Refinement systems and Martin-Löf type theory

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Types, Homotopy Type Theory, Verification

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What is a functor?

$$\begin{array}{c} \mathscr{E} \\ \Big\downarrow{\scriptstyle p} \\ \mathscr{B} \end{array}$$

Two alternative and complementary ways to look at it:

- ▷ a category $\mathscr{E}$ with infrastructure $\mathscr{B}$ [shadows]
- ▷ a category $\mathscr{B}$ with superstructure $\mathscr{E}$ [refinements]

What is a functor?

$$\begin{array}{c} \mathscr{E} \\ \big\downarrow{\scriptstyle p} \\ \mathscr{B} \end{array}$$

We write $R \sqsubset A$ and declare that

$$R \text{ is a refinement of } A$$

when $p(R) = A$.

What is a functor?

$$\begin{array}{c} \mathscr{E} \\ \Big\downarrow{\scriptstyle p} \\ \mathscr{B} \end{array}$$

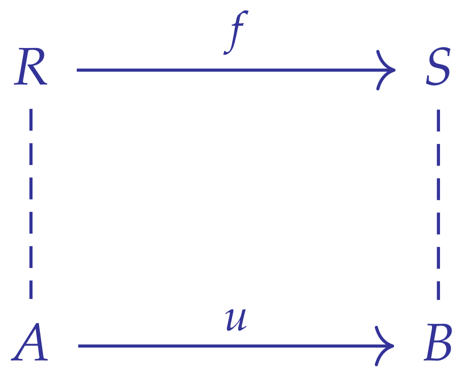
We write $f \sqsubset u$ and declare that

$$f : R \to S \quad \text{is a refinement of} \quad u : A \to B$$

when $p(f) = u$.

Note that $R \sqsubset A$ and $S \sqsubset B$ in that case.

What is a functor?



We write $f \sqsubset u$, and declare that

$$f : R \to S \quad \text{is a refinement of} \quad u : A \to B$$

when $p(f) = u$.

Note that $R \sqsubset A$ and $S \sqsubset B$ in that case.

Illustration

There is a functor

$$\Delta \quad : \quad \textbf{MultiCat} \quad \longrightarrow \quad \textbf{MonCat}$$

which transports every multicategory to a monoidal category.

Here, **MonCat** denotes the category of

monoidal categories and strict monoidal functors between them

Illustration

As a matter of fact, every multicategory $\mathscr{M}$ comes with a map

$$\mathscr{M} \longrightarrow 1$$

where 1 denotes the terminal multicategory.

This map of multicategory is transported to a strict monoidal functor ev and thus Δ defines a functor

$$\Delta \quad : \quad \textbf{MultiCat} \quad \longrightarrow \quad \textbf{MonCat}/\Delta$$

where Δ denotes the category of finite ordinals and monotone maps.

Illustration

This functor

$$\Delta\mathscr{M} \longrightarrow \Delta 1$$

retains the structure of the original multicategory

Indeed, a map in the multicategory $\mathscr{M}$

$$A_1, \cdots, A_n \longrightarrow B$$

is the same thing as a map in the category $\Delta\mathscr{M}$ refining the map

$$\bullet \quad \cdots \quad \bullet \longrightarrow \bullet$$

from $[n]$ to $[1]$ in the category Δ.

Polarities in linear logic

The left introduction of the sum

$$\frac{A \vdash C \qquad\qquad B \vdash C}{A \oplus B \vdash C}$$

is **invertible** in the sense that every proof

$$\frac{\begin{array}{c}\pi \\ \vdots\end{array}}{A \oplus B \vdash C}$$

factors uniquely as

$$\frac{\dfrac{\begin{array}{c}\pi_1 \\ \vdots\end{array}}{A \vdash C} \quad \dfrac{\begin{array}{c}\pi_2 \\ \vdots\end{array}}{B \vdash C}}{A \oplus B \vdash C}$$

up to proof equality.

Polarities in linear logic

This is nicely reflected in the categorical semantics by the fact that

$$A \oplus B$$

denotes the categorical sum of A and B.

