Segal condition meets computational effects

Paul-André Melliès

Laboratoire Preuves, Programmes, Systèmes
CNRS, Université Paris Diderot
Paris, France

***Abstract*—Every finitary monad T on the category of sets is described by an algebraic theory whose n-ary operations are the elements of the free algebra Tn generated by n letters. This canonical presentation of the monad (called its Lawvere theory) offers a precious guideline in the search for an intuitive presentation of the monad by generators and relations. Hence, much work has been devoted to extend this correspondence between monads and theories to situations of semantic interest, like enriched categories and countable monads. In this paper, we clarify the conceptual nature of these extended Lawvere theories by investigating the change-of-base mechanisms which underlie them. Our starting point is the Segal condition recently established by Weber for a general notion of monad with arities. Our first step is to establish the Segal condition a second time, by reducing it to the Linton condition which characterizes the algebras of a monad as particular presheaves over the category of free algebras. This reduction is achieved by a relevant change-of-base from the category of interest to its subcategory of arities. This conceptual approach leads us to an abstract notion of Lawvere theory with arities, which extends to every class of arity the traditional correspondence in Set between Lawvere theories and finitary monads. Finally, we illustrate the benefits of Lawvere's ideas by describing how the concrete presentation of the state monad recently formulated by Plotkin and Power is ultimately validated by a rewriting property on sequences of updates and lookups.**

***Keywords*-Computational effects; finitary monads; algebraic theories; Lawvere theories; state monad; higher dimensional algebra; nerve functor; Segal condition; monads with arities.**

I. INTRODUCTION

Mathematics is traditionally interested in numbers and spaces, and there is certainly a conceptual gap to fill in order to understand the mathematical nature of programming languages. Quite miraculously, this gap very often disappears when one climbs in abstraction, revealing beautiful landscapes where the conceptual tools of the two fields suddenly unify. One striking illustration is provided by the notion of computational monad introduced by Moggi [15] in order to describe a functional call-by-value language with effects. The notion of monad is intrinsically mathematical, and offers at the same time a concise and elegant way to describe a wide class of effects: nondeterminism, states, exceptions, interactive input/output, and continuations, see [1]. Another beautiful illustration is provided by the notion of sheaf on a Grothendieck topology (typically, the Schanuel topos) which offers a convenient setting to describe programming languages with local variables and fresh names [3].

It is fascinating to observe that the most promising links between mathematics and programming languages emerged at these somewhat himalayan heights. On the other hand, there is little doubt that this abstraction is only the preliminary stage of a much deeper unity of the two fields, including the most concrete and down-to-earth aspects of mathematics and software engineering. My main ambition in this paper is to illustrate this conceptual unity by revisiting the current state of the art on computational effects presented by operations and equations, in the light of a recent and unexpected connection with a fundamental tool of homotopy theory and higher dimensional algebra: the Grothendieck-Segal characterization of the simplicial nerve of a category.

The state monad, concretely

Given a computational monad capturing a particular notion of effect, typically the state monad

$$T(X) \quad = \quad (S \times X)^S$$

defined by a particular set S of states on the category of sets, one fundamental question is to understand how to present the monad by generators and relations. This question was recently solved in a very elegant way by Plotkin and Power [17] for a set S of states defined as

$$S \quad = \quad V^L$$

where L is a finite set of locations, and V is a countable set of values. A global store on a set A is defined there as a pair of functions

$$\begin{array}{lcccc} \texttt{lookup} & : & A^V & \longrightarrow & A^L \\ \texttt{update} & : & A & \longrightarrow & A^{L\times V} \end{array}$$

satisfying a series of basic equalities formulated in [17]. The extraordinary thing is that this notion of global store describes *exactly* the algebras of the state monad, in a very concrete way based on intuitive properties of lookups and updates in a store. However, the notion of global store

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defined in [17] is not algebraic in the usual sense, because the lookup and update operations have outputs with arity L and $V \times L$ respectively. It is not very difficult however to reformulate it as an algebraic theory, by defining a global store as a family of unary operations

$$\texttt{update}_{loc,val} \quad : \quad A \quad \longrightarrow \quad A$$

indexed by locations $loc \in L$ and values $val \in V$, together with a family of V-ary operations

$$\texttt{lookup}_{loc} \quad : \quad A^V \quad \longrightarrow \quad A$$

indexed by locations $loc \in L$. These operations should satisfy a series of equations easily deduced from [17] and which the interested reader will find expressed as a series of coherence diagrams in Section VI.

The fact that there exists such an algebraic theory for the state monad can be forecast by purely conceptual means, at least when the set of values V is finite. In that case, the set of states $S = V^L$ is finite, and the state monad is thus finitary, in the technical sense that it preserves filtered colimits in the category Set of sets and functions. It is well-known that every such finitary monad is described by an algebraic theory (called the Lawvere theory of the monad) whose n-ary operations are the elements of the free algebra Tn generated by n elements. In the case of the state monad, a n-ary operation is thus given by a set-theoretic function

$$S \quad \longrightarrow \quad S \times n. \tag{1}$$

It is instructive to stop at this point, and to look carefully at the description of the update and lookup operations as such set-theoretic functions, understood at the same time as maps in the Lawvere theory of the state monad:

$$\begin{array}{llcll} \texttt{update}_{loc,val} & : & S & \longrightarrow & S \\ & & state & \mapsto & state[loc := val] \\ \texttt{lookup}_{loc} & : & S & \longrightarrow & S \times V \\ & & state & \mapsto & (state, state(loc)) \end{array} \tag{2}$$

In their paper, Plotkin and Power [17] apply an advanced categorical argument (Beck theorem) in order to establish that the category of sets with global store is equivalent to the category of algebras of the state monad. We explain at the end of the paper (Section VII) how to deduce the property from a very simple and purely combinatorial argument based on the observation that the update and lookup operations present the Lawvere theory of the state monad by generators and relations. This means more specifically:

- that the update and lookup operations (2) generate all the operations (1) of the Lawvere theory,
- that the equations between the update and lookup operations formulated in Section VI are sufficient to reflect the equality between the operations (1) in the Lawvere theory.

These two fundamental facts will be established by applying basic rewriting techniques on the sequences of update and lookup operations.

Beyond finitary monads

The algebraic theory of global stores for a finite set V of values may be easily extended to a countable set of values... this requiring however to consider an operation $\texttt{lookup}_{loc}$ with countable arity V for every location loc. Of course, one needs to extend accordingly the original notion of Lawvere theory, in order to incorporate operations with countable arities. Although this may be done in a somewhat straightforward fashion, the question of arity is more subtle and more interesting than it seems, especially if one considers the enriched case investigated by Hyland and Power [6]. In fact, a purely conceptual and flexible notion of arity in algebraic theories is still missing, although it would be extremely useful in the daily practice of specifying and combining monadic effects. In this paper, we investigate that question starting from the notion of *monad with arities* recently introduced by Weber [21] in his work on the Segal condition, along a conceptual track in higher dimensional algebra opened by Berger [2] and Leinster [11]. We briefly explain this line of work here, starting from the Segal condition originally formulated by Grothendieck in order to characterize the simplicial nerve of a category.

Simplicial sets

The category of simplices Δ has the natural numbers $[n]$ seen as totally ordered sets $[n] = \{0, \ldots, n\}$ as objects, and the monotone functions between them as morphisms. There exists a fully faithful functor

$$i \quad : \quad \Delta \quad \longrightarrow \quad Cat \tag{3}$$

which embeds the category Δ into the category Cat of small categories and functors. The functor i transports every natural number n to the free category over the filiform graph

$$0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n$$

with n edges and $n + 1$ vertices. A simplicial set X is then defined as a presheaf over the category Δ, that is, as a family $(X_n)_{n \in \mathbb{N}}$ of sets, equipped with a function

$$X_f \quad : \quad X_q \longrightarrow X_p$$

for every monotone function $f : [p] \to [q]$. The definition is motivated by geometric intuitions: the point is that every simplicial set X describes a topological space (called its geometric realization) obtained by introducing a n-dimensional simplex for every element of X_n and gluing them together according to the gluing data provided by the "face" and "degeneracy" functions X_f.

Nerve of a category

Now, every functor

$$F \quad : \quad \mathcal{A} \quad \longrightarrow \quad \mathcal{B}$$

to a locally small category $\mathcal{B}$ induces a functor noted

$$\mathcal{B}(F,1) \quad : \quad \mathcal{B} \quad \longrightarrow \quad \widehat{\mathcal{A}} \tag{4}$$

which transports an object B of the category $\mathcal{B}$ to the presheaf $\mathcal{B}(F,B)$ over the category $\mathcal{A}$ defined as

$$\begin{array}{ccccc} \mathcal{B}(F,B) & : & \mathcal{A}^{op} & \longrightarrow & Set \\ & & A & \mapsto & \mathcal{B}(FA,B). \end{array}$$

The functor (3) induces in this way a functor

$$Cat(i,1) \quad : \quad Cat \quad \longrightarrow \quad \widehat{\Delta}$$

which transports every small category to a simplicial set, called its *nerve*. This nerve construction is extremely important, because it enables to see a category as a higher dimensional space, and to apply on it the marvelous tools of homotopy theory, see [13], [14] for details.

Segal condition

The Segal condition appears originally in a paper by Segal [19] where it is attributed to Grothendieck. The condition enables to characterize the simplical sets isomorphic to the nerve of a small category, starting from the observation that the diagram

$$\begin{array}{ccccc} & & [p] & & \\ & \overset{max}{\nearrow} & & \searrow & \\ [0] & & & & [p+q] \\ & \underset{min}{\searrow} & & \nearrow & \\ & & [q] & & \end{array}$$

defines a colimit diagram (that is, a pushout) in the category Δ, for every pair of natural numbers p and q, where $max(0) = p$ and $min(0) = 0$. The geometric intuition is that the graph $[p+q]$ is obtained by gluing together the graphs $[p]$ and $[q]$ on the terminal vertex $p \in [p]$ and initial vertex $0 \in [q]$. Now, the Segal condition reads as follows:

Theorem. *A simplicial set X is isomorphic to the nerve of a small category $\mathcal{C}$ precisely when the colimit diagram is transported to a limit diagram (that is, a pullback)*

$$\begin{array}{ccccc} & & X_p & & \\ & \overset{X_{max}}{\swarrow} & & \nwarrow & \\ X_0 & & & & X_{p+q} \\ & \underset{X_{min}}{\nwarrow} & & \swarrow & \\ & & X_q & & \end{array}$$

in the category Set of sets and functions.

In other words, the nerve of a category is characterized by the property that a $(p+q)$-dimensional simplex is the same thing as a pair (x, y) consisting of a p-simplex x and a q-simplex y whose extremal edges $X_{max}(x)$ and $X_{min}(y)$ coincide.

Segal condition reformulated

Let Δ_0 denote the subcategory of Δ with the same objects, and distance preserving functions $f : [m] \to [n]$ as morphisms:

$$\forall p \in [m], \quad f(p+1) \quad = \quad f(p) + 1.$$

Note that the category Δ_0 is at the same time a full subcategory of the category $Graph$ of oriented graphs, this defining a commutative diagram:

$$\begin{array}{ccc} \Delta & \xrightarrow{\ i\ } & Cat \\ \uparrow{\scriptstyle \ell} & & \uparrow{\scriptstyle Free} \\ \Delta_0 & \xrightarrow{\ i_0\ } & Graph \end{array} \tag{5}$$

where the functor $Free$ transports an oriented graph to its free category. Now, it appears that a simplicial set X satisfies the Segal condition if and only if there exists a graph G such that the functor

$$\Delta_0^{op} \xrightarrow{\ \ell^{op}\ } \Delta^{op} \xrightarrow{\ X\ } Set$$

is isomorphic to the functor

$$Graph(i_0, G) \quad : \quad n \quad \mapsto \quad Graph(i_0 n, G).$$

In this alternative formulation, the nerve X of a small category is characterized by the fact that its restriction to the category Δ_0 of filiform graphs describes (up to natural isomorphism) the set $Graph(i_0 n, G)$ of paths of length n of some graph G. Note that the Segal condition on X may be alternatively formulated as a sheaf condition for a particular Grothendieck topology on the category Δ_0, defining the structure of a Grothendieck topos on the category $Graph$, see the work by Berger [2] for details.

