Asynchronous games 3
An innocent model of linear logic

Paul-André Melliès

Equipe Preuves Programmes Systèmes
CNRS et Université Denis Diderot
Paris, France

Abstract

Since its early days, deterministic sequential game semantics has been limited to linear or polarized fragments of linear logic. Every attempt to extend the semantics to full propositional linear logic has bumped against the so-called Blass problem, which indicates (misleadingly) that a category of sequential games cannot be self-dual and cartesian at the same time. We circumvent this problem by considering (1) that sequential games are inherently positional; (2) that they admit internal positions as well as external positions. We construct in this way a sequential game model of propositional linear logic, which incorporates two variants of the innocent arena game model: the well-bracketed and the non well-bracketed ones.

Key words: Game semantics, linear logic, categorical models.

Foreword

This paper does not simply introduce the innocent model of propositional linear logic. It also explains in detail the conceptual stages which brought it to existence. We hope that this presentation will satisfy a categorically-minded audience. The paper is organized in six sections. We start by recalling André Joyal's category Y of Conway games and winning strategies (Section 1). We prove that the category Y does not have binary products (Section 2). This fact is well-known, but the proof does not appear anywhere in full details. We then reduce the *Blass problem* to the fact that the linear continuation monad $A \mapsto ((A \multimap \bot) \multimap \bot)$ is strong but not commutative on Conway games. Finally, after a crash course on asynchronous games (Section 4), we construct a linear continuation monad equivalent to the identity functor, by allowing internal positions in our games. This circumvents the Blass problem, and defines a model of linear logic (Section 5). We conclude (Section 6).

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1 Introduction: Conway games

Twenty-five years ago, André Joyal realized after a lecture by John H. Conway on surreal numbers, that he could construct a category Y with Conway games as objects, and winning strategies as morphisms, composed by sequential interaction. The construction appears in an article of 7 pages, written in French, and published in 1977 in the *Gazette des Sciences Mathématiques du Québec* [9]. Since it is extremely difficult to get a copy of the *Gazette* today, we find useful to recall below André Joyal's construction of the category Y of *Conway games*.

Before explaining the category, it may be worth discussing briefly what makes the category Y so interesting today. Two reasons at least. Historically, it is a precursor of game semantics for proof-theory and programming languages. Conceptually, it is a self-dual category of sequential games. We are particularly interested in this last point here. The categories of games considered today are generally symmetric monoidal closed, with a tensor product (noted $\otimes$) and a monoidal closure (noted $\multimap$). Except for a few exceptions, they are not self-dual. In contrast, the category Y is $*$-autonomous, that is, symmetric monoidal closed, with a *dualizing* object $\bot$ making the canonical morphism:

$$A \longrightarrow ((A \multimap \bot) \multimap \bot)$$

an isomorphism in the category Y, for every Conway game A. Since we are looking for game models of full propositional linear logic, and since linear logic is based on a duality between proofs and counter-proofs, we find extremely instructive to study the category Y more closely. For the reader's comfort, we will recast the original set-theoretic formulation of Conway games [9] in a graph-theoretic style. This choice is also made in the recent account of (money) games by André Joyal [10]. This may not be the best presentation, but it clarifies the connections with our own game-theoretic model of linear logic, given in Section 5.

Conway games. A Conway game is an oriented graph (V, E, λ) consisting of a set V of vertices, a set $E \subset V \times V$ of edges, and a function $\lambda : E \longrightarrow \{-1, +1\}$ associating a polarity -1 or $+1$ to every edge of the graph. The vertices are called the *positions* of the game, and the edges its *moves*. Intuitively, a move $m \in E$ is played by Player when $\lambda(m) = +1$ and by Opponent when $\lambda(m) = -1$. As is usual in graph-theory, we write $x \rightarrow y$ when $(x, y) \in E$, and call *path* any sequence of positions $s = (x_0, x_1, ..., x_k)$ in which $x_i \rightarrow x_{i+1}$ for every $i \in \{0, ..., k-1\}$. In that case, we write $s : x_0 \longrightarrow\!\!\!\!\rightarrow x_k$ to indicate that s is a path from the position x_0 to the position x_k.

In order to be a Conway game, the graph (V, E, λ) is required to verify two additional properties:

- the graph is rooted: there exists a position $*$ called the *root* of the game, such that for every other position $x \in V$, there exists a path from the root

$*$ to the position x:

$$* \to x_1 \to x_2 \to x_3 \ \cdots \ \to x_k \to x,$$

- the graph is well-founded: every sequence of positions

$$* \to x_1 \to x_2 \to x_3 \to \ \cdots$$

starting from the root is finite.

A path $s = (x_0, x_1, ..., x_k, x_{k+1})$ is called *alternating* when:

$$\forall i \in \{1, ..., k-1\}, \qquad \lambda(x_i \to x_{i+1}) = -\lambda(x_{i-1} \to x_i).$$

A *play* is defined as a path $s : * \longrightarrow\!\!\!\!\!\rightarrow x$ starting from the root. The set of plays of a Conway game A is denoted P_A.

Winning strategies. A *strategy* σ of the Conway game (E, V, λ) is defined as a set of alternating plays such that, for every positions x, y, z, z_1, z_2:

(i) the empty play $(*)$ is element of σ,

(ii) every play $s \in \sigma$ starts by an Opponent move, and ends by a Player move,

(iii) for every play $s : * \longrightarrow\!\!\!\!\!\rightarrow x$, for every Opponent move $x \to y$ and Player move $y \to z$,

$$* \xrightarrow{s}\!\!\!\!\!\rightarrow x \to y \to z \in \sigma \quad \Rightarrow \quad * \xrightarrow{s}\!\!\!\!\!\rightarrow x \in \sigma,$$

(iv) for every play $s : * \longrightarrow\!\!\!\!\!\rightarrow x$, for every Opponent move $x \to y$ and Player moves $y \to z_1$ and $y \to z_2$,

$$* \xrightarrow{s}\!\!\!\!\!\rightarrow x \to y \to z_1 \in \sigma \text{ and } * \xrightarrow{s}\!\!\!\!\!\rightarrow x \to y \to z_2 \in \sigma \quad \Rightarrow \quad z_1 = z_2.$$

Thus, a strategy is a set of plays closed under even-length prefix (Clause 3) and deterministic (Clause 4). A strategy σ is called *winning* when for every play $s : * \longrightarrow\!\!\!\!\!\rightarrow x$ element of σ and every Opponent move $x \to y$, there exists a position z and a Player move $y \to z$ such that the play

$$* \xrightarrow{s}\!\!\!\!\!\rightarrow x \to y \to z$$

is element of the strategy σ. Note that the position z is unique in that case, by determinism. We write $\sigma : A$ to mean that σ is a winning strategy of A.

Duality and tensor product. The dual $A^\perp$ of a Conway game $A = (V, E, \lambda)$ is the Conway game $A^\perp = (V, E, -\lambda)$ obtained by reversing the polarities of moves. The tensor product $A \otimes B$ of two Conway games A and B is the Conway game defined below:

- its positions are the pairs (x, y) noted $x \otimes y$ of a position x of the game A and a position y of the game B,
- its moves from a position $x \otimes y$ are of two kinds:

$$x \otimes y \rightarrow \begin{cases} u \otimes y \text{ if } x \rightarrow u, \\ x \otimes v \text{ if } y \rightarrow v, \end{cases}$$

- the move $x \otimes y \rightarrow u \otimes y$ is noted $(x \rightarrow u) \otimes y$ and has the polarity of the move $x \rightarrow u$ in the game A; the move $x \otimes y \rightarrow x \otimes v$ is noted $x \otimes (y \rightarrow v)$ and has the polarity of the move $y \rightarrow v$ in the game B.