Polarities in linear logic

Similarly, the left introduction of the tensor product

$$\frac{A, B \vdash C}{A \otimes B \vdash C}$$

is **invertible** in the sense that every proof

$$\frac{\begin{array}{c}\pi \\ \vdots\end{array}}{A \otimes B \vdash C}$$

factors uniquely as

$$\frac{\dfrac{\begin{array}{c}\pi \\ \vdots\end{array}}{A, B \vdash C}}{A \otimes B \vdash C}$$

up to proof equality.

Polarities in linear logic

A longstanding question... for me at least :-)

What is the **categorical** counterpart of this basic phenomenon?

One origin of the notion of **refinement system** with Noam...

Hermida's universal point of view

A monoidal category is the same thing as a multicategory where maps

$$f \quad : \quad A, B \quad \longrightarrow \quad C$$

are **representable** in the sense that there exists a map

$$A, B \quad \longrightarrow \quad A \otimes B$$

for every A, B such that every bilinear map f factors uniquely as

$$A, B \longrightarrow A \otimes B \longrightarrow C$$

One also needs to ask that representability is **contextual**.

Reformulated in the functorial language

A monoidal category is the same thing as a multicategory

$$\mathscr{M}$$

whose associated strict monoidal functor

$$\Delta\mathscr{M} \longrightarrow \Delta$$

is a left Grothendieck fibration such that the tensor product

$$f_1 \otimes f_2 \quad : \quad A_1 \otimes A_2 \quad \longrightarrow \quad B_1 \otimes B_2$$

of two left cartesian maps

$$f_1 \; : \; A_1 \; \longrightarrow \; B_1 \qquad\qquad f_2 \; : \; A_2 \; \longrightarrow \; B_2$$

is left cartesian.

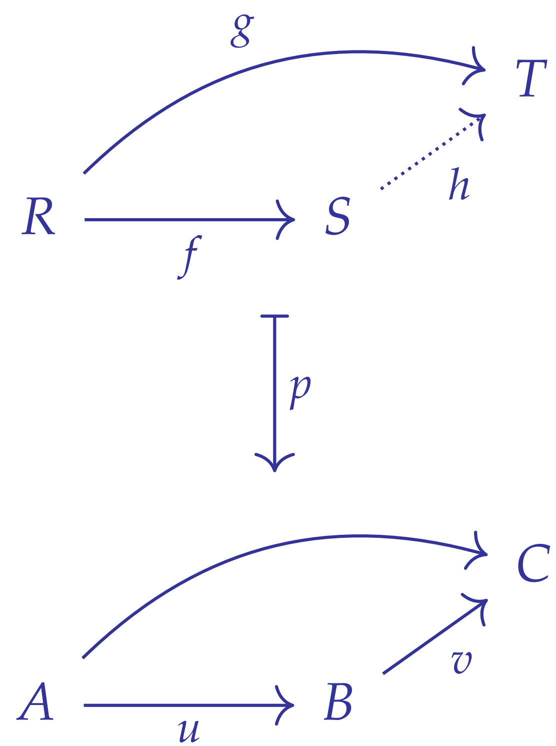
Left cartesian maps

A map $f : R \to S$ in $\mathscr{E}$ is left cartesian above $u : A \to B$ in $\mathscr{B}$ when the following property holds:

for every map $g : R \to T$

for every map $v : B \to C$
such that $g \sqsubset v \circ u$

there exists
a unique map $h : S \to T$
such that $h \circ f = g$
and $h \sqsubset v$.



Left Grothendieck fibrations

Definition. A functor

$$p \quad : \quad \mathscr{E} \longrightarrow \mathscr{B}$$

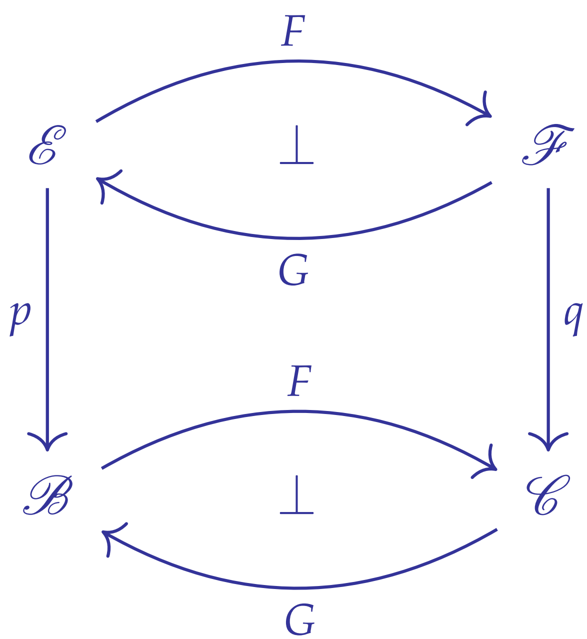
is a left Grothendieck fibration when there exists a left cartesian map