Linton condition

This alternative formulation of the Segal condition as a *representability* property (rather than as a *preservation-of-limit* property) provides the basic pattern of the present work, a precious guideline which will be reappear once and again in our investigation of the conceptual nature of algebraic theories. In order to understand the idea properly, it is wise to start from a striking analogy with the description by Linton [12] of the algebras of a monad T, dating back to the late 1960s. Recall that the Kleisli category $\mathcal{A}_T$ of a monad T on a category $\mathcal{A}$ has the same objects as the category $\mathcal{A}$, while its morphisms $A \to A'$ are the morphisms $A \to TA'$ of the category $\mathcal{A}$. The Kleisli category is equivalent to the category of free algebras of the monad T, this inducing a commutative diagram

$$\begin{array}{ccc} \mathcal{A}_T & \xrightarrow{\ i\ } & T\text{-}Alg \\ {\scriptstyle F}\uparrow & & \uparrow{\scriptstyle Free} \\ \mathcal{A} & \xrightarrow{\ id\ } & \mathcal{A} \end{array} \tag{6}$$

where F is the expected identity-on-object functor, and i is the comparison functor which transports an object A into the free algebra (TA, μ_A). The associated functor

$$T\text{-}Alg(i, 1) \quad : \quad T\text{-}Alg \quad \longrightarrow \quad \widehat{\mathcal{A}_T}$$

transports every algebra (A, h) to a presheaf over $\mathcal{A}_T$ which deserves the name of *nerve* of the algebra (A, h). Note moreover that the functor i is dense, this simply meaning that the induced functor $T\text{-}Alg(i, 1)$ is fully faithful. Now, Linton condition states that for every monad T,

Theorem [Linton] *A presheaf φ on the Kleisli category $\mathcal{A}_T$ is isomorphic to the nerve of an algebra if and only if the presheaf*

$$\mathcal{A}^{op} \xrightarrow{\ F^{op}\ } \mathcal{A}_T^{op} \xrightarrow{\ \varphi\ } Set$$

is representable in the category $\mathcal{A}$, this meaning that $\varphi \circ F^{op}$ is isomorphic to the presheaf $\mathbf{y}_A$ associated by the yoneda embedding to an object A of the category $\mathcal{A}$:

$$\mathbf{y}_A = \mathcal{A}(1, A) \quad : \quad A' \quad \mapsto \quad \mathcal{A}(A', A).$$

It is thus tempting to think of Linton condition as an extremal Segal condition where the functor i_0 in the commutative diagram (5) is replaced by the identity functor in the commutative diagram (6). Observe in particular that (6) is instantiated as

$$\begin{array}{ccc} FreeCat & \xrightarrow{\ i\ } & Cat \\ {\scriptstyle F}\uparrow & & \uparrow{\scriptstyle Free} \\ Graph & \xrightarrow{\ id\ } & Graph \end{array} \tag{7}$$

for the free category monad T on the category $Graph$.

Monads with arities

Once the connection with Linton condition established, the Segal condition reduces to understanding when the identity functor appearing in (6) may be replaced by a functor

$$i_0 \quad : \quad \Theta_0 \quad \longrightarrow \quad \mathcal{A}$$

describing a class of arities for the monad T. Although the connection with Linton condition does not appear in his work, this is precisely the question investigated by Weber [21] with the notion of monad with arities. The point is that every notion of arity i_0 induces a commutative diagram

$$\begin{array}{ccc} \Theta_T & \xrightarrow{\ i_T\ } & T\text{-}Alg \\ {\scriptstyle \ell}\uparrow & & \uparrow{\scriptstyle Free} \\ \Theta_0 & \xrightarrow{\ i_0\ } & \mathcal{A} \end{array}$$

where the category Θ_T is characterized by the fact that the functor ℓ is the identity on objects (hence, the category Θ_T has the same objects as the category Θ_0) and that the functor i_T is fully faithful (hence, the category Θ_T has the same morphisms as the category $T\text{-}Alg$, locally speaking). Weber formulates a series of sufficient conditions on the functor i_0 and on the monad T, such that the induced nerve functor

$$T\text{-}Alg(i_0, 1) \quad : \quad T\text{-}Alg \quad \longrightarrow \quad \widehat{\Theta_T}$$

satisfies a Segal condition, which states that the category $T\text{-}Alg$ is equivalent to the full subcategory of presheaves of Θ_T whose restriction along the functor ℓ is isomorphic to the restriction of a representable presheaf along the functor i_0. The resulting notion of monad with arities is extremely rich and flexible. Typically, a finitary monad on the category Set is the same thing as a monad with arity functor i_0 defined as the fully faithful functor

$$i_0 \quad : \quad Nat \quad \longrightarrow \quad Set \tag{8}$$

starting from the full subcategory of Set defined by the finite sets $\langle n \rangle = \{0, \ldots, n-1\}$. Similarly, a countable monad is the same thing as a monad with arity functor

$$i_0 \quad : \quad Count \quad \longrightarrow \quad Set \tag{9}$$

defined by extending the previous arity functor (8) with the countable set $\langle \omega \rangle = \{0, 1, 2, 3, \cdots\}$. More generally, any accessible monad on a locally presentable category $\mathcal{A}$ defines a monad with arities, with the arity functor i_0 then defined as the inclusion functor of a skeleton of the full subcategory of κ-presentable objects, for a regular cardinal κ.

Algebraic theories with arities

One main contribution of the present paper is (a) to improve marginally the original notion of monad with arities, by relaxing a cocompleteness hypothesis on the underlying category $\mathcal{A}$, and (b) to derive the Segal condition from the Linton condition in a nice and conceptual way, thanks to the discovery of a Beck-Chevalley property of the change-of-base operations. This analysis enables us (c) to formulate an abstract notion of Lawvere theory for every category $\mathcal{A}$ and every arity functor i_0, and (d) to establish a clean correspondence theorem, which states that the category $\mathbf{Law}(\mathcal{A}, i_0)$ of Lawvere theories is equivalent to the category $\mathbf{Mnd}(\mathcal{A}, i_0)$ of monads with arities i_0. This level of generality is achieved by replacing the familiar *preservation-of-limit* property of

Lawvere theories by a *preservation-of-representability* property inspired by the abstract definition of monad with arities.

Enriched Lawvere $\mathcal{A}$-theories

The notion of enriched Lawvere theory was introduced by Power [18] ten years ago. This notion has become extremely important in the semantic practice, at least because it enables to incorporate recursion and partiality into the study of monadic effects, see [5]. One must admit however that the notion of enriched Lawvere theory is technically involved, and one initial motivation of the present work was precisely to clarify its conceptual foundations, starting from a 2-categorical approach. It is only quite recently, in the course of writing that paper, that I discovered with great excitement that Nishizawa and Power [16] recently introduced the notion of enriched Lawvere $\mathcal{A}$-category, which contains (essentially) the same conceptual ingredients as the Segal condition formulated by Weber [21] at about the same time. This extraordinary convergence between two independent lines of research is another sign of the deep unity of the field, and of the relevance of the conceptual and unifying approach developed in the present paper.

Outline of the paper

After this long but necessary introduction, we recall in Section II the change-of-base operations on presheaves, followed by the notion of monad with arities in Section III. We then establish the Segal condition in Section IV, starting from Linton conditon and the observation of a Beck-Chevalley property on the change-of-base operations. We introduce in Section V an abstract notion of Lawvere theory with arities, and establish a correspondence theorem with monads with arities. Finally, we illustrate in Section VI and Section VII the concrete benefits of this trend of ideas on the global state monad, before concluding in Section VIII.

II. The three operations

The first step of the paper is to establish the Segal condition by a purely conceptual argument based on the change-of-base operations associated to a functor. These operations are so fundamental that we choose to describe them as early as possible in the article. The reader unaware of this categorical yoga inherited from Grothendieck [13], [14] should have a glimpse at the section, and jump to Section III where the notion of monad with arities is introduced. Just like rings are particular kinds of categories (with one object, enriched over the category of abelian groups) modules over a ring are particular kinds of presheaves. So, the idea is to extend to presheaves the classical operations on modules associated to a change-of-ring. Typically, every functor

$$F \quad : \quad \mathcal{A} \quad \longrightarrow \quad \mathcal{B}$$

induces a functor

$$F^* \quad : \quad \widehat{\mathcal{B}} \quad \longrightarrow \quad \widehat{\mathcal{A}}$$

defined by transporting every presheaf ψ to the presheaf $F^*(\psi)$ obtained by precomposition:

$$\mathcal{A}^{op} \quad \xrightarrow{F^{op}} \quad \mathcal{B}^{op} \quad \xrightarrow{\psi} \quad Set.$$

Whenever the category $\mathcal{A}$ is small (that is, when its objects define a set, rather than a class) the functor F^* has a left adjoint

$$\exists_F \quad : \quad \widehat{\mathcal{A}} \quad \longrightarrow \quad \widehat{\mathcal{B}}.$$

as well as a right adjoint

$$\forall_F \quad : \quad \widehat{\mathcal{A}} \quad \longrightarrow \quad \widehat{\mathcal{B}}$$

defined by transporting every presheaf φ to its left and right Kan extension along the functor $F^{op} : \mathcal{A}^{op} \to \mathcal{B}^{op}$. The logical notation for the adjoint functors is justified by the description of quantification in a topos: the functors $\exists_F$ and $\forall_F$ would be typically written $F_!$ and F_* in Grothendieck's notation.

III. Monads with arities

The notion of monad with arities was introduced by Weber [21] after a suggestion by Lack, this providing a concise and elegant account of the conceptual track opened by Berger [2] and Leinster [11] in higher dimensional algebra ; the notion of monad with arities has been also recently applied by Joyal and Kock [7] in order to define a nerve functor for compact symmetric multicategories (also called modular operads). As the reader will see below, our definition of monad with arities is slightly more liberal than the original one because we do not require that the underlying category $\mathcal{A}$ is cocomplete.

Fully faithful and dense functors

A functor $F : \mathcal{A} \to \mathcal{B}$ is fully faithful when the associated function

$$\mathcal{A}(A, A') \quad \longrightarrow \quad \mathcal{B}(FA, FA')$$

is a bijection for all objects A, A' of the category $\mathcal{A}$. A functor $F : \mathcal{A} \to \mathcal{B}$ is dense when the associated functor

$$\mathcal{B}(F, 1) \quad : \quad \mathcal{B} \quad \longrightarrow \quad \widehat{\mathcal{A}}$$

defined in (4) is fully faithful.

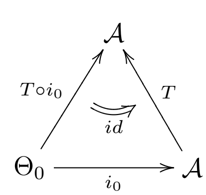
Monads with arities

A monad with arities consists of a monad (T, μ, η) on a category $\mathcal{A}$ together with a fully faithful and dense functor

$$i_0 \quad : \quad \Theta_0 \quad \longrightarrow \quad \mathcal{A} \qquad (10)$$

where Θ_0 is a small category, and such that:

1) the natural transformation



(11)

exhibits the functor T as a left Kan extension of the functor $T \circ i_0$ along the functor i_0,
2) the Kan extension (11) is preserved by the functor

$$\mathcal{A}(i_0, 1) \quad : \quad \mathcal{A} \quad \longrightarrow \quad \widehat{\Theta_0}.$$

Let us briefly discuss these two arity conditions on the monad. The first condition is somewhat expected: it captures very neatly the idea that the monad T is entirely defined by the functor $T \circ i_0$. This formulation is somewhat folklore: for instance, Kelly [8] characterizes in this way the finitary functors in a properly enriched setting.