Every play s of the tensor product $A \otimes B$ of two Conway games A and B may be projected to a play $s_{|A} \in P_A$ and to a play $s_{|B} \in P_B$. The Conway game $1 = (\emptyset, \emptyset, \lambda)$ has an empty set of positions and moves.

The category Y of Conway games. The category Y has Conway games as objects, and winning strategies of $A^{\perp} \otimes B$ as morphisms $A \longrightarrow B$. The identity strategy $\mathtt{id}_A : A^{\perp} \otimes A$ copycats every move received in one component A to the other component. The composite of two strategies $\sigma : A^{\perp} \otimes B$ and $\tau : B^{\perp} \otimes C$ is the strategy $\tau \circ \sigma : A^{\perp} \otimes C$ obtained by letting the strategies σ and τ react to a Player move in A or to an Opponent move in C, possibly after a series of internal exchanges in B.

A formal definition of identities and composition is also possible, but it requires to introduce a few notations. A play is called *legal* when it is alternating and when it starts by an Opponent move. The set of legal plays is denoted L_A. The set of legal plays of even-length, or equivalently ending by a Player move, is denoted L_A^{even}. The identity of the Conway game A is the strategy below:

$$\mathtt{id}_A \;=\; \{s \in L^{\text{even}}_{A^{\perp} \otimes A} \mid \forall t \in L^{\text{even}}_{A^{\perp} \otimes A}, t \text{ is prefix of } s \Rightarrow t_{|A^{\perp}} = t_{|A}\}.$$

The composite of two strategies $\sigma : A^{\perp} \otimes B$ and $\tau : B^{\perp} \otimes C$ is the strategy of $\tau \circ \sigma : A^{\perp} \otimes C$ below:

$$\tau \circ \sigma \;=\; \{s \in L^{\text{even}}_{A^{\perp} \otimes C} \mid \exists t \in P_{A \otimes B \otimes C}, t_{|A,B} \in \sigma, t_{|B,C} \in \tau, t_{|A,C} = s\}.$$

The tensor product between Conway games gives rise to a bifunctor on the category Y, which makes the category Y $*$-autonomous, that is, symmetric monoidal closed, with a dualizing object noted $\perp$. The category Y is more than just $*$-autonomous: it is compact closed, in the sense that there exists an isomorphism $(A \otimes B)^{\perp} \;\cong\; A^{\perp} \otimes B^{\perp}$ natural in A and B. As in any compact closed category, the dualizing object $\perp$ is isomorphic to the identity object of the monoidal structure, in that case the Conway game 1. Thus, the monoidal closure $A^{\perp} \otimes \perp$ is isomorphic to $A^{\perp}$, for every Conway game A.

2 Key observation: the category Y does not have binary products

The category Y has been rediscovered at the beginning of the 90's in the context of linear logic and programming language semantics. As a $*$-autonomous category, the category Y defines a model of Multiplicative Linear Logic (MLL). In this model, every closed formula F of MLL is interpreted as a Conway game $[F]$; and every proof π of the formula F is interpreted as a winning strategy $[\pi]$ of the Conway game $[F]$. This interpretation provides a precise and lively picture of proofs, understood as symbolic device interacting during cut-elimination.

Because MLL is only a small fragment of linear logic, many authors have tried to adapt the category Y in order to capture larger or more interesting fragments of the logic. One particularly resistant fragment is Multiplicative Additive Linear Logic (MALL) which is MLL extended with the *additive* connectives $\oplus$ and $\&$ and constants 0 and $\top$. Every $*$-autonomous category with finite products defines a model of MALL. Alas, the category Y does not have binary products. To our knowledge, the proof of this well-known fact appears nowhere in the litterature. We thus give it below, after introducing the subcategory Y^- of negative Conway games.

Negative Conway games. A Conway game A is called *negative* when every nonempty play of A starts by an Opponent move. The category Y^- is defined as the full subcategory of Y, whose objects are the negative Conway games. The category Y^- is symmetric monoidal closed. The symmetric monoidal structure is inherited from the category Y, while the monoidal closure of Y^- is slightly different. The category Y^- is introduced here because it has finite products. The terminal object of the category is the Conway game 1. The cartesian product of two negative Conway games A and B is the negative Conway game noted $A\&B$, and defined below:

- its set of positions is the disjoint sum of the set of positions of A and the set of positions of B, in which the two roots $*_A$ and $*_B$ of A and B are identified as the root $*_{A\&B}$ of $A\&B$. This construction is similar to *lifted sum* in domain theory,
- its Opponent moves from the root position $*_{A\&B}$ are of two kinds:

$$*_{A\&B} \to \begin{cases} x \text{ if } *_A \to x \text{ in the Conway game } A, \\ y \text{ if } *_B \to y \text{ in the Conway game } B, \end{cases}$$

- its moves from a position x in the component A are exactly the moves from x in the Conway game A, with the same polarity:

$$x \to y \text{ in the game } A\&B \iff x \to y \text{ in the game } A.$$

- its moves from a position x in the component B are exactly the moves from x in the Conway game B, with the same polarity.

It is not difficult to see that the game $A\&B$, equipped with the accurate projection strategies $A\&B \longrightarrow A$ and $A\&B \longrightarrow B$, defines a cartesian product of A and B in the category Y^-. The end of the section is devoted to the proof that:

Proposition 2.1 *The category Y does not have binary products.*

Proof. The forgetful functor $U : Y^- \longrightarrow Y$ has a right adjoint $\mathrm{Neg} : Y \longrightarrow Y^-$ which associates to every Conway game $A = (V, E, \lambda)$ the negative Conway game $\mathrm{Neg}(A) = (V', E', \lambda)$ obtained by removing every Player move starting from the root $*$:

$$E' = E \setminus \{(*, x) \in E \mid \lambda(* \to x) = +1\},$$

then removing every position in V not accessible from the root in the graph (V, E'):

$$V' = \{x \in V \mid \text{there exists a path in } (V, E') \text{ from the root to } x\}.$$

As a right adjoint, the functor Neg preserves limits. We proceed by contradiction, and suppose that every pair of Conway games A and B has a cartesian product noted $A \times B$ in the category Y. Then, the image $\mathrm{Neg}(A \times B)$ of this product is isomorphic to the cartesian product $\mathrm{Neg}(A)\&\mathrm{Neg}(B)$ in the category Y^-.

Now, a Conway game A is called *positive* when its dual $A^\perp$ is negative. We claim that the cartesian product of two positive Conway games A and B in Y, is a positive Conway game $A \times B$. Note that a Conway game A is positive iff $\mathrm{Neg}(A) = 1$. The negative game $\mathrm{Neg}(A \times B)$ associated to the product of two positive games A and B is equal to $\mathrm{Neg}(A)\&\mathrm{Neg}(B) = 1\&1 = 1$. The game $A \times B$ is thus positive, as claimed.

Let Y^+ denote the full subcategory of Y consisting of positive Conway games. Since Y^+ is a full subcategory of Y, we have just established that if Y has binary products, then Y^+ has binary products as well. We conclude our proof of Proposition 2.1 by showing that Y^+ does not have binary products.