$$\begin{array}{ccc} R & \xrightarrow{\quad f \quad} & push_u(R) \\ & \Big\downarrow{\scriptstyle p} & \\ A & \xrightarrow[\quad u \quad]{} & B \end{array}$$

for every refinement $R \sqsubset A$ and every map $u : A \to B$.

Left cartesian maps are preserved by left adjoints

Suppose given two refinements systems related by an adjunction



Key fact: Left cartesian maps are preserved by left adjoints...

The Grothendieck construction

From indexed categories to right fibrations

Indexed categories

We suppose given a category $\mathscr{B}$.

Definition.

An indexed $\mathscr{B}$-category is a pseudo-functor

$$\mathscr{P} \quad : \quad \mathscr{B}^{op} \longrightarrow \textbf{Cat}$$

which associates

- ▷ a category $\mathscr{P}_B$
- ▷ a functor $\mathscr{P}_u : \mathscr{P}_B \longrightarrow \mathscr{P}_A$

to every object B and to every map $u : A \longrightarrow B$ of the category $\mathscr{B}$.

The Grothendieck construction

The total category $\mathscr{E}$ is constructed as follows:

- ▷ its objects are the pairs

$$(A, R)$$

consisting of an object $A \in \mathscr{B}$ and of an object $R \in \mathscr{P}(A)$,

- ▷ its maps

$$f \quad : \quad (A, R) \longrightarrow (B, S)$$

are the pairs $f = (u, h)$ consisting of two maps

$$u \,:\, A \longrightarrow B \qquad \text{in the category} \quad \mathscr{B}$$
$$h \,:\, R \longrightarrow \mathscr{P}_u(S) \qquad \text{in the category} \quad \mathscr{P}(A)$$

The Grothendieck construction

Key observation:

The total category $\mathscr{E}$ comes equipped with a functor

$$p \quad : \quad \mathscr{E} \longrightarrow \mathscr{B}$$

which transports every object $(A, R) \in \mathscr{E}$ to the object $A \in \mathscr{B}$.

Moreover, the fiber

$$\mathscr{E}_A \quad = \quad p^{-1}(A)$$

above the object $A \in \mathscr{B}$ coincides with the category $\mathscr{P}_A$.

The Grothendieck construction

Fundamental and well-known fact.

The Grothendieck construction “glues” together the categories $\mathscr{P}_A$’s into a right fibration

$$p \quad : \quad \mathscr{E} \longrightarrow \mathscr{B}$$

The construction induces an equivalence of categories

the category of indexed $\mathscr{B}$-categories	$\cong$	**the category of right fibrations over $\mathscr{B}$**

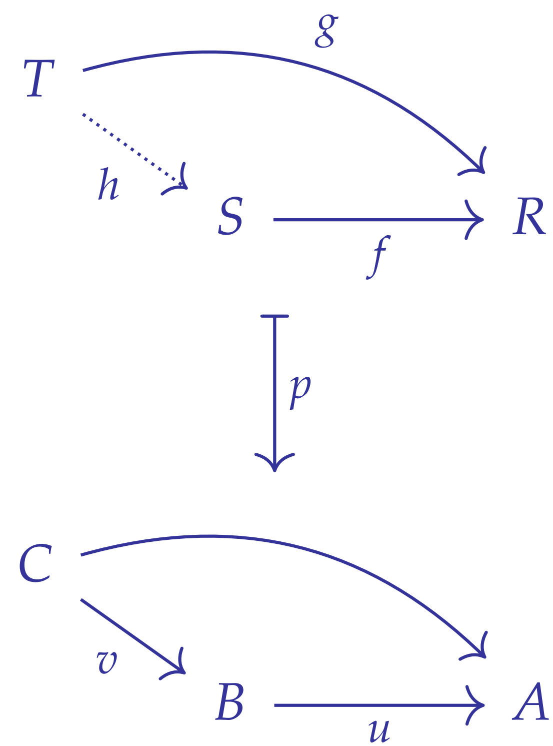
Right cartesian maps

A map $f : S \to R$ in $\mathscr{E}$ is right cartesian above $u : B \to A$ in $\mathscr{B}$ when the following property holds:

for every map $g : T \to R$

for every map $v : C \to B$
such that $g \sqsubset u \circ v$

there exists
a unique map $h : T \to S$
such that $f \circ h = g$
and $h \sqsubset v$.



Right Grothendieck fibrations

Definition. A functor

$$p \quad : \quad \mathscr{E} \longrightarrow \mathscr{B}$$

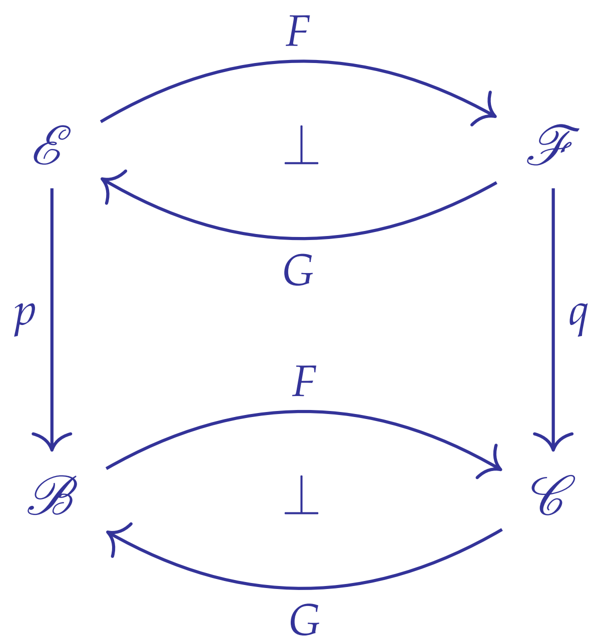
is a right Grothendieck fibration when there exists a right cartesian map

$$\begin{array}{ccc} pull_u(R) & \xrightarrow{\quad f \quad} & R \\ & \Big\downarrow{\scriptstyle p} & \\ B & \xrightarrow[\quad u \quad]{} & A \end{array}$$

for every refinement $R \sqsubset A$ and every map $u : A \to B$.

Right cartesian maps are preserved by right adjoints

Suppose given two refinements systems related by an adjunction



Dual fact: Right cartesian maps are preserved by right adjoints...

A Grothendieck construction

for general functors

An idea by Jean Bénabou

Displayed categories

We suppose given a category $\mathscr{B}$.

Definition.

A displayed $\mathscr{B}$-category is a normal lax functor

$$\mathscr{P} \quad : \quad \mathscr{B}^{op} \longrightarrow Dist$$

which associates

- ▷ a category $\mathscr{P}_A$
- ▷ a distributor $\mathscr{P}_u : \mathscr{P}_A^{op} \times \mathscr{P}_B \longrightarrow Set$

to every object A and to every map $u : A \longrightarrow B$ of the category $\mathscr{B}$.

The Grothendieck construction

The total category $\mathscr{E}$ is constructed as follows:

▷ its objects are the pairs

$$(A, R)$$

consisting of an object $A \in \mathscr{B}$ and of an object $R \in \mathscr{P}(A)$,

▷ its maps

$$f \quad : \quad (A, R) \longrightarrow (B, S)$$

are the pairs $f = (u, h)$ consisting of

a map	$u : A \longrightarrow B$	in the category	$\mathscr{B}$
an element	h	of the set	$\mathscr{P}_u(R, S)$

The Grothendieck construction

The fiber category $\mathscr{P}_A$ has the refinements of A as objects

$$R \sqsubset A$$

and the refinements of $id_A : A \to A$ as maps

$$h : R \to S \quad \sqsubset \quad id_A : A \to A$$

while the distributor $\mathscr{P}_u$ has its entries

$$\mathscr{P}_u(R, S)$$

defined as the sets of refinements of the map $u : A \to B$

$$h : R \to S \quad \sqsubset \quad u : A \to B$$

The Grothendieck construction

Fundamental and less-known fact.