The second arity condition is less expected, and it is certainly one main conceptual novelty of Weber's definition: it means that every colimit computed in $\mathcal{A}$ in order to reconstruct the monad T from the functor $T \circ i_0$ should be also seen as a colimit computed in Set by every arity n in the category Θ_0. This is typically the case when the category Θ_0 is the full subcategory of finitely presentable objects in a locally finitely presentable category $\mathcal{A}$, because all the colimits considered in $\mathcal{A}$ are filtered, and $\mathcal{A}(i_0, 1)$ preserves them. In that case, the two arity conditions on the monad T reduce to the first one, this probably explaining why the second arity condition never appeared in the literature.

A combinatorial formulation

One should also mention that the two arity conditions reduce to the fact that the functor $\mathcal{A}(i_0, 1) \circ T$ equipped with the identity transformation on the functor $\mathcal{A}(i_0, 1) \circ T \circ i_0$ defines a left Kan extension of that functor along the functor i_0. The reason is that the functor $\mathcal{A}(i_0, 1)$ is fully faithful, and thus reflects left Kan extensions. Hence, the arity conditions may be equivalently formulated by requiring that the canonical function

$$\int^{p \in \Theta_0} \mathcal{A}(i_0 n, T i_0 p) \times \mathcal{A}(i_0 p, A) \longrightarrow \mathcal{A}(i_0 n, TA)$$

is a bijection, for every object n of the category Θ_0 and every object A of the category $\mathcal{A}$. This should be understood as a unique decomposition property (modulo zig-zag) which states that every morphism

$$i_0 n \quad \longrightarrow \quad TA$$

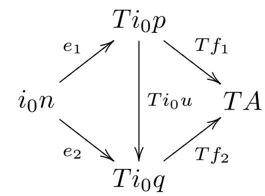
in the category $\mathcal{A}$ decomposes as

$$i_0 n \quad \xrightarrow{e} \quad T i_0 p \quad \xrightarrow{Tf} \quad TA$$

for a pair of morphisms $e : i_0 n \to T i_0 p$ and $f : i_0 p \to A$. And that, moreover, every two such factorizations are equivalent modulo the zig-zag relation $\sim$ defined as the transitive, symmetric and reflexive closure of the binary relation

$$(e_1, f_1) \quad \rightsquigarrow \quad (e_2, f_2)$$

which relates two factorizations (e_1, f_1) and (e_2, f_2) when there exists a morphism $u : p \to q$ of the category of arities Θ_0 making the diagram



commute in the category $\mathcal{A}$.

The state monad

It is instructive to understand from that point of view why the state monad T is finitary when the set of states S is finite. Recall that the finitary monads on the category Set are precisely the monad with arity functor i_0 described in (8). Hence, the state monad T is finitary because (a) every function

$$h \quad : \quad S \times [n] \quad \longrightarrow \quad S \times A$$

factors as

$$S \times [n] \quad \xrightarrow{e} \quad S \times [p] \quad \xrightarrow{S \times f} \quad S \times A$$

where the function $f : [p] \to A$ is defined as an injective enumeration of the finite image of h, and moreover (b) this factorization is unique modulo zig-zag. The Segal condition establishes then that the state monad may be presented by operations of finite arities and equations between them, as done in Section VI when $S = V^L$. On the other hand, the state monad is not finitary anymore when the set S is countable: it defines in that case a countable monad with arity functor i_0 defined as (9). This elementary example illustrates the flexibility of the notion of monad with arities.

IV. A conceptual proof of Segal condition

Our alternative proof of Segal condition starts with the definition of categories with arities, together with a notion of morphism between them. As we will see, one advantage of our argument (besides its conceptual simplicity) is that it does not require the hypothesis that the category $\mathcal{A}$ is cocomplete.

Categories with arities

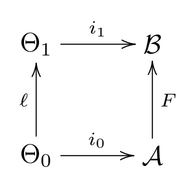
A category with arities $(\mathcal{A}, i_0)$ is defined as a fully faithful and dense functor

$$i_0 \quad : \quad \Theta_0 \quad \longrightarrow \quad \mathcal{A}$$

whose domain Θ_0 is a small category. A morphism between categories with arities

$$(F, \ell) \quad : \quad (\mathcal{A}, i_0) \quad \longrightarrow \quad (\mathcal{B}, i_1)$$

is defined as a pair of functors (F, ℓ) making the diagram



(12)

commute, and satisfying moreover the Beck-Chevalley condition which states that the natural transformation

$$\begin{array}{ccc} \widehat{\Theta_1} & \xrightarrow{\forall_{i_1}} & \widehat{\mathcal{B}} \\ {\scriptstyle \ell^*}\downarrow & \swarrow\!\!\!\!\swarrow & \downarrow{\scriptstyle F^*} \\ \widehat{\Theta_0} & \xrightarrow[\forall_{i_0}]{} & \widehat{\mathcal{A}} \end{array} \qquad (13)$$

defined as the mate (in a 2-categorical sense, see [9]) of the identity natural transformation $id : i_0^* \circ F^* \Rightarrow \ell^* \circ i_1^*$, is reversible. It is not difficult to deduce from the functorial properties of mateship that these morphisms compose, and thus define a category of categories with arities.

Segal condition

The Segal condition follows then quite immediately from two basic properties of these morphisms between categories with arities, together with Linton condition. The first property captures the very essence of Segal condition:

Proposition A. *For every morphism (F, ℓ) between categories with arities*

$$(F, \ell) \quad : \quad (\mathcal{A}, i_0) \quad \longrightarrow \quad (\mathcal{B}, i_1)$$

the adjunction $i_1^* \dashv \forall_{i_1}$ induces an adjunction between

- *the full subcategory $\mathcal{M}$ of presheaves of $\mathcal{B}$ whose restriction along F is representable in $\mathcal{A}$,*
- *the full subcategory $\mathcal{N}$ of presheaves of Θ_1 whose restriction along ℓ is representable along i_0.*

Moreover, this adjunction defines an equivalence between $\mathcal{M}$ and $\mathcal{N}$ when the functor F is essentially surjective.

Here, a presheaf of Θ_0 is called **representable along the functor** i_0 when it is isomorphic to the restriction along i_0 of a representable presheaf in $\mathcal{A}$. Note that this is equivalent to being isomorphic to a presheaf $\mathcal{A}(i_0, A)$ for some object A. Recall that a functor F is **essentially surjective** when there exists for every object B an object A such that FA is isomorphic to B. The second proposition establishes the existence of a morphism between categories with arities for every monad with arities:

Proposition B. *Every monad T with arity functor i_0 induces a commutative diagram*

$$\begin{array}{ccc} \Theta_T & \xrightarrow{i_T} & \mathcal{A}_T \\ {\scriptstyle \ell}\uparrow & & \uparrow{\scriptstyle F} \\ \Theta_0 & \xrightarrow{i_0} & \mathcal{A} \end{array} \qquad (14)$$

where the pair (F, ℓ) defines a morphism

$$(F, \ell) \quad : \quad (\mathcal{A}, i_0) \quad \longrightarrow \quad (\mathcal{A}_T, i_T)$$

of categories with arities.

Theorem [Segal condition]. *The canonical functor*

$$H \quad : \quad \Theta_T \xrightarrow{i_T} \mathcal{A}_T \longrightarrow T\text{-}Alg$$

induces an equivalence

$$T\text{-}Alg \xrightarrow{T\text{-}Alg(H,1)} \widehat{\Theta_T}$$

between the category T-Alg and the full subcategory of presheaves of Θ_T whose restriction along the functor ℓ is representable along the functor i_0.

V. Lawvere theories with arities

We introduce below a notion of Lawvere theory for a category with arities $(\mathcal{A}, i_0)$ and establish in that setting a clean correspondence theorem between theories and monads, generalizing the traditional correspondence between Lawvere theories and finitary monads in the category Set equipped with finite arities. It is interesting to notice that our definition proceeds in essentially the same way as the definitions of globular theory and of globular model by Berger (see definition 1.5 in [2]) in the particular case of the category of globular sets with arities defined as level trees.

Lawvere theories with arities

A Lawvere theory $\mathbb{L}$ on a category $\mathcal{A}$ with arities $i_0 : \Theta_0 \longrightarrow \mathcal{A}$ is defined as an identity-on-object functor

$$\mathbb{L} \quad : \quad \Theta_0 \quad \longrightarrow \quad \Theta_{\mathbb{L}}$$

such that $(\star)$ the endofunctor

$$\widehat{\Theta_0} \xrightarrow{\exists_{\mathbb{L}}} \widehat{\Theta_{\mathbb{L}}} \xrightarrow{\mathbb{L}^*} \widehat{\Theta_0}$$

transports every presheaf representable along i_0 to a presheaf representable along i_0. It is not difficult to see that:

Proposition C. *Every monad T with arity functor i_0 induces a Lawvere theory $\mathbb{L}_T : \Theta_0 \longrightarrow \Theta_T$.*

The property follows from the fact that the functor $\mathbb{L}^* \circ \exists_{\mathbb{L}}$ transports the presheaf $\mathcal{A}(i_0 A, 1)$ defined by an object A of the category $\mathcal{A}$ to a presheaf isomorphic to $\mathcal{A}(i_0 TA, 1)$.

Models of the theory

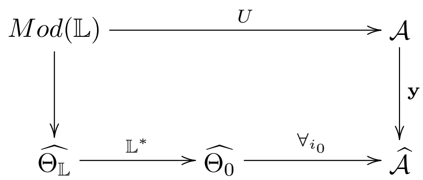
A model of the Lawvere theory $\mathbb{L}$ with arity functor i_0 is defined as a presheaf φ over $\Theta_{\mathbb{L}}$ whose restriction

$$\Theta_0^{op} \xrightarrow{\mathbb{L}^{op}} \Theta_{\mathbb{L}}^{op} \xrightarrow{\varphi} Set$$

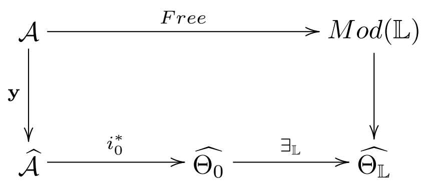
along $\mathbb{L}$ is representable along i_0. The category $Mod(\mathbb{L})$ is then defined as the full subcategory of presheaves of $\Theta_{\mathbb{L}}$ whose objects are the models of the theory $\mathbb{L}$. There exists a forgetful functor

$$U \quad : \quad Mod(\mathbb{L}) \quad \longrightarrow \quad \mathcal{A}$$

defined as the unique functor (up to natural isomorphism) making the diagram



commute, up to natural isomorphism. The preservation-of-representability property $(\star)$ required by our definition of Lawvere theory ensures that the functor U has a left adjoint $Free$ making the diagram



commute, up to natural isomorphism. This adjunction $Free \dashv U$ induces a monad T on the category $\mathcal{A}$ with the expected properties:

Proposition D. *The monad T has arity functor i_0 and induces a Lawvere theory $\mathbb{L}_T : \Theta_0 \to \Theta_T$ which coincides with the theory $\mathbb{L} : \Theta_0 \to \Theta_{\mathbb{L}}$.*

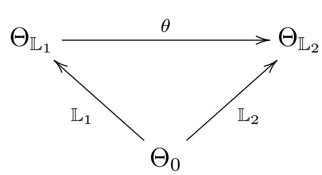
Note that, strictly speaking, the two categories Θ_T and $\Theta_{\mathbb{L}}$ are isomorphic, rather than equal.