Consider the negative game $\mathbb{B}$ interpreting the booleans, with four positions $*$, q, true, false, an Opponent move $* \to q$ and two Player moves $q \to \mathrm{true}$ and $q \to \mathrm{false}$. Let $X = \mathbb{B}^\perp$ denote the positive game obtained by dualizing $\mathbb{B}$. Consider two positive games A and B, and suppose that $A \times B$ is their cartesian product in Y^+. Let the morphism $\sigma_{\mathrm{true}} : X \longrightarrow A$ in the category Y^+ denote the smallest strategy of $\mathbb{B} \otimes A$ containing the play:

$$*_{\mathbb{B}} \otimes *_A \to q \otimes *_A \to \mathrm{true} \otimes *_A.$$

Similarly, let $\tau_{\mathrm{bool}} : X \longrightarrow B$ denote the smallest strategy of $\mathbb{B} \otimes B$ containing the play:

$$*_{\mathbb{B}} \otimes *_B \to q \otimes *_B \to \text{bool} \otimes *_B,$$

where bool is either the position true or false. Let $\langle \sigma_{\text{true}}, \tau_{\text{false}} \rangle : X \longrightarrow A \times B$ denote the unique morphism in Y^+ such that

(1) $$\sigma_{\text{true}} = \pi_1 \circ \langle \sigma_{\text{true}}, \tau_{\text{false}} \rangle \qquad \tau_{\text{false}} = \pi_2 \circ \langle \sigma_{\text{true}}, \tau_{\text{false}} \rangle$$

for $\pi_1 : A \times B \longrightarrow A$ and $\pi_2 : A \times B \longrightarrow B$ the projections associated to the Conway games A and B in Y^+. A careful inspection of (1) establishes that the strategy $\langle \sigma_{\text{true}}, \tau_{\text{false}} \rangle$ contains a play of the form:

$$*_{\mathbb{B}} \otimes *_{A\times B} \to q \otimes *_{A\times B} \to q \otimes x.$$

for a position x and a Player move $* \to x$ of the game $A \times B$; and that the strategy $\pi_1 : (A \times B)^{\perp} \otimes A$ contains a play of the form

$$*_{A\times B} \otimes *_A \to x \otimes *_A \to y_1 \otimes *_A$$

for a position y_1 and an Opponent move $* \to y_1$ of the game $A \times B$. Similarly, the strategy $\pi_2 : (A \times B)^{\perp} \otimes B$ contains a play of the form

$$*_{A\times B} \otimes *_B \to x \otimes *_A \to y_2 \otimes *_B$$

for a position y_2 and an Opponent move $* \to y_2$ of the game $A \times B$. Note that the two positions y_1 and y_2 may be equal in $A \times B$. Now, let $\nu : X \longrightarrow A \times B$ denote the smallest strategy containing the play:

$$*_{\mathbb{B}} \otimes *_{A\times B} \to q \otimes *_{A\times B} \to \text{true} \otimes *_{A\times B}.$$

And let $\nu' : X \longrightarrow A \times B$ denote the smallest strategy containing all the plays of the form:

$$*_{\mathbb{B}} \otimes *_{A\times B} \to q \otimes *_{A\times B} \to q \otimes x \to q \otimes y \to \text{true} \otimes y$$

for y a position such that $x \to y$ is an Opponent move of $A \times B$. The two equalities

$$\pi_1 \circ \nu = \sigma_{\text{true}} = \pi_1 \circ \nu' \qquad \pi_2 \circ \nu = \tau_{\text{true}} = \pi_2 \circ \nu'.$$

follow immediately from these definitions. So, there exists more than one morphism $X \longrightarrow A \times B$ making the cartesian diagrams commute for $\sigma_{\text{true}} : X \longrightarrow A$ and $\tau_{\text{true}} : X \longrightarrow B$. We conclude that the category Y^+ does not have binary products. This concludes the proof of Proposition 2.1. □

Remark: there is another more direct way to establish that the category Y does not have finite products, which is to show that the category Y does not have a terminal object. This alternative argument is less conclusive however, since it is possible to add *formally* an initial and a terminal object to the category Y, without breaking self-duality.

3 A categorical formulation of the Blass problem

We have just seen in section 2 that

- the category Y is $*$-autonomous but does not have finite products,
- its subcategory Y^- of negative Conway games is symmetric monoidal closed and has all products.

This explains why game semantics is generally more concerned with variants of the category Y^- than with variants of the category Y. Prima facie, self-duality is less important than cartesianity in order to interpret a programming language built on top of the λ-calculus. Besides, it is much simpler to interpret the exponential modality ! of linear logic in the category Y^- (or a variant) than in the category Y. By starting from the category Y^-, one obtains a model of Intuitionistic Linear Logic (ILL) whose categorical axiomatization ensures that the kleisli category associated to the comonad ! is cartesian closed, and thus defines a model of the simply-typed λ-calculus with products, see [16,6,12] among other works. By selecting among variants of the category Y^-, and among variants of the comonad, one generates a wide range of models of the λ-calculus, some of them capturing the essence of particular syntactic programming languages (cf. the full abstraction results).

The methodology is nice and fruitful. We claim however that the lack of self-duality of the category Y^- is a serious conceptual limitation of game semantics. Our ambition here is to clarify the foundations of the subject, by reunderstanding Y^- as part of a larger $*$-*autonomous* category Z with *products* and *coproducts*. In this section, we try to deduce the general shape of the category Z from a categorical reformulation of the so-called Blass problem. We proceed by keeping the symmetry between the category Y^- and its opposite category Y^+ as far as possible, in order to let unexpected structures emerge from the symmetry. This prepares Section 5, in which we construct a candidate for the category Z, a category of asynchronous games and innocent strategies.

First adjunction between lifting functors. We start our analysis by the so-called *lifting* functor $\Downarrow\ : Y^- \longrightarrow Y^+$ which associates to every negative Conway game A, the positive Conway game $\Downarrow A$ defined below:

- a position of $\Downarrow A$ is a position of A or a new position $*$,
- the only move from $*$ is the Player move $* \to *_A$ to the root $*_A$ of A,
- the moves from a position in A are the same as in A, with same polarities.

By duality, there is a lifting functor $\Uparrow\ : Y^+ \longrightarrow Y^-$ defined by the equation $\Uparrow A = (\Downarrow (A^\perp))^\perp$. Interestingly, the functor $\Downarrow$ is left adjoint to the functor $\Uparrow$. What this adjunction means on Conway games is very simple. Consider a negative Conway game A, and a positive Conway game B. The elements of $Y^+(\Downarrow A, B)$ and of $Y^-(A, \Uparrow B)$ are the winning strategies of $\Uparrow (A^\perp) \otimes B$ and the winning strategies of $A^\perp \otimes \Uparrow B$, respectively. Note that both $A^\perp$ and B

are positive Conway games. So, the plays starting by an Opponent move are the same in the Conway games $\Uparrow(A^{\perp}) \otimes B$ and $A^{\perp} \otimes \Uparrow B$: in each case, the dummy move followed by a play in $A^{\perp} \otimes B$. This induces a bijection between the set of strategies $Y^{+}(\Downarrow A, B)$ and $Y^{-}(A, \Uparrow B)$ which is natural in A and B. From this follows that $\Downarrow$ is left adjoint to $\Uparrow$. This adjunction induces a monad on Y^{-} and a comonad on Y^{+}, obtained by lifting every game twice. Note that variants of the monad on Y^{-} have been already observed, typically in the litterature on arena games.

Second adjunction between lifting functors. From now on, we focus on another adjunction $\uparrow \dashv \downarrow$ which follows from the adjunction $\Downarrow \dashv \Uparrow$, and which plays a fundamental role in the formulation of games as continuation passing style models. Note that the category Y^{+} has coproducts, since its opposite category Y^{-} has products. From this follows that the functor $\Downarrow$ factors as:

$$Y^{-} \xrightarrow{(1)} \Sigma Y^{-} \xrightarrow{(2)} Y^{+}$$

where ΣY^{-} is the free completion of Y^{-} with respect to coproducts. This completion is also called the *family construction* in [2]. We recall that:

- an object of ΣY^{-} is a family $\{A_i \mid i \in I\}$ of negative Conway games A_i, indexed by the elements of a set I,
- a morphism $\{A_i \mid i \in I\} \longrightarrow \{B_j \mid j \in J\}$ consists of a *reindexing function* $f : I \longrightarrow J$ and of a winning strategy $\sigma_i : A_i \longrightarrow B_{f(i)}$, for each index $i \in I$.

