction (see figure 2) as follows:

1. a path \(M \xrightarrow{u} P \xrightarrow{kl} P \xrightarrow{e} N\) is anti-standard when \(u\) and \(v\) are redexes and \(u;v\) is not standard,

2. a path \(M \xrightarrow{u} P \xrightarrow{f} Q \xrightarrow{v} N\) is anti-standard when \(u\) and \(v\) are redexes, \(\frac{u}{u'} \sim \frac{u}{u'}\) and \(M' \xrightarrow{u'} P' \xrightarrow{e} Q \xrightarrow{v} N\) is anti-standard.

\[
\begin{array}{c}
M \xrightarrow{u'} M' \\
\downarrow \sim \\
P \xrightarrow{e} Q
\end{array} \xrightarrow{e} Q \xrightarrow{v} N
\]

Figure 2. The inductive step in the definition of anti-standard paths

Lemma 4.4 ([1]) A path \(d;e\) is standard if and only if

- the path \(d\) is standard, and
- the path \(u;e\) is standard whenever \(u\) stops \(d\).

Corollary 4.5 If a redex \(u\) is internal, then there exists an anti-standard path \(M \xrightarrow{u} P \xrightarrow{e} Q \xrightarrow{v} N\).

Proof By definition, an internal redex is non external. Hence there is a standard path \(f : P \xrightarrow{e} Q\) such that \(u;f : M \xrightarrow{e} M'\) is not standard. By lemma 4.4, there is an anti-standard path in \(u;f\), but not in \(f\). We conclude.

Lemma 4.6 Suppose that \(V\) is open-stable and that \(u : M \xrightarrow{v} V\) is an internal redex. If \(V \in \mathcal{V}\), then \(M \in \mathcal{V}\).

\(^5\)Anti-standard paths were called conflicts in [GLM]. The terminology here is inspired by [K]'s anti-standard pairs.
4.3 A cone of minimal external paths

Proof By induction on the length of anti-standard paths \( M \xrightarrow{\alpha} V \xrightarrow{r} P \xrightarrow{e} Q \xrightarrow{u} N \) such that \( V \in \mathcal{V} \). If \( e \) is empty, then \( u,v \) is not standard, hence \( u,v \triangleright w,h \) for some redex \( w \) which contains \( u,v \), so that \( M \in \mathcal{V} \) by [head]. If \( e = \alpha \), then \( M \xrightarrow{\alpha} V \xrightarrow{r} P \cong M \xrightarrow{\alpha} M' \xrightarrow{\alpha} P \) and the path \( M' \xrightarrow{\alpha} P \xrightarrow{e} Q \xrightarrow{u} N \) is anti-standard.

\[
\begin{array}{c}
M \\
\xrightarrow{\alpha}
\end{array} \quad \begin{array}{c}
M' \\
\xrightarrow{\alpha}
\end{array} \quad \begin{array}{c}
P \\
\xrightarrow{e}
\end{array} \quad \begin{array}{c}
Q \\
\xrightarrow{u}
\end{array} \quad \begin{array}{c}
N
\end{array}
\]

By [closed], either \( M \in \mathcal{V} \), and we conclude, or \( P \in \mathcal{V} \). In that case, by induction, \( M' \in \mathcal{V} \) and by [stable], \( M \in \mathcal{V} \). We conclude. 

The main result of the subsection is a consequence of segmentation (theorem 4.3) and lemma 4.6.

**Lemma 4.7** Suppose that \( \mathcal{V} \) is open-stable. If \( m : M \rightarrow V \), \( m \in \mathcal{M} \) and \( V \in \mathcal{V} \), then \( M \in \mathcal{V} \).

4.3 A cone of minimal external paths

An important consequence of factorisation and Lemma 4.7 is that every path \( d : M \rightarrow W \) to \( W \in \mathcal{V} \) factors as \( d = \alpha ; m \) where \( e : M \rightarrow V \) is external and \( V \in \mathcal{V} \) is a value. Armed with this property we can focus our attention to the space of external path whose structure is much simpler than the global reduction space.