Correspondence theorem

A morphism $\mathbb{L}_1 \to \mathbb{L}_2$ between Lawvere theories $\mathbb{L}_1$ and $\mathbb{L}_2$ with the same arity functor i_0, is defined as an identity-on-object functor

$$\theta \quad : \quad \Theta_{\mathbb{L}_1} \quad \longrightarrow \quad \Theta_{\mathbb{L}_2}$$

making the diagram below commute:



This notion of morphism between Lawvere theories defines a category $\mathbf{Law}(\mathcal{A}, i_0)$ of Lawvere theories on the category with arities $(\mathcal{A}, i_0)$ whose definition is justified by the correspondence theorem below.

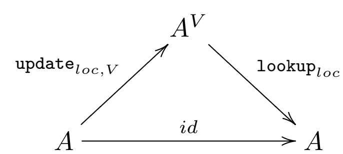
Theorem. *The category $\mathbf{Law}(\mathcal{A}, i_0)$ is equivalent to the category $\mathbf{Mnd}(\mathcal{A}, i_0)$ of monads with arities i_0.*

The proof of the correspondence theorem is purely 2-categorical, and simply requires a 2-category with Eilenberg-Moore and Kleisli objects [10], equipped with a Yoneda structure in the sense of Street and Walters [20], [22]. In particular, the result applies in exactly the same way to the enriched setting, by replacing the 2-category of categories, functors and natural transformations, by the 2-category of enriched categories, enriched functors and enriched natural transformations for a sufficiently nice category $\mathcal{V}$ of enrichment.

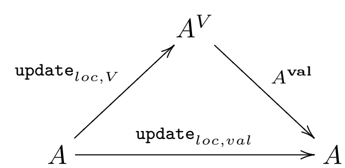
VI. Presentation of the state monad

We formulate the equational theory of global stores as a series of seven coherence diagrams, each of them providing the direct transcription of an equation in [17]. Note that the resulting commutative diagrams look simpler here than in the original presentation because the manipulation of locations (duplication, etc.) is done externally, rather than internally.

1. annihilation lookup − update: reading the value of a location loc and then updating the location loc with the obtained value is just like doing nothing.

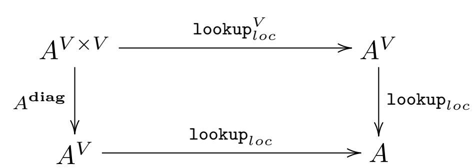


Here, the morphism $\mathtt{update}_{loc,V} : A \longrightarrow A^V$ is defined as the unique morphism making the diagram below commute

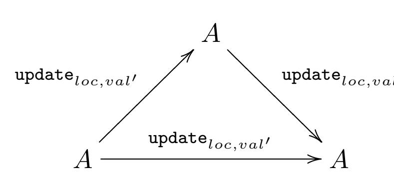


for every value $val \in V$, where $A^{\mathbf{val}} : A^V \longrightarrow A$ is the val-th projection of A^V over A.

2. interaction lookup − lookup: reading twice the same location loc is the same as reading it once.



3. interaction update − update: storing a value val and then a value val' at the same location loc is just like storing the value val' in the location.



4. interaction update − lookup: when one stores a value val in a location loc and then reads the location loc, one gets the value val.

$$\begin{array}{ccc} A^V & \xrightarrow{\texttt{lookup}_{loc}} & A \\ {\scriptstyle A^{\mathbf{val}}}\downarrow & & \downarrow{\scriptstyle \texttt{update}_{loc,val}} \\ A & \xrightarrow{\texttt{update}_{loc,val}} & A \end{array}$$

5. commutation lookup − lookup: The order of reading two different locations loc and loc' does not matter.

$$\begin{array}{ccccc} & A^{V\times V} & \xrightarrow{A^{\mathbf{swap}}} & A^{V\times V} & \\ {\scriptstyle \texttt{lookup}_{loc}}\swarrow & & & & \searrow{\scriptstyle \texttt{lookup}_{loc'}} \\ A^V & \xrightarrow{\texttt{lookup}^V_{loc'}} & A & \xleftarrow{\texttt{lookup}^V_{loc}} & A^V \end{array}$$

6. commutation update − update: the order of storing in two different locations loc and loc' does not matter.

$$\begin{array}{ccc} A & \xrightarrow{\texttt{update}_{loc,val}} & A \\ {\scriptstyle \texttt{update}_{loc',val'}}\downarrow & & \downarrow{\scriptstyle \texttt{update}_{loc',val'}} \\ A & \xrightarrow{\texttt{update}_{loc,val}} & A \end{array}$$

7. commutation update − lookup: the order of storing in a location loc and reading in a location loc' does not matter.

$$\begin{array}{ccc} A^V & \xrightarrow{\texttt{lookup}_{loc'}} & A \\ {\scriptstyle \texttt{update}^V_{loc,val}}\downarrow & & \downarrow{\scriptstyle \texttt{update}_{loc,val}} \\ A^V & \xrightarrow[\texttt{lookup}_{loc'}]{} & A \end{array}$$

VII. Presentation of the state monad revisited

We establish here that the algebraic presentation of objects with global store described in Section VI provides a presentation by generators and relations of the Lawvere theory $\mathbb{T}$ of the state monad. From this result follows immediately the result established by Plotkin and Power [17] stating that the category of objects with store is equivalent to the category of algebras of the state monad. Note that the result in [17] applies to any category with countable products and coproducts, but we focus here on the particular case of Set.

Let $\mathbb{S}$ denote the Lawvere theory generated by the object 1 and the family of operations

$$\texttt{update}_{loc,val} \ : \ 1 \to 1 \qquad \texttt{lookup}_{loc} \ : \ V \to 1$$

for $loc \in L$ and $val \in V$, together with the seven equations of Section VI.

Soundness

The interpretation of $\texttt{update}_{loc,val}$ and $\texttt{lookup}_{loc}$ described in the introduction satisfies the equations of a global store. This establishes the existence of an identity-on-object and product-preserving functor

$$I \quad : \quad \mathbb{S} \quad \longrightarrow \mathbb{T}.$$

There remains to establish that the functor I is fully faithful.

The functor I is full

In order to establish that point, one needs to show that every set-theoretic function $f : S \to S \times n$ is generated by a series of lookups and updates. This is not particularly difficult. The idea is to factor the function f as

$$S \quad \xrightarrow{g} \quad S \times V^L \quad \xrightarrow{h} \quad S \times n$$

where
1. the function g is the diagonal $S \to S \times S$ obtained by applying a lookup for each location $loc \in L$, one after the other,

2. the function h transports $(state_1, state_2)$ into $f(state_2)$. Here, the domain $S \times V^L$ should be understood as the sum of S taken $S = V^L$ times. This enables to define the function h as a family of constant functions

$$h_{state_2} = f(state_2) \quad : \quad S \quad \longrightarrow \quad S \times n$$

indexed by $state_2 \in V^L$, each constant function implemented as a series of updates writing the value $state(loc)$ into each location $loc \in L$, followed by an injection to the p-th component of $S \times n$:

$$S \quad \xrightarrow{state} \quad S \quad \xrightarrow{\mathbf{in}_p} \quad S \times n$$

where $f(state_2) = (state, p)$.

The functor I is faithful

This is the difficult and interesting part of the proof. Suppose given two terms u and v of the algebraic theory of global stores $u, v : n \to 1$ defining the same function

$$f \quad : \quad S \quad \longrightarrow \quad S \times n \qquad (15)$$

understood as an operation $n \to 1$ in the category $\mathbb{T}$. We need to show that the terms u and v are equal modulo the seven equations of the theory of global stores. The idea is to apply the first equation (annihilation) as many times as there are locations in L, in order to factorize the identity morphism in $\mathbb{S}$ as a sequence g of lookups, one for each location $loc \in L$, followed by a sequence f of updates writing in each location what has been just read:

$$id \quad : \quad 1 \quad \xrightarrow{h} \quad V^L \quad \xrightarrow{g} \quad 1.$$

Since $u = g \circ h \circ u$ and $v = g \circ h \circ v$, it is sufficient to establish that $h \circ u = h \circ v$ in order to conclude. Since the category $\mathbb{S}$ is cartesian, this amounts to the equality

$$\pi_{state} \circ h \circ u \quad = \quad \pi_{state} \circ h \circ v$$

for every projection $\pi_{state} : V^L \to 1$. Now, observe that the functor I transports the two maps $\pi_{state} \circ h \circ u$ and $\pi_{state} \circ h \circ v$ to the same *constant* operation $n \to 1$ of the theory $\mathbb{T}$. Observe also that $h_{state} = \pi_{state} \circ h : 1 \to 1$ is defined as a sequence of updates, one for each location, writing one after the other the value $state(loc)$ in each location $loc \in L$. The last part of the proof consists in removing the lookups appearing in $h_{state} \circ u$ and $h_{state} \circ v$ one after the other, by permuting them before updates thanks to equation 7. and removing them thanks to equation 4. The point is that every lookup in $h_{state} \circ u$ and $h_{state} \circ v$ reads a location previously updated in the term. Once every lookup removed from $h_{state} \circ u$ and $h_{state} \circ v$, there simply remains to remove the unnecessary updates by applying equation 6. to permute them and equation 3. to erase them. One obtains in this way a normal form for $h_{state} \circ u$ and $h_{state} \circ v$ consisting of a sequence of an update for each location $loc \in L$, the two normal forms for u and v coinciding modulo permutation of the updates by equation 6. This completes the proof that the functor I is faithful.

VIII. Conclusion and future works

We establish a general correspondence theorem between the notion of monad with arities defined by Weber [21] and an abstract notion of Lawvere theory with arities introduced here. The proofs are simple and conceptual, and clarify the change-of-base mechanisms which underlie the notion of Lawvere theory. Much progress has been made in the past decade in the art of combining monads [5], [4] this leading to the discovery of subtle issues about arities in enriched categories [6]. The present work is to a large extent motivated by the ambition to establish an appropriate 2-categorical framework to carry on this promising line of research. It is also part of a wider project of combining monadic effects with linear continuations, starting from the seminal work of Hyland, Levy, Plotkin and Power [4] and integrating diagrammatic techniques imported from game semantics. Finally, we believe that a conceptual understanding of these basic questions will contribute to the emergence of a semantic account of computational effects lying outside the scope of monadic effects, typically delimited continuations.

Acknowlegments

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Appendix

Appendix $\mathbb{I}$: Proofs of Proposition A and B.

The proofs of the two propositions are not particularly difficult conceptually: they are essentially based on a clear understanding of the meaning and properties of the three change-of-base operations on presheaves described in Section II. Size is a real torment however: because the category $\mathcal{A}$ is not supposed to be small, we cannot make the simplifying hypothesis that the functor

$$F^* \quad : \quad \widehat{\mathcal{A}_T} \quad \longrightarrow \quad \widehat{\mathcal{A}}$$

associated to the functor F to the kleisli category

$$F \quad : \quad \mathcal{A} \quad \longrightarrow \quad \mathcal{A}_T$$

has a left adjoint

$$\exists_F \quad : \quad \widehat{\mathcal{A}} \quad \longrightarrow \quad \widehat{\mathcal{A}_T}.$$

In fact, we only know that F^* has a right adjoint

$$\forall_F \quad : \quad \widehat{\mathcal{A}} \quad \longrightarrow \quad \widehat{\mathcal{A}_T}$$

defined as the inverse image functor

$$U^* \quad : \quad \widehat{\mathcal{A}} \quad \longrightarrow \quad \widehat{\mathcal{A}_T}$$

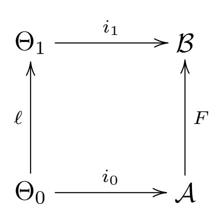
associated to the functor U right adjoint to F. This lack of a left adjoint $\exists_F$ makes the proof much more complicated than it should be, or becomes when the left adjoint $\exists_F$ happens to exist. There is a way to circumvent the difficulty however, provided by the notion of yoneda structure introduced by Street and Walters [20] and recently investigated by Weber [22]. We describe below the general proof, and let the astute reader reconstruct the simpler argument in the situation when the functor F^* happens to have a left adjoint $\exists_F$.