Dually, the lifting functor $\Uparrow$ factors as:

$$Y^{+} \xrightarrow{(3)} \Pi Y^{+} \xrightarrow{(4)} Y^{-}$$

where ΠY^{+} is the free completion of Y^{-} with respect to products. Note that the category ΠY^{+} is the opposite of the category ΣY^{-}.

By composing the resulting functors together, one obtains two new "lifting" functors $\uparrow$ and $\downarrow$ defined below:

$$\uparrow\ : \Sigma Y^{-} \xrightarrow{(2)} Y^{+} \xrightarrow{(3)} \Pi Y^{+}, \qquad \downarrow\ : \Pi Y^{+} \xrightarrow{(4)} Y^{-} \xrightarrow{(1)} \Sigma Y^{-}.$$

Our notation $\uparrow$ and $\downarrow$ for the lifting functors indicates already that we consider ΣY^{-} as a category of positive games, and ΠY^{+} as a category of negative games. Typically, we like to think of an object of ΣY^{-}, presented as a family $\{A_i \mid i \in I\}$ of negative games, as a positive game whose initial moves by Player are the indices $i \in I$. We come back to this point later in the Section.

Interestingly, the functor $\uparrow$ is left adjoint to the functor $\downarrow$. Indeed, consider a family $A = \{A_i \mid i \in I\}$ of negative Conway games, and a family $B = \{B_j \mid j \in J\}$ of positive Conway games. The family A is transported (or lifted) by $\uparrow$ to the singleton family consisting of the positive Conway game $\oplus_i \Downarrow A_i$, where $\oplus$ denotes the coproduct in Y^{+}. Dually, the family B is transported (or lifted) by $\downarrow$ to the singleton family consisting of the negative Conway game $\&_j \Uparrow B_j$. Now, we have a series of bijections between sets:

$$\begin{array}{rcll} \Sigma Y^-(A, \downarrow B) & \cong & \Pi_{i\in I} Y^-(A_i, \&_{j\in J} \Uparrow B_j) & \text{by definition of } \Sigma Y^-, \\ & \cong & \Pi_{(i,j)\in I\times J} Y^-(A_i, \Uparrow B_j) & \text{because } \& \text{ is product in } Y^-, \\ & \cong & \Pi_{(i,j)\in I\times J} Y^+(\Downarrow A_i, B_j) & \text{thanks to the adjunction } \Downarrow \dashv \Uparrow, \\ & \cong & \Pi_{j\in J} Y^+(\oplus_{i\in I} \Downarrow A_i, B_j) & \text{because } \oplus \text{ is coproduct in } Y^+, \\ & \cong & \Pi Y^+(\uparrow A, B) & \text{by definition of } \Sigma Y^+, \end{array}$$

whose naturality in A and B is easily established.

ΣY^- as a linear continuation category. As free completion of a symmetric monoidal closed category with products, the category ΣY^- is symmetric monoidal closed. The functor:

$$(- \circledast -) \ : \ \Sigma Y^- \times \Sigma Y^- \longrightarrow \Sigma Y^- \tag{2}$$

is defined on the families of positive Conway games, as follows:

$$\{A_i \mid i \in I\} \circledast \{B_j \mid j \in J\} = \{A_i \otimes B_j \mid (i,j) \in I \times J\}. \tag{3}$$

The monoidal closure $A \multimap\!\!\bullet\ B$ is defined as follows:

$$\{A_i \mid i \in I\} \multimap\!\!\bullet \{B_j \mid j \in J\} = \{\&_{i\in I}(A_i^{\perp} \otimes B_{f(i)}) \mid f \in I \to J\} \tag{4}$$

So, the initial Player moves of the Conway game $A \multimap\!\!\bullet\ B$ (equivalently, the indices of the family $A \multimap\!\!\bullet\ B$) are the set-theoretic functions f from the set I of initial Player moves in A, to the set J of initial Player moves in B. This way of defining the initial moves of $A \multimap\!\!\bullet\ B$ does not fit in with the general philosophy of game semantics, which is to avoid "extensional" constructions like set-theoretic function spaces. Quite fortunately, one may specialize the construction to the case where $B = \perp$ is the singleton family with the empty Conway game 1 as unique element. This defines what one calls a linear continuation category, that is, a symmetric monoidal category with finite coproducts distributive over the tensor product, and an exponentiable object $\perp$. Besides, the resulting endofunctor $A \mapsto (A \multimap\!\!\bullet \perp)$ of the category ΣY^- coincides with the endofunctor $A \mapsto \downarrow(A^{\perp})$.

ΣY^- and ΠY^+ as categories of central maps. We have indicated that we like to think of the category ΣY^- as a category of *positive* Conway games. This is justified by the existence of the functor $\Sigma Y^- \longrightarrow Y^+$ mentioned earlier, which transports every family $\{A_i \mid i \in I\}$ of negative games to the positive game with initial moves the indices $i \in I$, followed by the plays of A_i. The functor is faithful, and injective of objects. The category ΣY^- is thus isomorphic to its image in the category Y^+, which we note $Y^{+-}_{\bullet}$.

The category $Y^{+-}_{\bullet}$ may defined directly as follows. The objects of $Y^{+-}_{\bullet}$ are the Conway games in which:

- every initial move in a play is by Player,
- every second move in a play is by Opponent.

The morphisms $A \longrightarrow B$ of $Y^{+-}_{\bullet}$ are the winning strategies $\sigma : A^{\perp} \otimes B$ such that, for every Player move $*_A \rightarrow x$ in A, there exists a Player move $*_B \rightarrow y$ in B, such that the play $*_A \otimes *_B \rightarrow x \otimes *_B \rightarrow x \otimes y$ is element of the strategy σ.

Dually, the functor $\Pi Y^{+} \longrightarrow Y^{-}$ defines an isomorphism of categories $\Pi Y^{+} \cong Y^{-+}_{\bullet}$ where $Y^{-+}_{\bullet}$ is defined as the opposite category of $Y^{+-}_{\bullet}$. It is not difficult to see that the resulting functors:

$$\uparrow \, : Y^{+-}_{\bullet} \longrightarrow Y^{-+}_{\bullet}, \qquad \downarrow \, : Y^{-+}_{\bullet} \longrightarrow Y^{+-}_{\bullet}.$$

coincide with the lifting functors $\Uparrow$ and $\Downarrow$ restricted to the subcategories $Y^{+-}_{\bullet}$ and $Y^{-+}_{\bullet}$ of Y^{+} and Y^{-}, respectively. This justifies our notations for $\uparrow$ and $\downarrow$.

Now, let Y^{-+} denote the *full* subcategory of Y^{-} with the objects of $Y^{-+}_{\bullet}$, and let Y^{+-} denote the *full* subcategory of Y^{+} with the objects of $Y^{+-}_{\bullet}$. By construction, the category Y^{+-} is opposite to the category Y^{-+}.

There is a crucial observation to make here: the category Y^{-+} is the co-kleisli category over the category $Y^{-+}_{\bullet}$, induced by the comonad $\uparrow\downarrow$. It is not difficult indeed to check that the set $Y^{-+}(A, B)$ of morphisms between two negative Conway games A and B of Y^{-+}, is equal to the set $Y^{+-}_{\bullet}(\downarrow A, \downarrow B)$ of morphisms in the category $Y^{-+}_{\bullet}$. This implies that the category Y^{-+} is the *category of continuations* associated to the category $Y^{+-}_{\bullet}$.

The category Y^{-+} thus defines what Peter Selinger calls a (linear) *control category* in [17]. The category $Y^{-+}_{\bullet}$ is the category of *central maps* associated to this control category Y^{-+}. This is the key to understand together the family construction by Samson Abramsky and Guy McCusker in [2], the polarized presentation of games by Olivier Laurent in [11], the completeness theorem of continuation models for the $\lambda\mu$-calculus by Martin Hofmann and Thomas Streicher in [5], or the representation theorem of control categories as continuation models by Peter Selinger in [17].