**Lemma 4.8** Let \( \mathcal{V} \) be an open-stable set. Suppose that \( e_1 : M \rightarrow V_1 \) to \( V_1 \in \mathcal{V} \) and \( e_2 : M \rightarrow V_2 \) to \( V_2 \in \mathcal{V} \) are external paths, and that there are paths \( h_1 : V_1 \rightarrow N \) and \( h_2 : V_2 \rightarrow N \) such that \( e_1 ; h_1 \equiv e_2 ; h_2 \). Then there exists a path \( e : M \rightarrow V \) to \( V \in \mathcal{V} \) and paths \( i_1 : V \rightarrow V_1 \) and \( i_2 : V \rightarrow V_2 \) such that \( e_1 \equiv e \cdot i_1 \) and \( e_2 \equiv e \cdot i_2 \).

Proof See appendix.

From lemma 4.8 we obtain a cone of minimal paths from \( M \) to terms \( V_i \) \( \in \mathcal{V} \). Theorem 4.8 shows that this cone is the poly-reflection cone we are looking for.

**Minimal.** An external path \( e : M \rightarrow V \) is minimal when \( V \in \mathcal{V} \), and for every \( M \xrightarrow{f} W \xrightarrow{g} V \) with \( e \equiv f;g \) and \( W \in \mathcal{V} \) we have \( g = \text{id}_V \).

**Theorem 4.9 (fundamental theorem)** Let \( \mathcal{V} \) be an open-stable set of the ARS \( (\mathcal{G}, \triangleright) \). For every term \( M \), there exists a cone \( (e_i : M \rightarrow V_i)_{i \in I} \) of paths with each \( V_i \in \mathcal{V} \) such that for every path \( f : M \rightarrow W \) with \( W \in \mathcal{V} \), there exists one and only one \( i \in I \) such that \( f \equiv e_i ; h \) for some path \( h \).

\[
\begin{array}{c}
M \\
\xrightarrow{e_i}
\end{array} \quad \begin{array}{c}
V_i \\
\xrightarrow{h}
\end{array} \quad \begin{array}{c}
W
\end{array}
\]

Moreover, this path \( h \) is unique up to \( \equiv \).

Proof See appendix.

5 A categorical approach

We have just established a very useful rewriting theorem but we would like to go one step further and connect our work with mainstream denotational semantics. Our motto is that the fundamental theorem is really a stability theorem in the spirit of Berry [Be].

To understand this, assume that a functor \( [-] : \mathcal{C} \rightarrow \mathcal{S} \) interprets the terms of the ARS \( (\mathcal{G}, \triangleright) \) in a small category \( \mathcal{S} \). Assume now that the functor verifies a series of conditions corresponding in fact to open-stability of Section 2. We prove (see theorem 5.4) that the restriction of \( [-] \) to each slice category \( M \downarrow \mathcal{C} \) is a stable functor.

Stable functors introduced in [La, T] generalise the well-known concept of a functor with a left adjoint and extend Berry's definition of stability on \( \mathcal{D} \)-domains [Be]. A functor \( \mathcal{X} \rightarrow \mathcal{Y} \) is stable when its restriction to each slice \( \mathcal{X} \downarrow \mathcal{X} \) has a left adjoint, see [T] for alternative definitions.

**Notation.** If \( \mathcal{X} \) denotes a category and \( M \) an object of \( \mathcal{X} \), then \( \mathcal{X}^M \) denotes the slice category usually noted \( M \downarrow \mathcal{X} \). Similarly, if \( F : \mathcal{X} \rightarrow \mathcal{Y} \) is a functor, \( F^M : \mathcal{X}^M \rightarrow \mathcal{Y}^{FM} \) denotes the functor restricting \( F \) to the slice \( \mathcal{X}^M \).

We introduce the conditions we need on \( [-] \) to prove stability, theorem 5.4.

**Interpretation functor.** A functor \( [-] : \mathcal{C} \rightarrow \mathcal{S} \) is an interpretation functor when, for every two compatible redexes \( u \) and \( v \):
Lemma 5.2 The functor \([-]_S^M : \mathcal{C}_\mathbb{S}^M \rightarrow \mathcal{S}_\mathbb{S}^M\) is stable when \(S\) is small.