Proof of Proposition A.

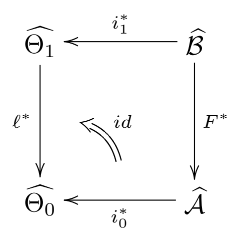
Step 1: Suppose given a morphism

$$(F, \ell) \quad : \quad (\mathcal{A}, i_0) \quad \longrightarrow \quad (\mathcal{B}, i_1)$$

between categories with arity, defining the commutative diagram

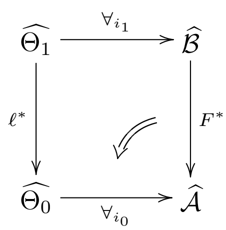


This induces a commutative diagram



From this follows that the functor i_1^* transports every presheaf ψ of $\mathcal{B}$ whose restriction along F is representable by an object A, to a presheaf $i_1^*\psi$ of Θ_1 whose restriction along ℓ is representable by the object $i_0 A$.

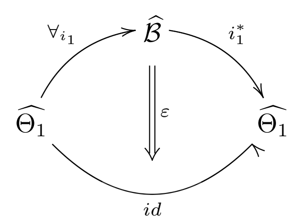
By definition, a morphism between categories with arity satisfies moreover a Beck-Chevalley property, stating that the induced natural transformation



is reversible. From this follows that the functor $\forall_{i_1}$ transports every presheaf φ of Θ_1 whose restriction along ℓ is representable by an object $i_0 A$, to a presheaf $\forall_{i_1}\varphi$ of $\mathcal{B}$ whose restriction along F is isomorphic to the presheaf $\forall_{i_0} \circ i_0 \circ \mathbf{y}(A)$. We then apply the result of the previous step, which states that the presheaf $\forall_{i_0} \circ i_0 \circ \mathbf{y}(A)$ is isomorphic to the presheaf $\mathbf{y}(A)$, and conclude that the functor $\forall_{i_1}$ transports every presheaf φ of Θ_1 whose restriction along ℓ is representable by an object $i_0 A$, to a presheaf $\forall_{i_1}\varphi$ of $\mathcal{B}$ whose restriction along F is representable by A.

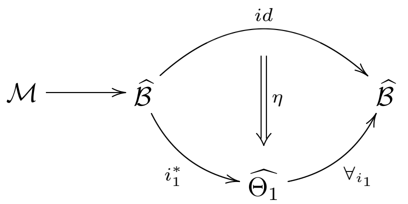
This establishes that the adjunction $i_1^* \dashv \forall_{i_1}$ between the presheaf categories of $\mathcal{A}$ and Θ_1 restricts to an adjunction between the full subcategory $\mathcal{M}$ of presheaves of $\mathcal{B}$ whose restriction along F is representable, and the full subcategory $\mathcal{N}$ of presheaves of Θ_1 whose restriction along ℓ is representable along i_0.

Observe moreover that the functor i_1 is fully faithful. The fact that i_1 is fully faithful may be equivalently formulated by saying that the counit



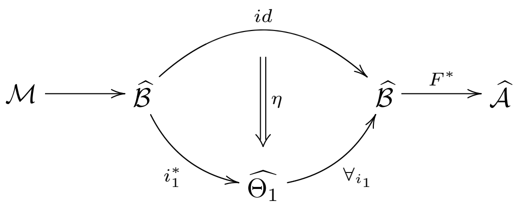
of the adjunction $i_1^* \dashv \forall_{i_1}$ is reversible. This is equivalent to asking that the functor $\forall_{i_1}$ is fully faithful, this establishing that the category $\mathcal{N}$ is a reflective subcategory of the category $\mathcal{M}$.

Step 2: Suppose that the functor F is essentially surjective, this meaning that for every object B of the category $\mathcal{B}$, there exists an object A of the category $\mathcal{A}$ such that FA is isomorphic to B. One way to establish that the adjunction $i_0^* \dashv \forall_{i_0}$ defines an equivalence between the two categories $\mathcal{M}$ and $\mathcal{N}$, is to show that the natural transformation

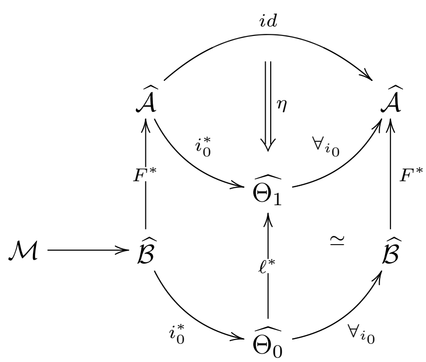


(16)

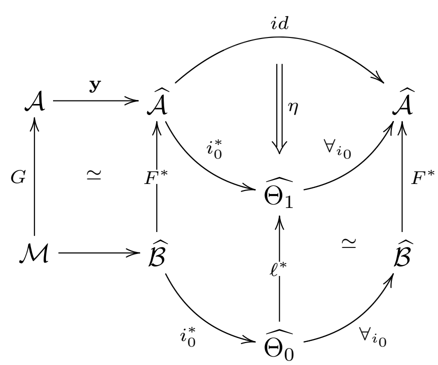
is reversible. Now the functor F^* is faithful because F is essentially surjective. Hence, in order to establish that (16) is reversible, it is sufficient to establish that the natural transformation



is reversible. The natural transformation may be decomposed in the following way:



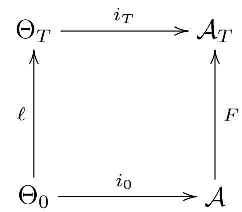
a diagram which may be completed as



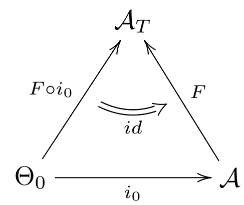
where the functor G is deduced from the definition of $\mathcal{M}$ as the category of presheaves ψ whose restriction along F is representable by an object defining $G\psi$ of the category $\mathcal{A}$. The first step of the proof has established that this natural transformation is reversible, because the functor i_0 is dense. This concludes the proof of Proposition A.

Proof of Proposition B.

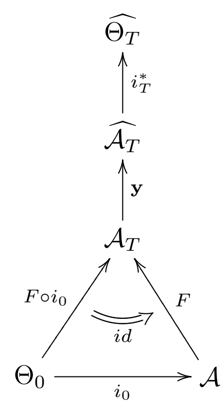
Step 1: Every monad with arity (T, i_0) induces a commutative diagram



where the category Θ_T is characterized by the fact that ℓ is an identity-on-object functor and i_T is a fully faithful functor. We will show at the last stage of the proof (step 4.) that the identity natural transformation



exhibits the functor F as a left kan extension of the functor $F \circ i_0$ along the functor i_0. To that purpose, we establish below that the identity natural transformation



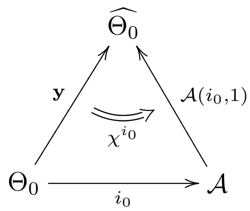
(17)

exhibits the functor

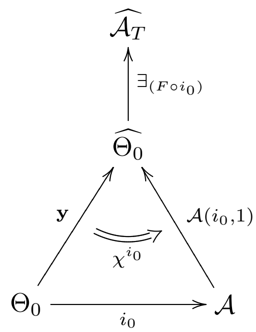
$$\mathcal{A}_T(i_T, 1) \circ F \quad = \quad i_T^* \circ \mathbf{y} \circ F$$

as a left kan extension of the functor $\mathcal{A}_T(i_T, 1) \circ F \circ i_0$ along the functor i_0. The proof of that fact is not particularly difficult, but it is pretty long if one wants to proceed carefully. On the other hand, we will see that the proof is nearly finished when the property is established.

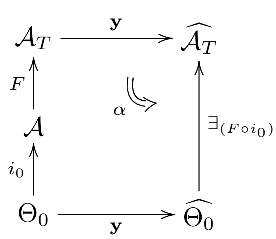
Now, one basic property of a yoneda structure is that the natural transformation χ^{i_0}



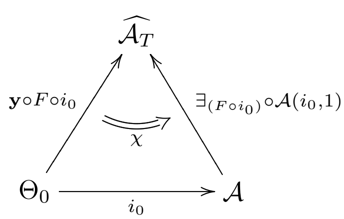
defines the functor $\mathcal{A}(i_0, 1)$ as a left kan extension of the yoneda embedding $\mathbf{y}$ along the functor i_0. The domain Θ_0 of the functor $F \circ i_0$ is small, this ensuring that the induced functor $(F \circ i_0)^*$ has a left adjoint functor $\exists_{(F \circ i_0)}$. Since left adjoint functors preserve kan extensions, the natural transformation



exhibits the functor $\exists_{(F \circ i_0)} \circ \mathcal{A}(i_0, 1)$ as a left kan extension of the functor $\exists_{(F \circ i_0)} \circ \mathbf{y}$ along the functor i_0. Now, the functor $\exists_{(F \circ i_0)}$ itself is defined in a yoneda structure as a particular left kan extension: namely, there exists a natural transformation α



which exhibits the functor $\exists_{(F \circ i_0)}$ as a left kan extension of the functor $\mathbf{y} \circ F \circ i_0$ along the functor i_0. Moreover, the transformation α is reversible because the yoneda embedding $\mathbf{y}$ on the category Θ_0 is fully faithful. Composing χ^{i_0} together with α, one obtains a natural transformation χ



which exhibits the functor

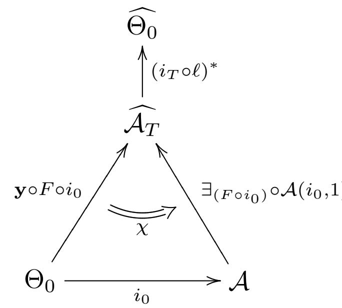
$$\exists_{(F \circ i_0)} \circ \mathcal{A}(i_0, 1)$$

as a left kan extension of the functor $\mathbf{y} \circ F \circ i_0$ along the functor i_0. The universality property of the left kan extension implies the existence of a natural transformation

$$\beta \quad : \quad \exists_{(F \circ i_0)} \circ \mathcal{A}(i_0, 1) \quad \Rightarrow \quad \mathbf{y} \circ F$$

such that β composed with χ is equal to the identity natural transformation on the functor $\mathbf{y} \circ F \circ i_0$.

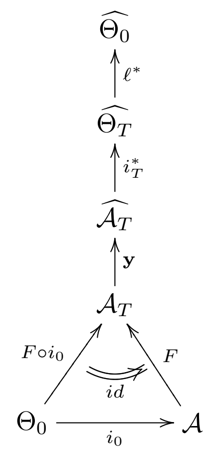
We want to show that the natural transformation β composed with i_T^* is reversible, and thus that the natural transformation (17) exhibits the functor $i_T^* \circ \mathbf{y} \circ F$ as a left kan extension. Because the left adjoint functors $(i_T \circ \ell)^*$ preserve kan extensions, the natural transformation



exhibits the functor

$$(i_T \circ \ell)^* \circ \exists_{(F \circ i_0)} \circ \mathcal{A}(i_0, 1)$$

as a left kan extension along the functor i_0. Now, observe that the identity natural transformation



is nothing but the left kan extension appearing in the definition of the monad with arity (T, i_0). Observe indeed that the functor

$$\mathcal{A}_T \xrightarrow{\ \mathbf{y}\ } \widehat{\mathcal{A}_T} \xrightarrow{\ i_T^*\ } \widehat{\Theta_T} \xrightarrow{\ \ell^*\ } \widehat{\Theta_0}$$

coincides with

$$\mathcal{A}_T \xrightarrow{\ \mathbf{y}\ } \widehat{\mathcal{A}_T} \xrightarrow{\ F^*\ } \widehat{\mathcal{A}} \xrightarrow{\ i_0^*\ } \widehat{\Theta_0}$$

and thus with

$$\mathcal{A}_T \xrightarrow{\ U\ } \mathcal{A} \xrightarrow{\ \mathbf{y}\ } \widehat{\mathcal{A}} \xrightarrow{\ i_0^*\ } \widehat{\Theta_0}$$

where U is the “forgetful functor” which transports every object A of the kleisli category $\mathcal{A}_T$ to the object TA of the category $\mathcal{A}$.