The adjunction $\uparrow \dashv \downarrow$ simulates synchronization. After this long discussion, we are ready to clarify the computational meaning of the adjunction $\uparrow \dashv \downarrow$. Suppose that A denotes a positive Conway game in $Y^{+-}_{\bullet}$, and B a negative Conway game in $Y^{-+}_{\bullet}$. Every element of $Y^{+-}_{\bullet}(A, \downarrow B)$ is a strategy σ of $A^{\perp} \otimes \downarrow B$ which waits for an Opponent move $m : *_A \rightarrow x$ in $A^{\perp}$, plays the dummy move in $\downarrow B$ after receiving m, waits for an Opponent move $n : *_B \rightarrow y$ in B, and carries on after receiving n. Symmetrically, every element of $Y^{-+}_{\bullet}(\uparrow A, B)$ is a strategy τ of $\downarrow A^{\perp} \otimes B$ which waits for an Opponent move $n : *_B \rightarrow y$ in B, plays the dummy move in $\downarrow A^{\perp}$ after receiving n, waits for an Opponent move $m : *_A \rightarrow x$ in $A^{\perp}$, and carries on after receiving m. In both cases, the strategy σ or τ waits for the two inputs $m : *_A \rightarrow x$ and $n : *_B \rightarrow y$, then carries on. In that way, the two strategies σ and τ implement the *synchronized* input of m in A and n in B: the strategy σ simulates synchronization of A and B by asking in A then in B (in the call-by-value order) whereas the strategy σ asks in B then in A (in the call-by-name order).

The Conway game $A \multimap B$. This discussion on synchronization has a categorical counterpart. The functor associated to the adjunction $\uparrow \dashv \downarrow$:

$$(Y_{\bullet}^{+-})^{\mathrm{op}} \times Y_{\bullet}^{-+} \longrightarrow \underline{\mathrm{Set}}. \tag{5}$$

factorizes as a functor on Conway games:

$$(- \multimap -) \;:\; (Y_{\bullet}^{+-})^{\mathrm{op}} \times Y_{\bullet}^{-+} \longrightarrow Y_{\bullet}^{-+} \tag{6}$$

postcomposed to the global element functor $Y_{\bullet}^{-+} \longrightarrow \underline{\mathrm{Set}}$ which associates to every negative Conway game its set of winning strategies. The Conway game $A \multimap B$ is defined just as $A^{\perp} \otimes B$ except that the initial Opponent moves are pairs (m, n) of a Player move in A and an Opponent move in B.

The Blass problem. The definition of the Conway game $A \multimap B$ coincides with the definition given by Andrea Blass in his game-theoretic account of linear logic [3]. Interestingly, the synchronization of the initial moves is precisely what leads (apparently) to the so-called *Blass problem*. The problem is the following one: there seems to be a natural way to build a category of negative and positive games, but this expected construction does work unfortunately, because it defines to a non-associative structure, see the comprehensive account by Samson Abramsky in [1].

The Blass problem may be reformulated categorically in the following way. As any profunctor, the functor (5) induces a category $Y_{\bullet}$ with Conway games of $Y_{\bullet}^{+-}$ and $Y_{\bullet}^{-+}$ as objects, and:

- the morphisms of $Y_{\bullet}^{+-}$ between two positive Conway games A and B,
- the morphisms of $Y_{\bullet}^{-+}$ between two negative Conway games A and B,
- the strategies of $A \multimap B$ from a positive game A to a negative game B,
- no morphism from a negative game A to a positive game B.

The composition law of the category $Y_{\bullet}$ is deduced from the composition laws of the categories $Y_{\bullet}^{+-}$ and $Y_{\bullet}^{-+}$, as well as from the functor (5). Associativity is ensured by the bifunctoriality of (5).

The Blass problem arises when one tries to replace the two categories $Y_{\bullet}^{+-}$ and $Y_{\bullet}^{-+}$ in the construction of $Y_{\bullet}$, by their kleisli categories Y^{+-} and Y^{-+}. Suppose indeed that one tries to compose a morphism $h_A : A' \longrightarrow A$ in the kleisli category Y^{+-}, a strategy $\sigma : A \multimap B$, and a morphism $\sigma : B \longrightarrow B'$ in the co-kleisli category Y^{-+}. This amounts to extending the functor (6) to a functor

$$(- \multimap -) \;:\; (Y^{+-})^{\mathrm{op}} \times Y^{-+} \longrightarrow Y^{-+}. \tag{7}$$

The Blass problem amounts to the fact that there is no such functor (7) but only a functor:

$$(- \multimap -) \;:\; (Y^{+-})^{\mathrm{op}} \otimes Y^{-+} \longrightarrow Y^{-+}. \tag{8}$$

where $(Y^{+-})^{\mathrm{op}} \otimes Y^{-+}$ is a variant of $(Y^{+-})^{\mathrm{op}} \times Y^{-+}$ without the interchange law between composition and tensor product, see [15] for a definition. In other words, the equality:

$$(\mathtt{id}_{A'} \multimap h_B) \circ (h_A \multimap \mathtt{id}_B) = (h_A \multimap \mathtt{id}_{B'}) \circ (\mathtt{id}_A \multimap h_B)$$

is not necessarily verified.

Now, observe, and this is the main point, that the functors (6) and (2) are related by a natural isomorphism $A \multimap B \cong (A \circledast B^{\perp})^{\perp}$. Thus, extending the functor $\multimap$ from the categories $Y_{\bullet}^{+-}$ and $Y_{\bullet}^{-+}$ to their kleisli categories Y^{+-} and Y^{-+}, is just like extending the bifunctor $\circledast$ from $Y_{\bullet}^{+-}$ to its kleisli category Y^{+-}. This enables to apply this well-known fact of the theory of monads, see [8,15], that the functor $\circledast$ defines a premonoidal structure on Y^{+-} because the linear continuation monad $\downarrow\uparrow$ on the category $Y_{\bullet}^{+-}$ is strong but not commutative.

Towards the category Z. We have just reduced Blass problem to the property that the linear continuation monad $A \mapsto ((A \multimap\!\bullet \perp) \multimap\!\bullet \perp)$ is strong but not commutative. This provides us with a recipe to get a model of linear logic: find an analogue of the category $Y_{\bullet}^{+-}$ in which the linear continuation monad $A \mapsto ((A \multimap\!\bullet \perp) \multimap\!\bullet \perp)$ would be *commutative*. More than that: in order to obtain a $*$-autonomous category, we want this linear continuation monad to be equivalent (as a monad) to the identity. The category of asynchronous games introduced in Section 5 is designed precisely for that purpose.

4 A crash course on asynchronous game semantics

In this section, we recall the definitions of asynchronous games and innocent strategy given in [14]. We call these games *simple games* in order to prepare Section 5 in which they appear as components of more general games. The original definition of asynchronous game given in [14] is also adapted in three ways. First, we consider asynchronous games with a well-founded event structure, in order to relate them to Conway games. This is only a detail of presentation, since all our definitions apply to non well-founded asynchronous games. We also add an incompatibility relation # between the moves of the game, in order to interpret the additive connectives and constants of linear logic. Finally, we associate a *payoff* in $\{-\infty, -1, +1, \infty\}$ to every position of the game, in order to distinguish between Player positions (with positive payoff) and Opponent positions (with negative payoff) as well as between internal positions (with infinite payoff) and external positions (with unit payoff).