The Grothendieck construction “glues” together
the categories $\mathscr{P}_A$’s into a functor

$$p \quad : \quad \mathscr{E} \longrightarrow \mathscr{B}$$

The construction induces an equivalence of categories

the category of displayed $\mathscr{B}$-categories	$\cong$	the category of general functors over $\mathscr{B}$

Functors as type refinement systems

An idea developed with N. Zeilberger (2013)

Refinement systems

Basic idea: every functor can be seen as a proof system.

Definition. A **typing judgement**

$$P \underset{u}{\Longrightarrow} Q$$

is a triple consisting of two refinements

$$P \sqsubset A \qquad\qquad Q \sqsubset B$$

and a map

$$A \xrightarrow{\;u\;} B$$

in the basis category $\mathscr{B}$.

Composition rule

The composition rule

$$\frac{P \underset{u}{\Longrightarrow} Q \quad Q \underset{v}{\Longrightarrow} R}{P \underset{u;v}{\Longrightarrow} R} \;;$$

is the basic rule of the refinement system.

Illustration

For instance, the rule of ”covariant subtyping”

$$\frac{P \underset{c}{\Longrightarrow} Q \quad Q \Longrightarrow R}{P \underset{c}{\Longrightarrow} R}$$

is derivable from the derivation tree

$$\dfrac{\dfrac{P \underset{c}{\Longrightarrow} Q \quad Q \Longrightarrow R}{P \underset{c;\mathrm{id}}{\Longrightarrow} R}\;;}{P \underset{c}{\Longrightarrow} R}\;\sim$$

where we use the equality $u; id = u$ for the $\sim$-rule.

Right cartesian maps in refinement systems

One requires the existence of a refinement

$$\frac{c : A \to B \quad Q \sqsubset B}{\mathbf{pull}_c Q \sqsubset A}$$

and of the left and right introduction rules:

$$\frac{}{\mathbf{pull}_c Q \underset{c}{\Longrightarrow} Q}\ L\mathbf{pull}_c \qquad \frac{P \underset{d;c}{\Longrightarrow} Q}{P \underset{d}{\Longrightarrow} \mathbf{pull}_c Q}\ R\mathbf{pull}_c$$

Right cartesian maps in refinement systems

One requires in addition that the equations defined by the β-rule:

$$\frac{\dfrac{P \overset{\beta}{\underset{d;c}{\Longrightarrow}} Q}{P \underset{d}{\Longrightarrow} \mathbf{pull}_c Q}\ R\mathbf{pull}_c \qquad \dfrac{}{\mathbf{pull}_c Q \underset{c}{\Longrightarrow} Q}\ L\mathbf{pull}_c}{P \underset{d;c}{\Longrightarrow} Q}\ ; \qquad = \quad P \overset{\beta}{\underset{d;c}{\Longrightarrow}} Q$$

and by the η-rule are satisfied:

$$P \overset{\eta}{\underset{d}{\Longrightarrow}} \mathbf{pull}_c Q \quad = \quad \dfrac{\dfrac{P \overset{\eta}{\underset{d}{\Longrightarrow}} \mathbf{pull}_c Q \qquad \dfrac{}{\mathbf{pull}_c Q \underset{c}{\Longrightarrow} Q}\ L\mathbf{pull}_c}{P \underset{d;c}{\Longrightarrow} Q}\ ;}{P \underset{d}{\Longrightarrow} \mathbf{pull}_c Q}\ R\mathbf{pull}_c$$

Illustration: right adjoints preserve right cartesian maps

In order to establish the property, one builds two derivations

$$\dfrac{}{G[\mathbf{pull}_c Q] \underset{G[c]}{\Longrightarrow} G[Q]}\ L\mathbf{pull}_{G[c]} \qquad \dfrac{P \underset{d;G[c]}{\Longrightarrow} G[Q]}{P \underset{d}{\Longrightarrow} G[\mathbf{pull}_c Q]}\ R\mathbf{pull}_{G[c]}$$

satisfying the β and η laws.