This establishes that the natural transformation β composed with the functor i_T^* followed by the functor ℓ^* is reversible. Now, the functor ℓ^* reflects isomorphisms because the functor ℓ is one-to-one on objects. We conclude that the natural transformation $i_T^* \circ \beta$

$$i_T^* \circ \exists_{(F \circ i_0)} \circ \mathcal{A}(i_0, 1) \;\Rightarrow\; \mathcal{A}_T(i_T, 1) \circ F \qquad (18)$$

is reversible, this establishing that our previous diagram (17) exhibits the functor $\mathcal{A}_T(i_T, 1) \circ F$ as a left kan extension along the functor i_0.

Step 2: The equality

$$i_T \circ \ell \quad = \quad F \circ i_0$$

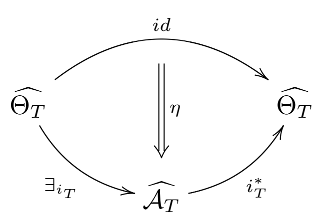
induces an isomorphism

$$\exists_{i_T} \circ \exists_\ell \quad = \quad \exists_F \circ \exists_{i_0}.$$

which may be then composed with (18). This induces a natural isomorphism

$$i_T^* \circ \exists_{i_T} \circ \exists_\ell \circ \mathcal{A}(i_0, 1) \;\Rightarrow\; \mathcal{A}_T(i_T, 1) \circ F.$$

Another way to express that the functor i_T is fully faithful is to say that the unit

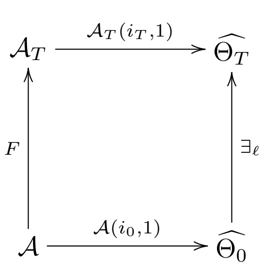


(19)

of the adjunction $\exists_{i_T} \dashv i_T^*$ is reversible. Composing the two natural transformations induces a reversible natural transformation

$$\exists_\ell \circ \mathcal{A}(i_0, 1) \;\Rightarrow\; \mathcal{A}_T(i_T, 1) \circ F$$

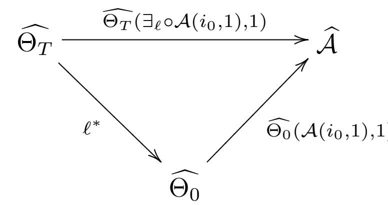
Step 3: We have just established that the diagram

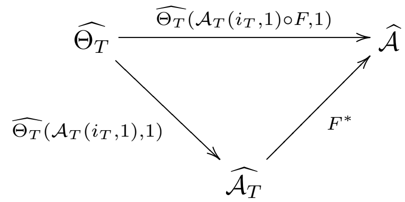


commutes up to a natural isomorphism. From this we deduce a natural isomorphism

$$\widehat{\Theta_T}(\exists_\ell \circ \mathcal{A}(i_0, 1), 1) \quad \cong \quad \widehat{\Theta_T}(\mathcal{A}_T(i_T, 1) \circ F, 1)$$

Now, the two diagrams

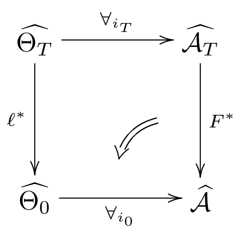




commute up to isomorphism. We then apply the two equalities

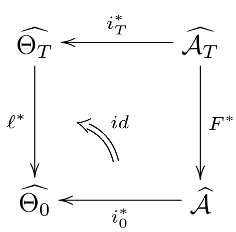
$$\forall_{i_0} \quad = \quad \widehat{\Theta_0}(\mathcal{A}(i_0, 1), 1)$$
$$\forall_{i_T} \quad = \quad \widehat{\Theta_T}(\mathcal{A}_T(i_T, 1), 1)$$

to deduce a natural isomorphism



(20)

A careful check establishes then that the natural transformation (20) just constructed coincides with the mate of the identity natural transformation

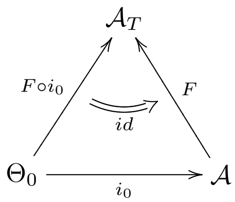


This establishes the Beck-Chevalley condition required by the definition of morphism between categories with arity.

Step 4: At this stage, there only remains to show that i_T is dense in order to establish the proposition. One follows essentially the same argument as in the proof of Proposition A in order to establish that fact. The main point to observe is that the restriction along F of a presheaf representable by an object A of the category $\mathcal{A}_T$ is representable by the object TA of the category $\mathcal{A}$. This concludes the proof of Proposition B. Let us add one more fact however: density of i_T means that the functor

$$\mathcal{A}_T(i_T, 1) \quad : \quad \mathcal{A}_T \quad \longrightarrow \quad \widehat{\Theta_T}$$

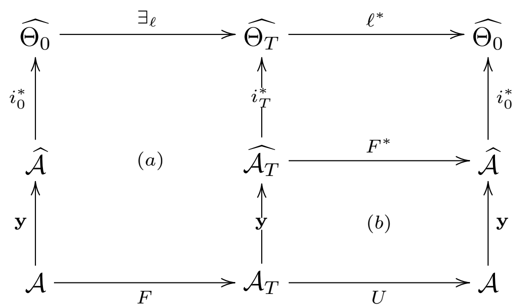
is fully faithful. This implies our initial claim that the identity natural transformation



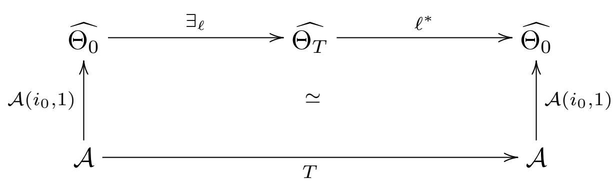
exhibits the functor F as a left kan extension of the functor $T \circ i_0$ along the functor i_0.

APPENDIX III : PROOFS OF PROPOSITION C AND D.

Proof of Proposition C. We establish here that every monad T with arity i_0 induces a Lawvere theory $\ell : \Theta_0 \to \Theta_T$ with the same arity i_0.



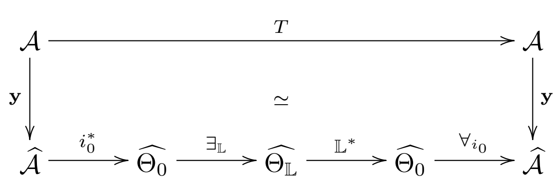
Note that the existence of a reversible natural transformation (a) has been established above, while the reversible transformation (b) is deduced from the adjunction $F \dashv U$. Note in particular that the diagram establishes the existence of a reversible natural transformation



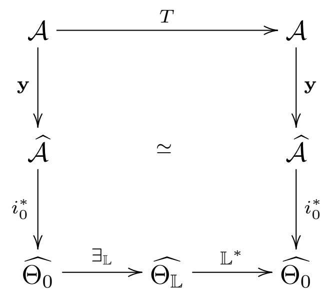
Proof of Proposition D. We establish below that every Lawvere theory

$$\mathbb{L} \quad : \quad \Theta_0 \quad \longrightarrow \quad \Theta_{\mathbb{L}}$$

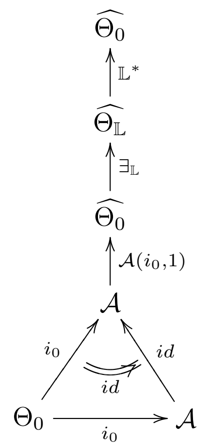
with arity i_0 induces a monad T with the same arity i_0. The main difficulty is to establish that the monad T induced by the adjunction $Free \dashv U$ has arity i_0. By definition, there exists a monad morphism



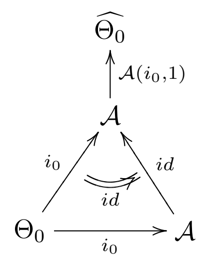
By definition, the functor i_0 is fully faithful. From this follows that the diagram



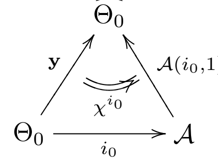
commutes up to reversible natural transformation. The functors $\mathbb{L}^*$ and $\exists_{\mathbb{L}}$ are left adjoint functors. Hence, in order to establish that the identity natural transformation



exhibits the functor $\mathbb{L}^* \circ \exists_{\mathbb{L}} \circ \mathcal{A}(i_0, 1)$ as a left kan extension of $\mathbb{L}^* \circ \exists_{\mathbb{L}} \circ \mathcal{A}(i_0, 1) \circ i_0$ along the functor i_0, it is sufficient to establish that the identity natural transformation

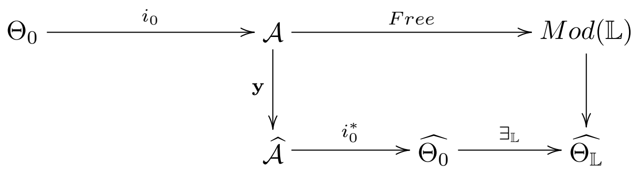


exhibits the functor $\mathcal{A}(i_0, 1)$ as left kan extension of the functor $i_0 \circ \mathcal{A}(i_0, 1)$ along the functor i_0. This follows from the fact that i_0 is fully faithful, this implying that the canonical natural transformation

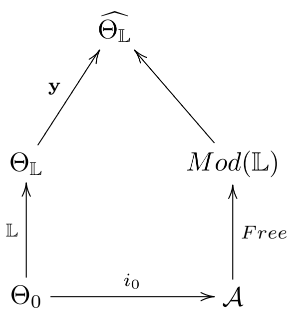


is reversible. The fact that the previous diagram describes a left kan extension follows quite immediately. This establishes that the functor T induced by the Lawvere arity $\mathbb{L}$ has the same arity i_0.

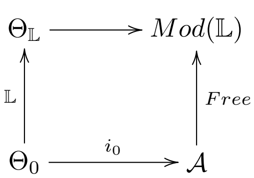
There remains to show that the Lawvere theory induced by the monad T coincides with $\mathbb{L}$. We will show that $\Theta_{\mathbb{L}}$ is the full subcategory of $Mod(\mathbb{L})$ given by the objects of Θ_0. First of all,



We have already shown that $i_0^* \circ \mathbf{y} \circ i_0$ is isomorphic to $\mathbf{y}$ because the functor i_0 is fully faithful. This implies that the functor above is equal to $\mathbf{y} \circ \mathbb{L}$.



commutes up to natural isomorphism. From this, we conclude that there exists a fully faithful functor $\Theta_{\mathbb{L}} \to Mod(\mathbb{L})$ such that



commutes strictly. This completes the proof of Proposition D.

APPENDIX III : DISCUSSION ON ALGEBRAIC THEORIES.