Proof (sketch) \(\mathcal{C}_\mathbb{S}^M\) has pullbacks and cofiltered colimits, hence all small wide pullbacks, which are moreover preserved by \([-]_S^M\) thanks to the [stable] condition. We conclude that \([-]_S^M\) is stable with the smallness of \(S\) and the solution set condition, see [T], theorem 1.3.2.

Lemma 5.3 (Barr) Given

\[
\begin{array}{ccc}
A' & \xrightarrow{v'} & B' \\
\downarrow & & \downarrow \\
A & \xrightarrow{v} & B
\end{array}
\]

exhibiting \(A'\) as a coreflective subcategory of \(A\) and \(B'\) as a coreflective subcategory of \(B\), and supposing that both functors from \(A\) to \(B\) are the same up to natural isomorphism: if \(V\) has a left adjoint, so has \(V'\).

Theorem 5.4 (stability) Assume that \([-]_\Sigma\) is an interpretation functor from \(C\) to a small category \(S\). Then every functor \([-]_\Sigma^M : [\mathcal{C}]^M \rightarrow \mathcal{S}_\mathbb{S}^M\) is stable.

Proof We prove that \([-]_\Sigma^M : [\mathcal{E}]^M \rightarrow \mathcal{S}_\mathbb{S}^M\) is stable. This means that for every external path \(e : M \rightarrow N\), the “slice” functor \([-]_\Sigma e : [\mathcal{E}]^M \rightarrow [\mathcal{E}]^N\) is a right adjoint — where \([\mathcal{E}]^M\) and \([\mathcal{E}]^N\) are the slices \([\mathcal{E}]^M \downarrow e\) and \([\mathcal{E}]^N \downarrow e\) of \([\mathcal{E}]^M\) and \([\mathcal{E}]^N\).

We use the following diagram with the obvious denotations for \(\mathcal{C}_\Sigma = (\mathcal{C}_\mathbb{S})^M \downarrow e\) and \([\mathcal{E}]^M\):

\[
\begin{array}{ccc}
[\mathcal{E}]^M & \xrightarrow{v} & [\mathcal{S}]^M \\
\downarrow & & \downarrow \\
\mathcal{C}_\Sigma & \xrightarrow{U^\Sigma_\mathbb{S}} & \mathcal{S}_\Sigma \\
\downarrow & & \downarrow \\
\mathcal{C}_\Sigma & \xrightarrow{[-]_\Sigma^M} & \mathcal{S}_\mathbb{S}^M
\end{array}
\]

The functors \(U^\Sigma_\mathbb{S}\) and \(G^\Sigma\) are the slices of the forgetful functor \(U_\mathbb{S} : [\mathcal{C}] \rightarrow \mathcal{C}_\mathbb{S}\) and the functor \(G : \mathcal{C}_\mathbb{S} \rightarrow [\mathcal{E}]\) obtained by precomposing \(F : [\mathcal{C}] \rightarrow [\mathcal{E}]\) with the forgetful functor \(V : \mathcal{C}_\mathbb{S} \rightarrow [\mathcal{C}]\) of \(\mathcal{C}_\mathbb{S}\). We explicit the adjunction \(U^\Sigma_\mathbb{S} \dashv G^\Sigma\) as follows: letting \((f_1, f_2)\) be a couple of \([\mathcal{E}]\)-morphisms \(M \xrightarrow{f_1} P \xrightarrow{f_2} N\) and \((g_1, g_2)\) of \(\mathcal{C}_\mathbb{S}\)-morphisms \(M \xrightarrow{g_1} Q \xrightarrow{g_2} N\) such that \(U_\mathbb{S}(f_1; f_2) = g_1; g_2 = e\) in \(\mathcal{C}_\mathbb{S}\), there are two bijections, both natural in \(f_1, f_2\) and \(g_1, g_2\):

\[
\mathcal{C}_\mathbb{S}(U^\Sigma_\mathbb{S}(f_1; f_2), (g_1, g_2)) \cong [\mathcal{E}]^M((U^\Sigma_\mathbb{S}(f_1; f_2), V^\Sigma(g_1, g_2))
\]