Event structures. An *event structure* $(\mathcal{M}, \leq, \#)$ is a partially ordered set $(\mathcal{M}, \leq)$ of *events* equipped with a binary symmetric irreflexive relation $\#$ verifying:

- the set $m \downarrow = \{n \in \mathcal{M} \mid n \leq m\}$ is *finite* for every event $m \in \mathcal{M}$,

- $m \# n \leq p$ implies $m \# p$ for every events $m, n, p \in \mathcal{M}$.

Two events $m, n \in \mathcal{M}$ are called *incompatible* when $m \# n$, and *compatible* otherwise. Two moves m and n are called *independent* when they are compatible, and different. We write $m \mathrel{\mathrm{I}} n$ in that case.

Positions. A *position* of an event structure A is a *finite* downward closed subset of $(\mathcal{M}_A, \leq_A)$, consisting of pairwise compatible events. The set of positions of A is denoted $\mathcal{D}_A$.

The positional graph. Every event structure A induces a graph $\mathcal{G}_A$ whose nodes are the positions $x, y \in \mathcal{D}_A$, whose edges $m : x \longrightarrow y$ are the events verifying $y = x + \{m\}$, where $+$ indicates a disjoint union, that is, $y = x \cup \{m\}$ and the move m is not element of x. An event structure is called *well-founded* when the graph $\mathcal{G}_A$ is well-founded.

Simple asynchronous games. A *simple game* $A = (\mathcal{M}_A, \leq_A, \#_A, \lambda_A, \kappa_A)$ is a well-founded event structure $(\mathcal{M}_A, \leq_A, \#_A)$ whose events are called the *moves* of the game, equipped with a *polarity* function $\lambda_A : \mathcal{M}_A \longrightarrow \{-1, +1\}$ on moves, and a *payoff* function $\kappa_A : \mathcal{D}_A \longrightarrow \{-\infty, -1, +1, +\infty\}$ on positions. A move with polarity $+1$ (resp. -1) is called a Player (resp. Opponent) move. A Player (resp. Opponent) position is a position with payoff in $\{+1, +\infty\}$ (resp. in $\{-1, -\infty\}$). An external (resp. internal) position is a position with payoff in $\{+1, -1\}$ (resp. in $\{+\infty, -\infty\}$).

The underlying Conway game. The positional graph attached to the simple game A defines a Conway game $\mathcal{G}_A$, in which the polarity of a move $x \to y$ is given by the polarity of the underlying move m such that $y = x + \{m\}$ in the simple game A. For simplicity, we write P_A instead of $P_{\mathcal{G}_A}$ for the set of plays of $\mathcal{G}_A$. There is more structure in $\mathcal{G}_A$ than in a usual Conway game, since every position has a payoff, and moves may be *permuted* in plays, as explained below. The set of external positions of $\mathcal{G}_A$ is denoted $\mathcal{D}_A^\circ$.

Homotopy. Given two paths $s, s' : x \twoheadrightarrow y$ in $\mathcal{G}_A$, we write $s \sim^1 s'$ when the paths s and s' are of length 2, with $s = m \cdot n$ and $s' = n \cdot m$ for two moves $m, n \in \mathcal{M}_A$. The *homotopy equivalence* $\sim$ between paths is defined as the least equivalence relation containing $\sim^1$, and closed under composition. We also use the notation $\sim$ in our diagrams to indicate that two (necessarily independent) moves m and n are permuted. The word *homotopy* is justified mathematically by the work on *directed homotopy* by Philippe Gaucher and Eric Goubault [4]. Indeed, every asynchronous game defines a *directed* simplicial set, in which directed homotopy between paths coincides with our permutation equivalence $\sim$.

Strategy. A strategy σ of a simple asynchronous game is a strategy of the underlying Conway game $\mathcal{G}_A$, such that, moreover, every play $s : * \twoheadrightarrow x$

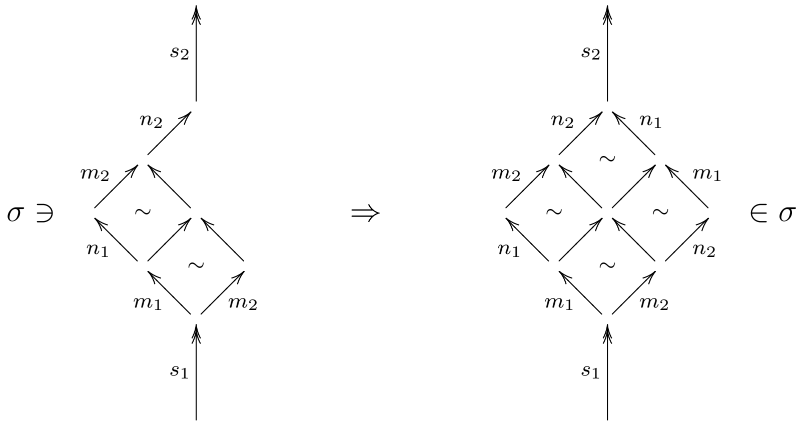


Fig. 1. Backward consistency

in the strategy σ has its target position x of positive payoff: $+1$ or $+\infty$. A strategy σ of A is winning when it is winning in the underlying Conway game $\mathcal{G}_A$. We write $\sigma : A$ when σ is a winning strategy of the simple asynchronous game A.

Innocence. We reformulate in [14] the usual notion of innocence found in arena games, as follows. A strategy σ is called *innocent*, when it is *side consistent* and *forward consistent* in the following sense.

Backward consistency. A strategy σ is *backward consistent* (see Figure 1) when for every play $s_1 \in P_A$, for every path s_2, for every moves $m_1, n_1, m_2, n_2 \in \mathcal{M}_A$, it follows from

$$s_1 \cdot m_1 \cdot n_1 \cdot m_2 \cdot n_2 \cdot s_2 \in \sigma \text{ and } n_1 \text{ I } m_2 \text{ and } m_1 \text{ I } m_2$$

that

$$n_1 \text{ I } n_2 \text{ and } m_1 \text{ I } n_2 \text{ and } s_1 \cdot m_2 \cdot n_2 \cdot m_1 \cdot n_1 \cdot s_2 \in \sigma.$$

Forward consistency. A strategy σ is *forward consistent* (see Figure 2) when for every play $s_1 \in P_A$ and for every moves $m_1, n_1, m_2, n_2 \in \mathcal{M}_A$, it follows from

$$s_1 \cdot m_1 \cdot n_1 \in \sigma \text{ and } s_1 \cdot m_2 \cdot n_2 \in \sigma \text{ and } m_1 \text{ I } m_2 \text{ and } m_2 \text{ I } n_1$$

that

$$m_1 \text{ I } n_2 \text{ and } n_1 \text{ I } n_2 \text{ and } s_1 \cdot m_1 \cdot n_1 \cdot m_2 \cdot n_2 \in \sigma.$$

Positional strategy. A strategy $\sigma : A$ is called *positional* when for every two plays $s_1, s_2 : *_A \twoheadrightarrow x$ in the strategy σ, and every path $t : x \twoheadrightarrow y$ of $\mathcal{G}_A$, one has:

$$s_1 \sim s_2 \text{ and } s_1 \cdot t \in \sigma \;\Rightarrow\; s_2 \cdot t \in \sigma.$$

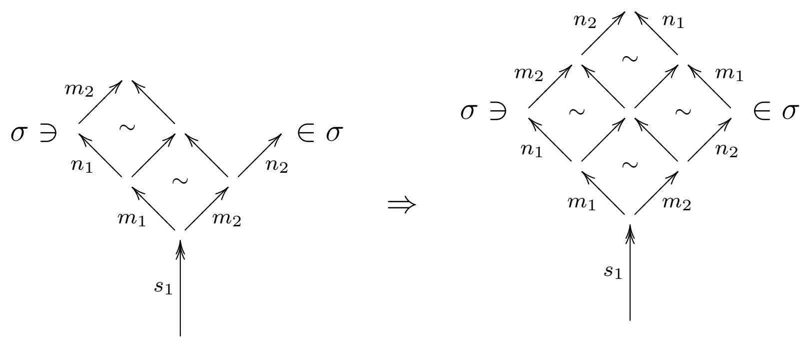


Fig. 2. Forward consistency

Proposition 4.1 ([14]) *Every innocent strategy σ is positional.*

Note that every positional strategy is characterized by the set of positions of $\mathcal{D}_A$ it reaches, defined as $\sigma^\bullet = \{x \in \mathcal{D}_A, \exists s \in \sigma, s : *_A \longrightarrow\!\!\!\!\!\rightarrow x\}$.