The construction is immediate for the left rule:

$$\dfrac{\dfrac{}{\mathbf{pull}_c Q \underset{c}{\Longrightarrow} Q}\ L\mathbf{pull}_c}{G[\mathbf{pull}_c Q] \underset{G[c]}{\Longrightarrow} G[Q]}\ G$$

Illustration: right adjoints preserve right cartesian maps

The right rule is a little bit more involved to establish:

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  \dfrac{P \underset{\iota}{\Longrightarrow} GF[P]}{}\;\iota
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{P \underset{d;G[c]}{\Longrightarrow} G[Q]}{F[P] \underset{F[d];FG[c]}{\Longrightarrow} FG[Q]}\;F
          \quad
          \dfrac{}{FG[Q] \underset{o}{\Longrightarrow} Q}\;o
        }{F[P] \underset{F[d];FG[c];o}{\Longrightarrow} Q}\;;
      }{F[P] \underset{F[d];o;c}{\Longrightarrow} Q}\;\sim_1
    }{F[P] \underset{F[d];o}{\Longrightarrow} \mathbf{pull}_c Q}\;R\mathbf{pull}_c
  }{GF[P] \underset{GF[d];G[o]}{\Longrightarrow} G[\mathbf{pull}_c Q]}\;G
}{
  \dfrac{P \underset{\iota;GF[d];G[o]}{\Longrightarrow} G[\mathbf{pull}_c Q]}{P \underset{d}{\Longrightarrow} G[\mathbf{pull}_c Q]}\;\sim_2
}\;;
$$

The bifibrational Day construction

Monoidal closed bifibrations at work

Grothendieck bifibrations

Definition. A functor

$$p \quad : \quad \mathscr{E} \longrightarrow \mathscr{B}$$

is a bifibration when it is a right fibration and a left fibration.

Remark:

A bifibration is the same thing as a fibration where every functor

$$pull_u \quad : \quad \mathscr{E}_B \longrightarrow \mathscr{E}_A$$

for $u : A \to B$ comes equipped with a left adjoint functor

$$push_u \quad : \quad \mathscr{E}_A \longrightarrow \mathscr{E}_B$$

The Day tensor product

Remarkable fact. Every presheaf category

$$[\mathscr{A}^{op}, Set]$$

associated to a monoidal category $(\mathscr{A}, \otimes, I)$ is monoidal closed.

The Day tensor product $\varphi \otimes_{Day} \psi$ is defined by the coend formula

$$a \quad \mapsto \quad \int^{a_1, a_2} \mathscr{A}(a, a_1 \otimes a_2) \times_{Set} \varphi(a_1) \times_{Set} \psi(a_2)$$

while the Day implication $\varphi \multimap_{Day} \psi$ is defined by the end formula

$$a_2 \quad \mapsto \quad \int_{a_1} \varphi(a_1) \Rightarrow_{Set} \psi(a_1 \otimes a_2)$$

The presheaf bifibration

Consider the category **Psh** whose objects are pairs

$$(\mathscr{A}, \varphi)$$

consisting of a category $\mathscr{A}$ and of a contravariant presheaf

$$\varphi \quad \in \quad [\mathscr{A}^{op}, Set]$$

and whose maps

$$(F, \theta) \quad : \quad (\mathscr{A}, \varphi) \longrightarrow (\mathscr{B}, \psi)$$

are pairs consisting of a functor

$$F \quad : \quad \mathscr{A} \longrightarrow \mathscr{B}$$

and of a natural transformation

$$\theta \quad : \quad \varphi \Longrightarrow \psi \circ F^{op}$$

The presheaf bifibration

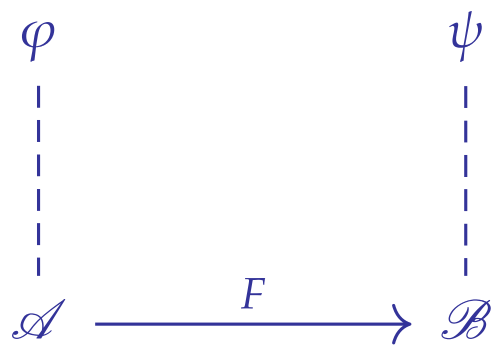
Well-known fact. The canonical functor

$$p \quad : \quad \mathbf{Psh} \longrightarrow \mathbf{Cat}$$

defines a bifibration.