When the category $\mathcal{A}$ is cocomplete, a natural question is whether the condition $(\star)$ defining our notion of algebraic theory is equivalent to requiring that the functor

$$\Theta_0 \xrightarrow{\mathbb{L}} \Theta_{\mathbb{L}} \xrightarrow{\Theta_{\mathbb{L}}(\mathbb{L},1)} \widehat{\Theta_0}$$

transports every arity (that is, every object p in the category Θ_0) to a presheaf representable along i_0. It appears that this is not the case, as we show with the following example. The category *Graph* is a category of contravariant presheaves on the category Γ defined as:

$$[0] \underset{t}{\overset{s}{\rightrightarrows}} [1]$$

Now the category Γ_{+-} is defined as the category

$$[0] \underset{t}{\overset{s}{\rightrightarrows}} [1] \circlearrowright inv$$

together with the equations:

$$inv \circ inv = id \qquad inv \circ s = t \qquad inv \circ t$$

The identity-on-object functor

$$F \quad : \quad \Gamma \quad \longrightarrow \quad \Gamma_{+-}.$$

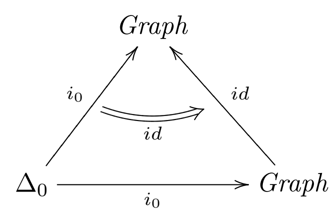
induces a monad T on the category of graphs, defined as

$$Graph \xrightarrow{\exists_F} Graph_{+-} \xrightarrow{F^*} Graph$$

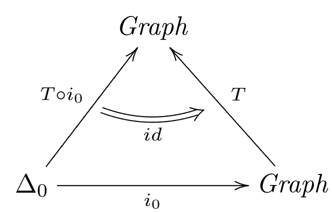
where $Graph_{+-}$ denotes the category of contravariant presheaves on the category Γ_{+-}. The monad T transports every graph G to the graph TG with the same vertices, and a pair of edges

$$u^+ \quad : \quad x \to y \qquad\qquad u^- \quad : \quad y \to x$$

for every edge $u : x \to y$ in the original graph G. By density, the identity cell



exhibits the identity functor as a left Kan extension of the functor i_0 along itself. Now, the monad T is left adjoint to the functor $F^* \circ \forall_F$, and thus preserves this left Kan extension. From this follows that the identity cell



(21)

exhibits the monad T as a left Kan extension of the functor $T \circ i_0$ along the functor i_0. An important observation is that this left Kan extension is *not* preserved by the nerve functor

$$Graph(i_0, 1) \quad : \quad Graph \quad \longrightarrow \quad \widehat{\Delta_0}$$

This point is established as follows. Consider the graph

$$G \quad = \quad A \underset{v}{\overset{u}{\rightrightarrows}} B$$

and the path

$$A \xrightarrow{u^+} B \xrightarrow{v^-} A$$

in the graph TG. This path is represented by a morphism

$$[2] \longrightarrow TG.$$

It is not difficult to see that this morphism does not factor in the following way:

$$[2] \xrightarrow{\;e\;} T[p] \xrightarrow{\;Th\;} TG$$

since $p = 0$ or $p = 1$ in all the morphisms

$$[p] \xrightarrow{\;h\;} G\ .$$

This establishes that the left Kan extension (21) is not preserved by the nerve functor, and thus, that the monad T is *not* a monad with arities the subcategory Δ_0 of finite filiform graphs in the category *Graph*.

Now, let Δ_T denote the full subcategory of filiform graphs in the Kleisli category induces by the monad T. Hence, the category Δ_T has the same objects as the category Δ_0 and its morphisms are defined as:

$$\Delta_T([p],[q]) \quad = \quad Graph(i_0[p], Ti_0[q])$$

The identity-on-object functor

$$\mathbb{L} \quad : \quad \Delta_0 \longrightarrow \Delta_T$$

induces a functor

$$\Delta_0 \xrightarrow{\;\mathbb{L}\;} \Delta_T \xrightarrow{\;\Delta_T(\mathbb{L},1)\;} \widehat{\Delta_0} \tag{22}$$

which transports every object $[q]$ of Δ_0 to the presheaf:

$$[p] \quad \mapsto \quad Graph(i_0[p], Ti_0[q])$$

Note that this presheaf is the restriction along the functor i_0 of a presheaf on the category *Graph* represented by the object $Ti_0[q]$. As such, and by definition, the presheaf is representable along the functor i_0.

At this point, we are ready to explain why the functor $\mathbb{L}$ does not define an algebraic theory with arities i_0 in our sense, formulated in Section V. Recall that the identity cell

$$\begin{array}{ccc} & \widehat{\Delta_0} & \\ {}^{\mathbb{L}}\nearrow & \Downarrow id & \nwarrow^{\mathbb{L}^* \circ \exists_{\mathbb{L}}} \\ \Delta_0 & \xrightarrow[\mathbf{y}]{} & \widehat{\Delta_0} \end{array}$$

exhibits the functor

$$\widehat{\Delta_0} \xrightarrow{\;\exists_{\mathbb{L}}\;} \widehat{\Delta_T} \xrightarrow{\;\mathbb{L}^*\;} \widehat{\Delta_0}$$

as a left Kan extension of $\mathbb{L}$ along the Yoneda embedding. This left Kan extension is preserved by the left adjoint **colim** to the nerve functor. By construction, the theory $\mathbb{L}$ factors as

$$\Delta_0 \xrightarrow{\;i_0\;} Graph \xrightarrow{\;T\;} Graph \xrightarrow{\;\Delta_0(i_0,1)\;} \widehat{\Delta_0}$$

up to natural isomorphism, and that

$$Graph \xrightarrow{\;\Delta_0(i_0,1)\;} \widehat{\Delta_0} \xrightarrow{\;\mathbf{colim}\;} Graph$$

is naturally isomorphic to the identity. From this follows that there exists a reversible cell

$$\begin{array}{ccc} & Graph & \\ {}^{T\circ i_0}\nearrow & \Rightarrow & \nwarrow^{\mathbf{colim}\circ\mathbb{L}^* \circ \exists_{\mathbb{L}}} \\ \Delta_0 & \xrightarrow[\mathbf{y}]{} & \widehat{\Delta_0} \end{array}$$

which exhibits the functor

$$\widehat{\Delta_0} \xrightarrow{\;\exists_{\mathbb{L}}\;} \widehat{\Delta_T} \xrightarrow{\;\mathbb{L}^*\;} \widehat{\Delta_0} \xrightarrow{\;\mathbf{colim}\;} Graph$$

as a left Kan extension of $T \circ i_0$ along the Yoneda embedding. This implies that the diagram commutes

$$\begin{array}{ccccc} \widehat{\Delta_0} & \xrightarrow{\exists_{\mathbb{L}}} & \widehat{\Delta_T} & \xrightarrow{\mathbb{L}^*} & \widehat{\Delta_0} \\ \uparrow{\scriptstyle \Delta_0(i_0,1)} & & & & \downarrow{\scriptstyle \mathbf{colim}} \\ Graph & & \xrightarrow{\quad T \quad} & & Graph \end{array}$$

up to natural isomorphism, because the functor T is a left Kan extension of $T \circ i_0$ along the functor i_0, and moreover, the Yoneda embedding factors as

$$\Delta_0 \xrightarrow{\;i_0\;} Graph \xrightarrow{\;\Delta_0(i_0,1)\;} \widehat{\Delta_0}$$

up to natural isomorphism. Now, suppose that the functor $\mathbb{L}$ is an algebraic theory in the sense of Section V, and thus, that the functor

$$\widehat{\Delta_0} \xrightarrow{\;\exists_{\mathbb{L}}\;} \widehat{\Delta_T} \xrightarrow{\;\mathbb{L}^*\;} \widehat{\Delta_0}$$

transports every presheaf representable along i_0 to a presheaf representable along i_0. This means that there exists a functor F such that

$$\begin{array}{ccccc} \widehat{\Delta_0} & \xrightarrow{\exists_{\mathbb{L}}} & \widehat{\Delta_T} & \xrightarrow{\mathbb{L}^*} & \widehat{\Delta_0} \\ \uparrow{\scriptstyle \Delta_0(i_0,1)} & & & & \uparrow{\scriptstyle \Delta_0(i_0,1)} \\ Graph & & \xrightarrow{\quad F \quad} & & Graph \end{array}$$

commutes up to natural isomorphism... By postcomposing the diagram with the functor **colim**, one obtains that the functor F is equal to the functor T, up to natural isomorphism. Hence, the diagram

$$\begin{array}{ccccc} \widehat{\Delta_0} & \xrightarrow{\exists_{\mathbb{L}}} & \widehat{\Delta_T} & \xrightarrow{\mathbb{L}^*} & \widehat{\Delta_0} \\ \uparrow{\scriptstyle \Delta_0(i_0,1)} & & & & \uparrow{\scriptstyle \Delta_0(i_0,1)} \\ Graph & & \xrightarrow{\quad T \quad} & & Graph \end{array}$$

commutes up to natural isomorphism... this meaning that the left Kan extension (21) is preserved by the nerve functor. This contradicts our preliminary observation, and we conclude that $\mathbb{L}$ is not an algebraic theory with arities provided by the category Δ_0 of finite filiform graphs in *Graph*.

APPENDIX $\mathbb{IV}$: ALGEBRAIC THEORIES (CONTINUED).

We have seen that every algebraic theory with arities i_0

$$\mathbb{L} \quad : \quad \Theta_0 \quad \longrightarrow \quad \Theta_{\mathbb{L}}$$

induces a monad T with the same arities, which makes the diagram

$$\begin{array}{ccccc} \widehat{\Theta_0} & \xrightarrow{\exists_{\mathbb{L}}} & \widehat{\Theta_{\mathbb{L}}} & \xrightarrow{\mathbb{L}^*} & \widehat{\Theta_0} \\ \uparrow{\scriptstyle \mathcal{A}(i_0,1)} & & \simeq & & \uparrow{\scriptstyle \mathcal{A}(i_0,1)} \\ \mathcal{A} & & \xrightarrow{\quad T \quad} & & \mathcal{A} \end{array}$$

commute up to natural isomorphism. The composite

$$\Theta_0 \xrightarrow{i_0} \mathcal{A} \xrightarrow{\mathcal{A}(i_0,1)} \widehat{\Theta_0}$$

is isomorphic to Yoneda embedding because the functor i_0 is fully faithful. From this follows that the functor t defined as

$$\Theta_0 \xrightarrow{t} \mathcal{A} \quad = \quad \Theta_0 \xrightarrow{i_0} \mathcal{A} \xrightarrow{T} \mathcal{A}$$

makes the diagram

$$\begin{array}{ccccc} \widehat{\Theta_0} & \xrightarrow{\exists_{\mathbb{L}}} & \widehat{\Theta_{\mathbb{L}}} & \xrightarrow{\mathbb{L}^*} & \widehat{\Theta_0} \\ \uparrow{\scriptstyle \mathbf{y}} & & \simeq & & \uparrow{\scriptstyle \mathcal{A}(i_0,1)} \\ \Theta_0 & & \xrightarrow{\quad t \quad} & & \mathcal{A} \end{array}$$

commute, up to natural isomorphism. Observe moreover that the definition of a monad with arities i_0 implies that the left Kan extension of the functor t along the functor i_0 is preserved by the nerve functor $\mathcal{A}(i_0, 1)$.

We would like to characterize algebraic theories in this way when the nerve functor

$$\mathcal{A}(i_0, 1) \quad : \quad \mathcal{A} \quad \longrightarrow \quad \widehat{\Theta_0}$$

has a left adjoint, which will be denoted **colim**, with a reversible counit:

$$\varepsilon \quad : \quad Id \quad \longrightarrow \quad \mathbf{colim} \circ \mathcal{A}(i_0, 1)$$

This typically happens when the category $\mathcal{A}$ has small colimits, because the functor i_0 is dense. It is worth noticing that given an algebraic theory $\mathbb{L}$ with arities i_0, the associated monad T may be simply defined in that case as the composite functor

$$\mathcal{A} \xrightarrow{\mathcal{A}(i_0,1)} \widehat{\Theta_0} \xrightarrow{\exists_{\mathbb{L}}} \widehat{\Theta_{\mathbb{L}}} \xrightarrow{\mathbb{L}^*} \widehat{\Theta_0} \xrightarrow{\mathbf{colim}} \mathcal{A}$$

because the functor $\mathbf{colim} \circ \mathcal{A}(i_0, 1)$ is isomorphic to the identity.