(3)
and

\[ [F]^e(U^e(f_1, f_2), V^e(g_1, g_2)) \cong [E]^e(f_1, f_2, G^e(g_1, g_2)) \tag{4} \]

Bijection (3) is coming from the fact that \( f_1 \) is external and \( g_1 \) is standard, using figure 1(D). Bijection (4) is the adjunction \( U^e : F^e \rightharpoonup E^e \) projected on the slices \([C]^e\) and \([E]^e\). Composing the two bijections, we obtain \( U^e \rightharpoonup G^e \).

The functor \([-]_{C}^e\) has a left adjoint by lemma 5.2, so we use Barr’s lemma and deduce that \([-]_{E}^e\) has a left adjoint too. The functor \([-]_{E}^M\) is therefore stable, and so is \([-]_{C}^M\).

\section*{Example 1}

Take the ARS \((G, \triangleright)\) and the domain \(S\) of Böhm trees. The functor \([-]\) which associates to every \( \lambda \)-term its “immediate” Böhm tree is an interpretation functor. Each restriction \([-]_{C}^M\) is therefore stable.

\section*{Example 2}

Qualitative domains can be seen as simple forms of ARSs. A qualitative domain \([G]\) is a couple \((E, X)\) consisting of a set \(X\) called the web and a set \(E\) of subsets of \(X\) such that:

1. if \(e \in E\) and \(e' \subseteq e\), then \(e' \in E\),
2. \(E\) is closed under directed unions.

The definition of the ARS \((E, X)_{ARS} = (G, \triangleright)\) corresponding to a qualitative domain \((E, X)\) is straightforward:

1. \(G\)'s vertices are the finite subsets of \(X\) elements of \(E\); \(G\)'s arrows \(e \rightarrow e'\) are the triples \((e, x, e')\) where \(x \in X\), and \(e'\) is the disjoint union of \(e\) and the singleton \(\{x\}\),
2. \(f \triangleright g\) when \(f = v; u'\) and \(g = u; v'\) with \(u = (e, x, e_1)\), \(v = (e, y, e_2)\), \(u' = (e_2, x, e')\), \(v' = (e_1, y, e')\) and \(x \neq y\).

The category \([C]\) corresponding to \((E, X)_{ARS}\) is an order category isomorphic to the slice category \([C]^e\).

So, the theorem 5.4 tells here that a continuous function \([-]_{C}\) from \((E, X)\) to \((F, Y)\) is stable (in the usual sense) if and only if \([-]_{C}\) verifies [stable] as a functor on \((E, X)_{ARS}\), or put another way, if and only if

\[ [e] = [e \cup x] \cap [e \cup y] \]

for every \(e \in E\) and every two elements \(x \neq y\) of \(X \setminus e\) such that \(e \cup \{x, y\} \in E\).

\section*{Example 3}

Let \(\Sigma = \{ \perp \rightarrow T \} \) be the Sierpinsky category. Interpretation functors \([\cdot] : C \rightarrow \Sigma\) are in one-to-one correspondence with reduction-closed open-stable sets on \((G, \triangleright)\). In fact, the stability of \([\cdot]_{C}^M\) implies the existence of a poly-reflective cone \((e_i : M \rightarrow V_i)_{i \in I}\) from \(M\) to the inverse image of \(T\) — hence the fundamental theorem 4.9 in the case of reduction-closed open-stable sets.

\section*{6 Related works and open problems}

A result related to ours appears in [PSS] where a class of monotone input/output automata is studied and shown to compute stable functions when it verifies a local stability property. The result can be interpreted in the light of theorem 4.9 by projecting the universal cone \((e_i : M \rightarrow V_i)_{i \in I}\) to the subautomata of non-input events. The projected cone is in general not universal (see example 7 in Section 2) but in the case of deterministic systems like [PSS]'s, it is.