The presheaf bifibration

Indeed, in the following situation



the pull and the push along the functor F are defined as

$$\begin{array}{rclcrcl} pull_F(\psi) & = & \psi \circ F^{op} & : & a & \mapsto & \psi(Fa) \\ push_F(\varphi) & = & \mathrm{Lan}_F(\varphi) & : & b & \mapsto & \int^a \mathscr{B}(b, Fa) \times_{Set} \varphi(a) \end{array}$$

A cartesian bifibration

Additional fact. The category **Psh** is cartesian and the functor

$$p \quad : \quad \mathbf{Psh} \longrightarrow \mathbf{Cat}$$

transports cartesian products of **Psh** into cartesian products of **Cat**.

Indeed, the cartesian product of two presheaves on $\mathscr{A}$ and $\mathscr{B}$

$$(\mathscr{A}, \varphi) \times (\mathscr{B}, \psi)$$

is given by the presheaf on the category $\mathscr{A} \times \mathscr{B}$

$$(\mathscr{A} \times \mathscr{B}, \varphi \times \psi) \quad : \quad (a, b) \quad \mapsto \quad \varphi(a) \times_{Set} \psi(b)$$

obtained by computing the pointwise cartesian product in Set.

A cartesian closed bifibration

Key observation [MZ-2013]

The category **Psh** is cartesian closed and the functor

$$p \quad : \quad \textbf{Psh} \longrightarrow \textbf{Cat}$$

preserves the cartesian closed structure.

Indeed, given two presheaves on categories $\mathscr{A}$ and $\mathscr{B}$

$$(\mathscr{A}, \varphi) \Rightarrow (\mathscr{B}, \psi)$$

is defined as the presheaf on the category $\mathscr{A} \Rightarrow \mathscr{B}$

$$(\mathscr{A} \Rightarrow \mathscr{B}, \varphi \Rightarrow \psi) \quad : \quad F \quad \mapsto \quad \int_{a \in \mathscr{A}} \varphi(a) \Rightarrow_{Set} \psi(Fa)$$

The bifibrational Day construction

Proposition [MZ-2013]

If $\mathscr{E} \to \mathscr{B}$ is a monoidal closed bifibration, then every monoid

$$(A, m : A \otimes A \to A, e : 1 \to A) \quad \in \quad \mathscr{B}$$

in the basis determines a monoidal closed structure on the fiber $\mathscr{E}_A$, where the tensor and implication are defined for all $R, S \sqsubset A$ by

$$\begin{array}{rcl} R \otimes_A S & = & push_m(R \otimes S) \\ R \multimap_A S & = & pull_{curry(m)}(R \multimap S) \end{array}$$

and the tensor unit is defined by $1_A = push_e 1$.

In the case of the presheaf bifibration

The Day tensor product

As it should, the push formula defines the Day tensor product:

$$push_{\otimes}(\varphi \times \psi) \;:\; a \;\mapsto \int^{(a_1,a_2)\in\mathscr{A}\times\mathscr{A}} \mathscr{A}(a, a_1 \otimes a_2) \times_{Set} \varphi(a_1) \times_{Set} \psi(a_2)$$

In the case of the presheaf bifibration

The Day implication

Here, the recipe is to apply the end formula

$$\varphi \Rightarrow \psi \quad : \quad (F : \mathscr{A} \to \mathscr{A}) \quad \mapsto \quad \int_{a \in \mathscr{A}} \varphi(a) \Rightarrow_{Set} \psi(Fa)$$

to the left currification functor

$$curry(\otimes)(a_2) \ : \ \left(a_1 \mapsto a_1 \otimes a_2\right) \quad : \quad \mathscr{A} \to \mathscr{A}$$

and then to pull back to obtain the expected presheaf

$$pull_{curry(\otimes)}(\varphi \Rightarrow \psi) \quad : \quad a_2 \mapsto \int_{a \in \mathscr{A}} \varphi(a) \Rightarrow_{Set} \psi(a \otimes a_2)$$

over the category $\mathscr{A}$

The bifibrational Day construction

Methodological point

It is possible to formulate the property in a purely proof-theoretic way.