So, let us suppose from now on that there exists an adjunction $\mathbf{colim} \dashv \mathcal{A}(i_0, 1)$ with a reversible counit ε, and consider a functor

$$\mathbb{L} \quad : \quad \Theta_0 \quad \longrightarrow \quad \Theta_{\mathbb{L}}$$

Suppose moreover that the induced functor

$$\widehat{\Theta_0} \xrightarrow{\exists_{\mathbb{L}}} \widehat{\Theta_{\mathbb{L}}} \xrightarrow{\mathbb{L}^*} \widehat{\Theta_0}$$

transports every representable presheaf into a presheaf representable along i_0. This last hypothesis may be alternatively formulated by requiring that there exists a functor

$$t \quad : \quad \Theta_0 \quad \longrightarrow \quad \mathcal{A}$$

making the diagram

$$\begin{array}{ccccc} \widehat{\Theta_0} & \xrightarrow{\exists_{\mathbb{L}}} & \widehat{\Theta_{\mathbb{L}}} & \xrightarrow{\mathbb{L}^*} & \widehat{\Theta_0} \\ \uparrow{\scriptstyle \mathbf{y}} & & \simeq & & \uparrow{\scriptstyle \mathcal{A}(i_0,1)} \\ \Theta_0 & & \xrightarrow{\quad t \quad} & & \mathcal{A} \end{array} \qquad (23)$$

commute up to natural isomorphism. Our hypothesis on ε implies that the functor t is isomorphic to the composite functor

$$\Theta_0 \xrightarrow{\mathbf{y}} \widehat{\Theta_0} \xrightarrow{\exists_{\mathbb{L}}} \widehat{\Theta_{\mathbb{L}}} \xrightarrow{\mathbb{L}^*} \widehat{\Theta_0} \xrightarrow{\mathbf{colim}} \mathcal{A}$$

Now, the identity 2-cell

$$\begin{array}{ccc} & \widehat{\Theta_0} & \\ {\scriptstyle \mathbf{y}} \nearrow & \overset{id}{\Rightarrow} & \nwarrow {\scriptstyle \mathcal{A}(i_0,1)} \\ \Theta_0 & \xrightarrow[i_0]{} & \mathcal{A} \end{array}$$

exhibits the nerve functor $\mathcal{A}(i_0, 1)$ as a left Kan extension of $\mathbf{y}$ along i_0, and this left Kan extension is preserved by the left adjoint functor

$$\widehat{\Theta_0} \xrightarrow{\exists_{\mathbb{L}}} \widehat{\Theta_{\mathbb{L}}} \xrightarrow{\mathbb{L}^*} \widehat{\Theta_0} \xrightarrow{\mathbf{colim}} \mathcal{A}$$

This means in particular that the functor T defined as

$$\mathcal{A} \xrightarrow{\mathcal{A}(i_0,1)} \widehat{\Theta_0} \xrightarrow{\exists_{\mathbb{L}}} \widehat{\Theta_{\mathbb{L}}} \xrightarrow{\mathbb{L}^*} \widehat{\Theta_0} \xrightarrow{\mathbf{colim}} \mathcal{A}$$

is a left Kan extension of the functor t, exhibited by the identity 2-cell

$$\begin{array}{ccc} & \mathcal{A} & \\ {}^{t}\nearrow & \overset{\Leftarrow}{id} & \nwarrow^{T} \\ \Theta_0 & \xrightarrow[i_0]{} & \mathcal{A} \end{array}$$

We claim that the functor $\mathbb{L}$ defines an algebraic theory with arities i_0 precisely when the left Kan extension is preserved by the nerve functor $\mathcal{A}(i_0, 1)$. One direction has been established by the discussion above: when the functor $\mathbb{L}$ defines an algebraic theory with arities i_0, the nerve functor $\mathcal{A}(i_0, 1)$ preserves the left Kan extension T by definition of a monad with arities. The other direction reduces to showing that the diagram

$$\begin{array}{ccccc} \widehat{\Theta_0} & \xrightarrow{\exists_{\mathbb{L}}} & \widehat{\Theta_{\mathbb{L}}} & \xrightarrow{\mathbb{L}^*} & \widehat{\Theta_0} \\ {\scriptstyle \mathcal{A}(i_0,1)}\uparrow & & \simeq & & \uparrow{\scriptstyle \mathcal{A}(i_0,1)} \\ \mathcal{A} & & \xrightarrow{\quad T \quad} & & \mathcal{A} \end{array} \qquad (24)$$

commutes up to natural isomorphism when the left Kan extension is preserved by the functor $\mathcal{A}(i_0, 1)$. So, suppose that this property holds, and that the functor $\mathcal{A}(i_0, 1) \circ T$ is indeed a left Kan extension of the functor $\mathcal{A}(i_0, 1) \circ t$ along the functor i_0. In order to establish that the diagram above commutes up to natural isomorphism, it is thus sufficient to establish that the functor

$$\mathcal{A} \xrightarrow{\mathcal{A}(i_0,1)} \widehat{\Theta_0} \xrightarrow{\exists_{\mathbb{L}}} \widehat{\Theta_{\mathbb{L}}} \xrightarrow{\mathbb{L}^*} \widehat{\Theta_0} \qquad (25)$$

is a left Kan extension of the functor $\mathcal{A}(i_0, 1) \circ t$ along the functor i_0. The property follows from three facts. First of all, we know from (23) that the functor $\mathcal{A}(i_0, 1) \circ t$ is isomorphic to the composite functor

$$\Theta_0 \xrightarrow{\mathbf{y}} \widehat{\Theta_0} \xrightarrow{\exists_{\mathbb{L}}} \widehat{\Theta_{\mathbb{L}}} \xrightarrow{\mathbb{L}^*} \widehat{\Theta_0} \qquad (26)$$

Then, the identity 2-cell

$$\begin{array}{ccc} & \widehat{\Theta_0} & \\ {}^{\mathbf{y}}\nearrow & \overset{\Leftarrow}{id} & \nwarrow^{\mathcal{A}(i_0,1)} \\ \Theta_0 & \xrightarrow[i_0]{} & \mathcal{A} \end{array}$$

exhibits the nerve functor $\mathcal{A}(i_0, 1)$ as a left Kan extension of $\mathbf{y}$ along i_0. Finally, this left Kan extension is preserved by the left adjoint functor

$$\widehat{\Theta_0} \xrightarrow{\exists_{\mathbb{L}}} \widehat{\Theta_{\mathbb{L}}} \xrightarrow{\mathbb{L}^*} \widehat{\Theta_0}$$

This establishes that the left Kan extension of the functor (26) along i_0 is equal to the functor (25). This implies the existence of a reversible 2-cell (24) since a left Kan extension is unique up to reversible 2-cell.

From this, we conclude that an identity-to-object functor

$$\mathbb{L} \quad : \quad \Theta_0 \quad \longrightarrow \quad \Theta_{\mathbb{L}}$$

defines an algebraic theory with arities i_0 precisely when (1) the induced functor

$$\widehat{\Theta_0} \xrightarrow{\exists_{\mathbb{L}}} \widehat{\Theta_{\mathbb{L}}} \xrightarrow{\mathbb{L}^*} \widehat{\Theta_0}$$

transports the representable presheaves to presheaves representable along i_0, and (2) the left Kan extension

$$\begin{array}{ccc} & \mathcal{A} & \\ & \uparrow{\scriptstyle \mathbf{colim}} & \\ & \widehat{\Theta_0} & \\ & \uparrow{\scriptstyle \mathbb{L}^*} & \\ & \widehat{\Theta_{\mathbb{L}}} & \\ & \uparrow{\scriptstyle \exists_{\mathbb{L}}} & \\ & \widehat{\Theta_0} & \\ {}^{\mathbf{y}}\nearrow & \overset{\Leftarrow}{id} & \nwarrow^{\mathcal{A}(i_0,1)} \\ \Theta_0 & \xrightarrow[i_0]{} & \mathcal{A} \end{array}$$

is preserved by the nerve functor $\mathcal{A}(i_0, 1)$.

Remark: The condition (2) may be alternatively formulated by requiring that the left Kan extension

$$\begin{array}{ccc} & \widehat{\Theta_0} & \\ & \uparrow{\scriptstyle \mathbb{L}^*} & \\ & \widehat{\Theta_{\mathbb{L}}} & \\ & \uparrow{\scriptstyle \exists_{\mathbb{L}}} & \\ & \widehat{\Theta_0} & \\ {}^{\mathbf{y}}\nearrow & \overset{\Leftarrow}{id} & \nwarrow^{\mathcal{A}(i_0,1)} \\ \Theta_0 & \xrightarrow[i_0]{} & \mathcal{A} \end{array}$$

is preserved by the monad

$$\widehat{\Theta_0} \xrightarrow{\mathbf{colim}} \mathcal{A} \xrightarrow{\mathcal{A}(i_0,1)} \widehat{\Theta_0}$$

on the presheaf category $\widehat{\Theta_0}$.

Remark.: The two conditions (1) and (2) may be unified, and stated more concisely, by simply requiring that the functor defined as

$$\mathcal{A} \xrightarrow{\mathcal{A}(i_0,1)} \widehat{\Theta_0} \xrightarrow{\exists_{\mathbb{L}}} \widehat{\Theta_{\mathbb{L}}} \xrightarrow{\mathbb{L}^*} \widehat{\Theta_0}$$

is isomorphic to the result of postcomposing itself with the monad

$$\widehat{\Theta_0} \xrightarrow{\mathbf{colim}} \mathcal{A} \xrightarrow{\mathcal{A}(i_0,1)} \widehat{\Theta_0}$$

Remark.: When $\mathcal{A}$ is a presheaf category on a full and dense subcategory Θ of the category Θ_0, the category $\mathcal{A}$ is equipped with the arity functor

$$i_0 \quad : \quad \Theta_0 \quad \longrightarrow \quad \mathcal{A}$$

defined as the nerve

$$\Theta(i,1) \quad : \quad \Theta_0 \quad \longrightarrow \quad \widehat{\Theta}$$

induced by the fully faithful functor

$$i \quad : \quad \Theta \quad \longrightarrow \quad \Theta_0$$

embedding the full subcategory Θ inside Θ_0. In that case, the nerve functor

$$\mathcal{A}(i_0,1) \quad : \quad \mathcal{A} \quad \longrightarrow \quad \widehat{\Theta_0}$$

coincides with the functor

$$\forall_i \quad : \quad \widehat{\Theta} \quad \longrightarrow \quad \widehat{\Theta_0}.$$

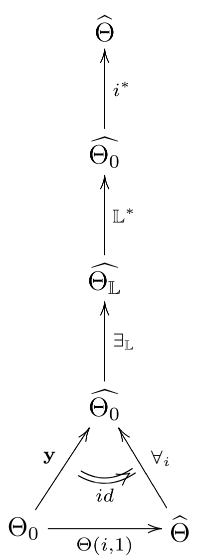
Hence, an identity-on-object functor

$$\mathbb{L} \quad : \quad \Theta_0 \quad \longrightarrow \quad \Theta_{\mathbb{L}}$$

is an algebraic theory with arities i_0 precisely when (1) the induced monad

$$\widehat{\Theta_0} \xrightarrow{\exists_{\mathbb{L}}} \widehat{\Theta_{\mathbb{L}}} \xrightarrow{\mathbb{L}^*} \widehat{\Theta_0}$$

transports representable presheaves into presheaves representable along i_0, and (2) the left Kan extension



is preserved by the functor

$$\forall_i \quad : \quad \widehat{\Theta} \quad \longrightarrow \quad \widehat{\Theta_0}$$