5 An innocent model of propositional linear logic

Lifting of a simple games. The lifting $\Downarrow A$ of any simple game A is the simple game defined by lifting the set of moves $\mathcal{M}_A$ with an Opponent move m, and giving the internal and Player payoff $+\infty$ to the root $*_{\Downarrow A}$ of the simple game $\Downarrow A$. The operation $\Uparrow A$ is defined dually.

Tensor product of simple games. The tensor product $A \otimes B$ of two simple games

$$A = (\mathcal{M}_A, \leq_A, \#_A, \lambda_A, \kappa_A) \quad \text{and} \quad B = (\mathcal{M}_B, \leq_B, \#_B, \lambda_B, \kappa_B)$$

is defined by a disjoint sum of polarized event structures

$$(\mathcal{M}_A + \mathcal{M}_B, \leq_A + \leq_A, \#_A + \#_B, \lambda_A + \lambda_B).$$

The underlying Conway game of $A \otimes B$ is thus equal to the tensor product of the underlying Conway games of A and B. The payoff $\kappa_{A \otimes B}(x \otimes y)$ of a position $x \otimes y$ is given by the table below, in which the payoffs $\kappa_A(x)$ and $\kappa_B(y)$ appear in the first row and first column.

$\otimes$	$-\infty$	-1	$+1$	$+\infty$
$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
-1	$-\infty$	$-\infty$	-1	$+\infty$
$+1$	$-\infty$	-1	$+1$	$+\infty$
$+\infty$	$-\infty$	$+\infty$	$+\infty$	$+\infty$

Note that the table is symmetric in A and B, and that the tensor product of an internal position with another position is always internal.

Asynchronous games. An asynchronous game is a pair $\{\pi \mid A_i \mid i \in I\}$ consisting of a polarity $\pi \in \{+1, -1\}$ and of a family $(A_i)_{i \in I}$ of simple games indexed by I. A *position* of A is defined as a position of any of the simple games A_i. The *component* of a position of A is the simple game A_i in which it appears. A position of A is called *initial* when it is the root of its component A_i. Every initial position in A is required to have a positive payoff when $\pi = +1$, and a negative payoff when $\pi = -1$. An asynchronous game is called *negative* when $\pi = -1$ and *positive* when $\pi = +1$.

Lifting. The lifting of a negative game $A = \{-1 \mid A_i \mid i \in I\}$ is the positive game $\downarrow A = \{+1 \mid \&_{i \in I} \Downarrow A_i\}$ consisting of a unique simple game $\&_{i \in I} \Downarrow A_i$ with polarized event structure the disjoint sum of the polarized event structures of the $\Downarrow A_i$'s, with all moves in $\Downarrow A_i$ and $\Downarrow A_j$ incompatible when $i \neq j$. Note that the underlying Conway game of $\&_{i \in I} \Downarrow A_i$ is the cartesian product of the underlying Conway game of each $\Downarrow A_i$ in the category Y^-.

Multiplicatives. The tensor product $A \circledast B$ of two *positive* games $A = \{+1 \mid A_i \mid i \in I\}$ and $B = \{+1 \mid B_j \mid j \in J\}$ is defined by synchronizing the initial positions of A and B:

$$A \circledast B \overset{\text{def}}{=} \{+1 \mid A_i \otimes B_j \mid (i,j) \in I \times J\} \tag{9}$$

The tensor product of *positive* and *negative* asynchronous games is deduced from (9) as:

$A \circledast B$	$\overset{\text{def}}{=}$	$A \circledast \downarrow B$	when A is positive and B is negative,
$A \circledast B$	$\overset{\text{def}}{=}$	$(\downarrow A) \circledast B$	when A is negative and B is positive,
$A \circledast B$	$\overset{\text{def}}{=}$	$(\downarrow A) \circledast (\downarrow B)$	when A and B are negative.

The $⅋$-product of two asynchronous games A and B is deduced by the de Morgan equality: $A ⅋ B \overset{\text{def}}{=} (A^\perp \circledast B^\perp)^\perp$ where duality is defined as expected. Note that the asynchronous game $A \circledast B$ is always positive, and that the asynchronous game $A ⅋ B$ is always negative.

Strategies. A strategy σ of a negative asynchronous game $\{-1 \mid A_i \mid i \in I\}$ is defined as a family $\{\ \sigma_i \mid i \in I\}$ of strategies σ_i of the simple game A_i. The strategy σ is innocent (resp. winning) when each strategy σ_i is innocent (resp. winning).

External equivalence. The main idea underlying our model is that two innocent strategies should be identified when they meet the same *external* positions. The set of *external* positions of the strategy $\sigma = \{\ \sigma_i \mid i \in I\}$ on a negative asynchronous game $\{-1 \mid A_i \mid i \in I\}$ is the family $\sigma^\circ = \{\ \sigma_i^\bullet \cap \mathcal{D}_{A_i}^\circ \mid i \in$

$I\}$. Two innocent strategies σ and τ of an asynchronous game A are called *externally equivalent* when $\sigma^\circ = \tau^\circ$. We write this $\sigma \simeq_A \tau$.

The category Z. The category Z has asynchronous games as objects, and $\simeq$-equivalence classes of winning innocent strategies of $A^\perp \parr B$ as morphisms from A to B.

Proposition 5.1 *The category Z is $*$-autonomous and has all products.*

We indicate briefly how morphisms $\sigma : A \longrightarrow B$ behave in the category Z, depending on the polarity of the two games A and B.

Case 1: the games A and B are positive. In that case, the morphism σ is a strategy (modulo $\simeq$) of $A^\perp \parr \downarrow B$ which thus waits for an initial position $(x, *)$ in $A^\perp \parr \downarrow B$, then plays either $(x, *) \to (x, y)$ where y is an initial position of B, or $(x, *) \to (x', *)$ where x' is a position of payoff $-\infty$ in A. The fact that the position x' is necessarily internal in A follows from the requirement that, if played by σ, the position $(x', *)$ is necessarily of positive payoff. Since the payoff of the position $*$ is $-\infty$ in $\downarrow B$, the payoff of the position x' has to be $+\infty$.

This has one remarkable consequence. Call *external* any asynchronous game with no internal position. By the previous discussion, a morphism between two *external* positive games A and B behaves in the same way as a central map on Conway games, discussed in Section 3. That is, after receiving the initial position x of A, the strategy σ plays necessarily an initial position y in B. In that sense, the category Z is a category of *central* strategies.

At this point, it is worth stressing that the monoidal closure of two external positive games A and B is not external nor positive any more: it is the negative game $A^\perp \parr B$, also equal to $A \parr \uparrow(B^\perp)$. The initial positions of this game are the initial positions of A, understood as internal positions in $\uparrow(A^\perp) \parr B$. Each initial position $(x, *)$ is followed by a Player move in A to an internal position $(x', *)$ or by a Player move in B to an external position (x, y).

This improves the set-theoretic definition of monoidal closure (4) in a very satisfactory way, since the definition of (4) is simply replaced by "commuting" the order in which Player and Opponent appear in the game $A \multimap\!\!\bullet\, B$. By way of illustration, consider three external and positive games A, B, C. Any strategy in $A \otimes B \longrightarrow C$ waits for a pair (x, y) of initial positions in A and B, then plays an initial position in C. Exactly the same can be said of a strategy $B \longrightarrow A^\perp \parr C$, which would not be the case using definition (4).