The idea is to construct the three derivations

$$\frac{S_1 \longrightarrow T_1 \quad S_2 \longrightarrow T_2}{S_1 * S_2 \longrightarrow T_1 * T_2}\ M* \qquad \frac{S * T \longrightarrow U}{S \longrightarrow T -\!\!* U}\ R^{-*} \qquad \frac{}{(T -\!\!* U) * T \longrightarrow U}\ L^{-*}$$

and establish that they satisfy the required equations

Note: here, we simply treat the case of magmas instead of monoids

The bifibrational Day construction

The β-equation

$$\dfrac{\dfrac{\dfrac{S * T \xrightarrow{\beta} U}{S \longrightarrow T \mathrel{-\!\!*} U}\,R^{-\!*} \qquad \dfrac{}{T \longrightarrow T}\,I}{S * T \longrightarrow (T \mathrel{-\!\!*} U) * T}\,M* \qquad \dfrac{}{(T \mathrel{-\!\!*} U) * T \longrightarrow U}\,L^{-\!*}}{S * T \longrightarrow U}\,C \qquad \sim \qquad S * T \xrightarrow{\beta} U$$

and η-equation

$$S \xrightarrow{\eta} T \mathrel{-\!\!*} U \qquad \sim \qquad \dfrac{\dfrac{\dfrac{S \xrightarrow{\eta} T \mathrel{-\!\!*} U \qquad \dfrac{}{T \longrightarrow T}\,I}{S * T \longrightarrow T \mathrel{-\!\!*} U * T}\,M* \qquad \dfrac{}{(T \mathrel{-\!\!*} U) * T \longrightarrow U}\,L^{-\!*}}{S * T \longrightarrow U}\,C}{S \longrightarrow T \mathrel{-\!\!*} U}\,R^{-\!*}$$

The three derivations

The three derivations are constructed in the following way:

$$\dfrac{\dfrac{\dfrac{S_1 \longrightarrow T_1 \quad S_2 \longrightarrow T_2}{S_1 \cdot S_2 \longrightarrow T_1 \cdot T_2}\,M \qquad \dfrac{}{T_1 \cdot T_2 \underset{\circledast}{\longrightarrow} T_1 * T_2}\,R\circledast}{S_1 \cdot S_2 \underset{\circledast}{\longrightarrow} T_1 * T_2}\,C}{S_1 * S_2 \longrightarrow T_1 * T_2}\,L\circledast$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{S \cdot T \underset{\circledast}{\longrightarrow} S * T}\,R\circledast \qquad S * T \longrightarrow U}{S \cdot T \underset{\circledast}{\longrightarrow} U}\,C}{S \underset{\rho[\circledast]}{\longrightarrow} \overset{U}{\underset{T}{\circ\!\!-}}}\,R^{\circ\!-}}{S \longrightarrow T \mathbin{-\!\!*} U}\,R\rho[\circledast]^*$$

The three derivations

$$
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  \dfrac{
    \dfrac{}{T \mathrel{-\!\!*} U \xrightarrow[\rho[\circledast]]{} \overset{U}{\underset{T}{\circ\!\!-}}}\; L\rho[\circledast]^{*}
    \qquad
    \dfrac{}{T \longrightarrow T}\; I
  }{
    (T \mathrel{-\!\!*} U) \cdot T \xrightarrow[\rho[\circledast]\cdot -]{} \overset{U}{\underset{T}{\circ\!\!-}} \cdot T
  }\; M
  \qquad
  \dfrac{}{\overset{U}{\underset{T}{\circ\!\!-}} \cdot T \xrightarrow[\olessthan]{} U}\; L^{\circ\!-}
}{
  (T \mathrel{-\!\!*} U) \cdot T \xrightarrow[(\rho[\circledast]\cdot -);\olessthan]{} U
}\; C
$$

$$
\dfrac{
  \dfrac{(T \mathrel{-\!\!*} U) \cdot T \xrightarrow[(\rho[\circledast]\cdot -);\olessthan]{} U}{(T \mathrel{-\!\!*} U) \cdot T \xrightarrow[\circledast]{} U}\; \sim
}{
  (T \mathrel{-\!\!*} U) * T \longrightarrow U
}\; L\circledast
$$