Other “linear” systems with intrinsic stability are Mazurkiewicz trace languages and asynchronous transition systems, see [WN]. In fact, stability of such true concurrency models is behind their interpretation as event structures.

To conclude, let us mention [GK] where a theory of needed computations is developed in a conflict-free duplicating abstract framework. [GK] contains the interesting idea of describing sets of head-normal forms from their frontier property, and we reuse the idea in Section 2 with our notion of open-stable set.

We list a few open problems:

1. the fundamental theorem offers a new perspective for the analysis of normalisation properties, see for instance [M, 2] and its treatment of the \(\lambda\)–calculus. A connection with the existing normalisation results on well-balanced [Tr] and almost orthogonal [R] rewriting systems would clarify the field.

2. our axiomatisation is at an early stage of development. In particular graph rewriting systems like DAGs (directed acyclic graph) are not described directly but using their translation as trees, see [CK, M] for details. Also, associative-
commutative rules are not modelled as they should be, as isomorphisms in the Lévy category.

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References


Figure 3. Case A. in lemma 4.8.

7 Appendix

Proof of lemma 4.8

We first characterise starts and stops as crossing and co-crossing redexes.

CROSS. A redex $u : M \rightarrow P$ crosses a path $e : M \rightarrow N$ to a redex $v : N \rightarrow P$ when:

1. $u = v$ and $e = \text{id}_M$,
2. $e = e'v'$ and $v' \rightarrow u$ and $u'$ crosses $e'$ to $v$.

Co-CROSS. A redex $u : Q \rightarrow N$ co-crosses a path $e : M \rightarrow N$ to a redex $v : P \rightarrow M$ when:

1. $u = v$ and $e = \text{id}_P$,
2. $e = e'v$ and $v' \rightarrow u'$ and $v'$ co-crosses $e'$ to $u$.

Lemma 7.1 ([1]) A redex $u : M \rightarrow P$ starts a path $d : M \rightarrow N$ if and only if $d$ factors as $d = d_1v; d_2$ and $u$ crosses $d_1$ to $v$.

Lemma 7.2 ([1]) A redex $u : Q \rightarrow N$ stops a path $d : M \rightarrow N$ if and only if $d$ factors as $d = d_1v; d_2$ and $u$ co-crosses $d_2$ to $v$.

Lemma 7.3 Suppose that $V_1, V_2$ and $W$ are $\mathcal{V}$-values and that a path $e : V_1 \rightarrow V_2$ and a redex $v : W \rightarrow V_2$. If $v$ co-crosses $e$ to a redex $u : M \rightarrow V_1$, then $M \in \mathcal{V}$.

Proof. Proceed as in lemma 4.6, by induction on the length of $e$.

Proof (of lemma 4.8) By induction on the couple $(n_1, n_2)$ lexicographically ordered, where $n_1$ and $n_2$ are the lengths of $e_1$ and $e_2$. Let $u$ be the first reduction step of $e_1$ so that $e_1 = u; E_1$ for some path $E_1$. Note that $h_1$ and $h_2$ can be chosen standard (take $\downarrow h_1$ and $\downarrow h_2$) so that $e_1; h_1 \approx e_2; h_2$. With this choice $u$ starts $e_2; h_2$, and by lemma 7.1 one of two following cases occurs:

A. either $u$ starts $e_2$, hence $M \xrightarrow{u} M' \xrightarrow{E_2} V_2 \approx M \xrightarrow{e_2} V_2$ for some path $E_2$,

B. or $u$ crosses $e_2$ to a redex $v$ which starts $h_2$. In particular $M \xrightarrow{u} M' \xrightarrow{E_2} W_2 \approx M \xrightarrow{e_2} V_2 \xrightarrow{v} W_2$ and $V_2 \xrightarrow{v} W_2 \xrightarrow{h_2} N \approx V_2 \xrightarrow{h_2} N$ for some paths $E_2$ and $H_2$.