Case 2: the games A and B are negative. The situation is just dual to the previous one.

Case 3: the game A is positive and the game B is negative. The strategy σ (modulo $\simeq$) waits for a pair of an initial position in A and an initial position in B, then plays a move in A or in B.

Case 4: the game A is negative and the game B is positive. In that case, the strategy σ (modulo $\simeq$) plays an initial position in A or an initial position in B. Note that this initial position has to be internal. Indeed, the strategy σ is forbidden to play an external position x in A or y in B because the resulting position $(x, *)$ or $(*, y)$ would be of payoff $-\infty$ in the negative game $\uparrow(A)^{\perp} \parr \uparrow B$. From this follows that there is no strategy from a negative game A to a positive game B when A and B are external.

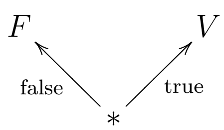
All this ensures that every positive game A is isomorphic to the negative game $\uparrow A$. From this follows that Z has all products, since the full subcategory of negative games in Z is easily shown to have products, in the same way as the categories Y^- or ΠY^+.

A model of linear logic. The category Z may be equipped with an exponential modality $!$ constructed according to the group theoretic ideas of [13]. One obtains that:

Proposition 5.2 *The category Z together with the exponential modality defines a model of propositional linear logic, in the sense of [16,6,12].*

Besides, the category Z incorporates two well-known variants of the innocent arena game model: the well-bracketed and the non well-bracketed ones. More precisely, there are structure preserving functors F (resp. G) from the category of arena games and well-bracketed (resp. non well-bracketed) innocent strategies, to the category Z. The two functors F and G differ mainly in the way they translate the boolean arena (noted `bool`).

The two asynchronous games $F(\texttt{bool})$ and $G(\texttt{bool})$ are very similar. Both are negative, and have a unique component, consisting of a simple game with an Opponent position $*$ at root, two external Player positions V, F, and two Player moves true : $x \to V$ and false : $x \to F$. Intuitively, Opponent plays the initial position $*$, then Player answers either V (for Vrai) or F (for Faux). The two games are represented as follows:



The two games $F(\texttt{bool})$ and $G(\texttt{bool})$ differ only in the value of the payoff function at the root position. The position $*$ is internal (with payoff $-\infty$) in the game $F(\texttt{bool})$ and external (with payoff -1) in the game $G(\texttt{bool})$. It is worth noting that the game $F(\texttt{bool})$ is isomorphic to the game $1 \oplus 1$, which is the expected interpretation of booleans. And that the game $G(\texttt{bool})$ is equal to the game $1 \oplus 1$ lifted by an *affine* variant of the exponential modality, this defining the (linear) *continuation passing style* interpretation of booleans.

Interestingly, the functor G is full and faithful, and translates every arena game to an asynchronous game in which every position is external. On the other hand, the functor F is full, but not faithful. This is nicely illustrated by considering the left and right implementations of the `and` operator of type

$$X = \texttt{bool} \multimap \texttt{bool} \multimap \texttt{bool}.$$

Each left and right implementation is interpreted respectively as a strategy σ_1 and σ_2 in the category of arena games and well-bracketed strategies. The two strategies $F(\sigma_1)$ and $F(\sigma_2)$ are identified in the asynchronous game $F(X)$ because they hit the same set of external positions. The two strategies $G(\sigma_1)$ and $G(\sigma_2)$ are *not* identified in the asynchronous game $G(X)$ because all the positions in $G(X)$ are external. Intuitively, the external positions track the "terminal states" in the game $F(X)$, and all the "intermediate states" in the game $G(X)$.

6 Conclusion and future work

By imposing the isomorphism $A \cong \downarrow\uparrow A$ in a category of sequential games, we identify enough strategies, and obtain a model of propositional linear logic. We conjecture that the resulting category Z (or a close variant) provides a fully complete model of propositional linear logic. We would like to understand also how this category Z is related to the free bicompletion of the singleton category, with respect to limits and colimits, characterized and popularized by André Joyal.

Acknowledgements. The main part of Section 3 was developed in collaboration with Peter Selinger, during the winter 2000-2001. We have learned recently that a similar line of research was developed by Robin Cockett and Robert Seely, on polarized categories and modules. There is also the work by Martin Hyland and Andrea Schalk on abstract games in [7], which provides alternative models of propositional linear logic. There remains to understand how these works are related to the work presented here.

References

[1] Samson Abramsky. Sequentiality vs. concurrency in games and logic. Report Research Report RR-01-08, Oxford University, Programming Research Group, April 2001.

[2] Samson Abramsky and Guy McCusker. Call-by-value games. In Mogens Nielsen and Wolfgang Thomas, editors, *Proceedings of CSL'97*, volume 1414 of *Lecture Notes in Computer Science*. Springer, 1998.

[3] Andrea Blass. A games semantics for linear logic. *Annals of Pure and Applied Logic*, 56:183–220, 1992.

[4] Eric Goubault. Geometry and concurrency: A user's guide. *Mathematical Structures in Computer Science*, 10(4), August 2000.

[5] M. Hofmann and T. Streicher. Completeness of continuation models for $\lambda\mu$-calculus. *Information and Computation*, 179(2):332–355, 2002.

[6] Martin Hyland. *Game Semantics*. Publications of the Newton Institute. Cambridge University Press, 1997.

[7] Martin Hyland and Andrea Schalk. Games on graphs and sequentially realizable functionals. In *Logic in Computer Science 02*, pages 257–264, Kopenhavn, July 2002. IEEE Computer Society Press.

[8] Bart Jacobs. Semantics of weakening and contraction. *Annals of Pure and Applied Logic*, 69:73–106, 1994.

[9] André Joyal. Remarques sur la théorie des jeux à deux personnes. *Gazette des Sciences Mathématiques du Québec*, 1(4):46–52, 1977.

[10] André Joyal. Free lattices, communication and money games. In *Logic and scientific methods. Volume one of the proceedings of the tenth international congress of logic, methodology and philosophy of science, Florence, Italy, Synth. Libr. 259*, pages 29–68. Dordrecht: Kluwer Academic Publishers, 1997.

[11] Olivier Laurent. Polarized games (extended abstract). In *Proceedings of the seventeenth annual symposium on Logic In Computer Science*, pages 265–274, Copenhagen, July 2002. IEEE Computer Society Press.

[12] Paul-André Melliès. Categorical models of linear logic revisited. Prépublication électronique PPS//03/09//n°22 (pp), Laboratoire Preuves, Programmes et Systèmes, March 2003. To appear in *Theoretical Computer Science*.

[13] Paul-André Melliès. Asynchronous games 1: a group-theoretic formulation of uniformity. Prépublication électronique PPS//04/06//n°31 (pp), Laboratoire Preuves, Programmes et Systèmes, 2004.

[14] Paul-André Melliès. Asynchronous games 2: the true concurrency of innocence. In Philippa Gardner and Nobuko Yoshida, editors, *Proceedings of CONCUR '04*, Lecture Notes in Computer Science. Springer Verlag, September 2004.

[15] John Power and Edmund Robinson. Premonoidal categories and notions of computation. *Mathematical Structures in Computer Science*, 7(5):453–468, 1997.

[16] Robert Seely. Linear logic, $*$-autonomous categories and cofree coalgebras. In *Applications of categories in logic and computer science*, number 92. Contemporary Mathematics, 1989.

[17] Peter Selinger. Control categories and duality: on the categorical semantics of the lambda-mu calculus. *Mathematical Structures in Computer Science*, 11(2):207–260, 2001.