If case A. occurs, $E_1 : M' \rightarrow V_1$ and $E_2 : M' \rightarrow V_2$ are external by right-decomposition of external paths, see figure 1(B). Moreover, $E_1; h_1 \approx E_2; h_2$ by figure 1(D) and $u; E_1; h_1 \approx u; E_2; h_2$. By induction hypothesis, there is a path $e' : M' \rightarrow V$ to some $V \in \mathcal{V}$ and paths $i_1 : V \rightarrow V_1$ and $i_2 : V \rightarrow V_2$ such that $e'; i_1 \approx E_1$ and $e'; i_2 \approx E_2$. We conclude that $e = u; e'$ verifies the properties $e; i_1 \approx e_1$ and $e; i_2 \approx e_2$.

In case B., observe before anything else that $u; E_2; h_2 \approx u; E_1; h_1$ implies with figure 1(D) that $E_2; h_2 \approx E_1; h_1$. Now, suppose that there exists a term $W \in \mathcal{V}$ so that $e_2$ factors as $e_2 \approx f; g$ where $M \xrightarrow{f} W \xrightarrow{g} V_2$ and $g$ is non empty. We call this case B1. By the factorisation theorem, $f$ factors as $f \equiv e_2; m_f$, with $e_2$ external and $m_f$ internal. Note that the path $f$ is standard because $e_2$ is standard. The series of equivalence

$$f \approx \downarrow f \approx \downarrow e_2; m_f \approx e_2; \downarrow m_f$$

shows that the length of $e_2; m_f$ is less than the length of $f$, hence strictly less than the length of $e_2$. Moreover, the target of $e_2; m_f : M \rightarrow W_2$ is a term $W_2 \in \mathcal{V}$, and if $h_f$ denotes $\downarrow m_f; g; h_2$, then $e_2; h_f \approx \approx e_2; h_2$, hence $e_2; h_f \approx \approx e_1; h_1$. By induction hypothesis applied to $e_1 : M \rightarrow V_1$ and $e_f : M \rightarrow W_2$, there
exists a path $E_f : M \to W_f$ to $W_f \in \mathcal{V}$ and two paths $j_1 : W_f \to V_1$ and $j_2 : W_f \to W_2$ such that $E_f ; j_1 \equiv e_1$ and $E_f ; j_2 \equiv e_f$. We conclude that $e = E_f$, $i_1 = j_1$ and $i_2 = j_2 \upharpoonright_{M_2} g$ verify the required properties $e; i_1 \simeq e_1$ and $e; i_2 \simeq e_2$.

We still have to consider case B2, when $f; g \simeq e_2$ and $g$ non empty implies that $f$'s target is not in $\mathcal{V}$. Now, let $w : V_2 \to V_3$ be the last reduction step of $e_2$. By [closed], either $V_3 \in \mathcal{V}$ or $W_2 \notin \mathcal{V}$. The first case has been treated in case B1 and we treat the second case here. First of all, the path $E_2$ is standard because $w; E_2; H_2 \simeq e_2; h_2$ and $e_2; h_2$ is standard, but $E_2$ is not necessarily external. By the factorisation theorem $E_2$ factors as $E_2 \equiv E_3; M_5$ where $E_3 : M' \to W_3$ is external and $M_3 : W_3 \to W_2$ is internal. By lemma 4.7, $W_3 \in \mathcal{V}$. By the series of equivalence

$$
E_3; \downharpoonright_{M_3} H_2 \\
\simeq \downharpoonright_{E_3; M_3} H_2 \quad \text{because } E_3 \text{ is external},
$$

$$
\simeq \downharpoonright_{E_1; H_1} \quad \text{because } E_2; H_2 \equiv E_1; H_1,
$$

$$
\simeq E_1; \downharpoonright_{H_1} \quad \text{because } E_1 \text{ is external}.
$$

the induction hypothesis can be applied on $E_1 : M' \to V_1$ and $E_3 : M' \to W_3$ to show that there exists a path $E : M' \to W$ to $W \in \mathcal{V}$ and paths $J_1 : W \to V_1$ and $J_2 : W \to W_2$ such that $E; J_1 \simeq E_1$ and $E ; J_2 \simeq E_3$. If $j_1 = J_1$ and $j_2 = J_2 \downharpoonright M_3$, we deduce that $E; j_1 \simeq E_1$ and $E; j_2 \simeq E_3$ with the series of equivalence

$$
E; j_2 \\
= E_3; \downharpoonright_{M_3} \quad \text{because } E; j_2 \equiv E_3,
$$

$$
\simeq \downharpoonright_{E_3; M_3} \quad \text{because } E_3 \text{ is external},
$$

$$
\simeq E_2 \quad \text{because } E_2 \text{ is standard}.
$$

Note that the redex $v$ stops $w; E_2 \simeq w; E; j_2$, hence by lemma 7.2 $v$ co-crosses $j_2$ to some redex $w : W_F \to W$ and this redex $w$ co-crosses $E$ to the redex $u$. The definition of co-crossing implies the existence of paths $F$ and $G$ such that $u$ crosses $F$ to $w$ and $w$ crosses $G$ to $v$ — see figure 5. Here, $W_F \in \mathcal{V}$ because $V_2 \in \mathcal{V}$, $W_2 \in \mathcal{V}$ and $W \in \mathcal{V}$, see lemma 7.3. By figure 1(D) and the series of equivalence

$$
F; G; v \simeq F; w; j_2 \simeq u; E; j_2 \simeq u; E; e_2 \simeq v
$$

we deduce $F; G \simeq e_2$. We are in case B2, and $W_F \in \mathcal{V}$, hence $G$ must be empty, therefore $j_2$ is empty too, and consequently $E \simeq E_2$. We conclude that $e = e_2$, $i_1 = v; j_1$ and $i_2 = \text{id}_W v$ verify $e; i_1 \simeq e_1$ and $e; i_2 \simeq e_2$.

This concludes the proof. ■

**Proof of theorem 4.9**

**Proof** We show that the cone $(e_i : M \to V_i)_{i \in E}$ of all minimal paths from $M$ to $\mathcal{V}$ verifies the enumerated property. Suppose that $f : M \to V$ is a path to a term $V \in \mathcal{V}$. We apply the factorisation theorem to factor $f$ as $f \equiv e; m$ so that, by lemma 4.7, $e : M \to W$ transforms $M$ into a term $W \in \mathcal{V}$. The path $e$ is external. Among the external paths $E : M \to P$ such that $E ; F \simeq e$ with a target $P$ in $\mathcal{V}$, there is a path whose length is minimal. By construction, this path is minimal and therefore a member of the cone. Let $e_i : M \to V_i$ be this path. The path $f$ factors as $f \equiv e_i ; h$ for some path $h$.

To show the unicity of $i$ let $j \in I$ be another indice such that factors $f \equiv e_j ; h'$. Lemma 4.8 tells us that there exists a path $E : M \Rightarrow W$ to $W \in \mathcal{V}$, and paths $k_1 : W \Rightarrow V_i$, $k_2 : W \Rightarrow V_j$ such that $E; k_1 \simeq e_i$ and $E; k_2 \simeq e_j$. By factorisation, $E$ factors as $e_E; m_E$ where $e_E$ is external and $m_E$ is internal. By lemma 4.7, $E$'s target is in $\mathcal{V}$. Also, $e_E; m_E; k_1 \equiv e_i$ and $e_E; m_E; k_2 \equiv e_j$ implies that $e_E; \downharpoonright_{m_E} k_1 \simeq e_i$ and $e_E; \downharpoonright_{m_E} k_2 \simeq e_j$. By minimality of $e_i$ and $e_j$, $e_E \simeq e_i$ and $e_E \simeq e_j$. Hence $i = j$.

We only have to check that $h$ is unique up to $\equiv$. Suppose that $e_i ; h \equiv e_i ; h'$. Then $e_i ; \downharpoonright_h \simeq e_i ; \downharpoonright_{h'}$ because $e_i$ is external, hence $\downharpoonright_h \equiv \downharpoonright_{h'}$ by figure 1(D). We conclude that $h \equiv \downharpoonright_h \equiv \downharpoonright_{h'}$. ■