Template games:

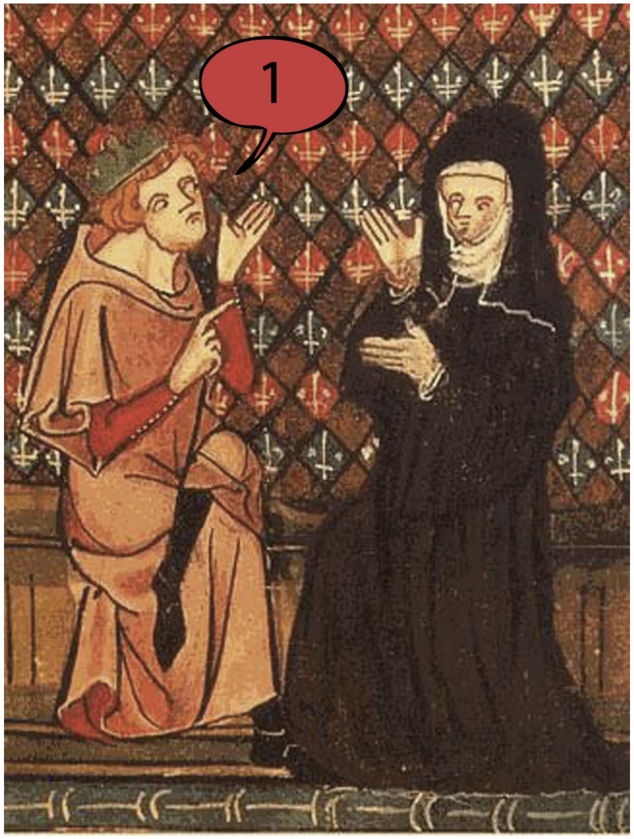
a homotopy model of differential linear logic

Paul-André Melliès

Institut de Recherche en Informatique Fondamentale (IRIF)
CNRS & Université de Paris

⫝̸ Séminaire Logique et Interactions ⫝̸
Institut de Mathématiques de Marseille ⫝̸ 16 March 2022

Understanding logic in space and time

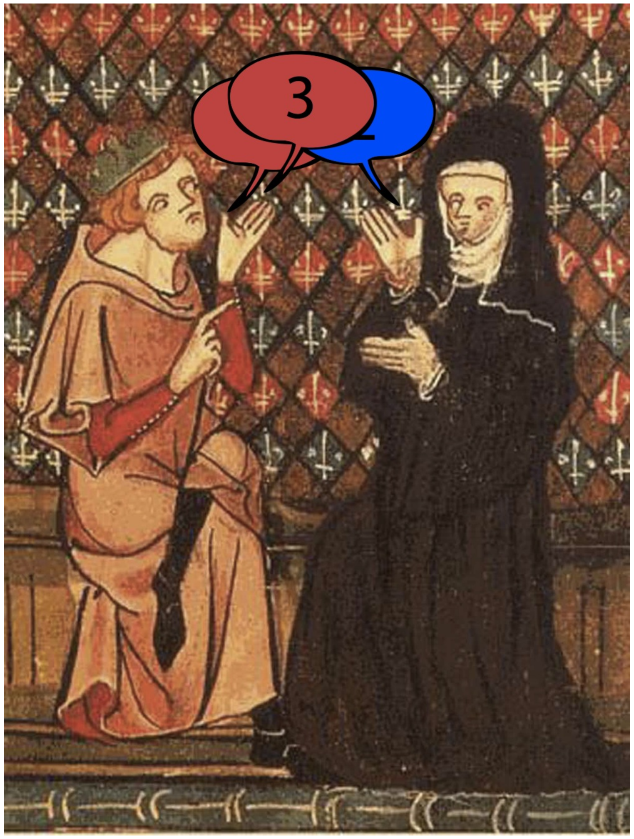


What are the principles at work in a dialogue game?

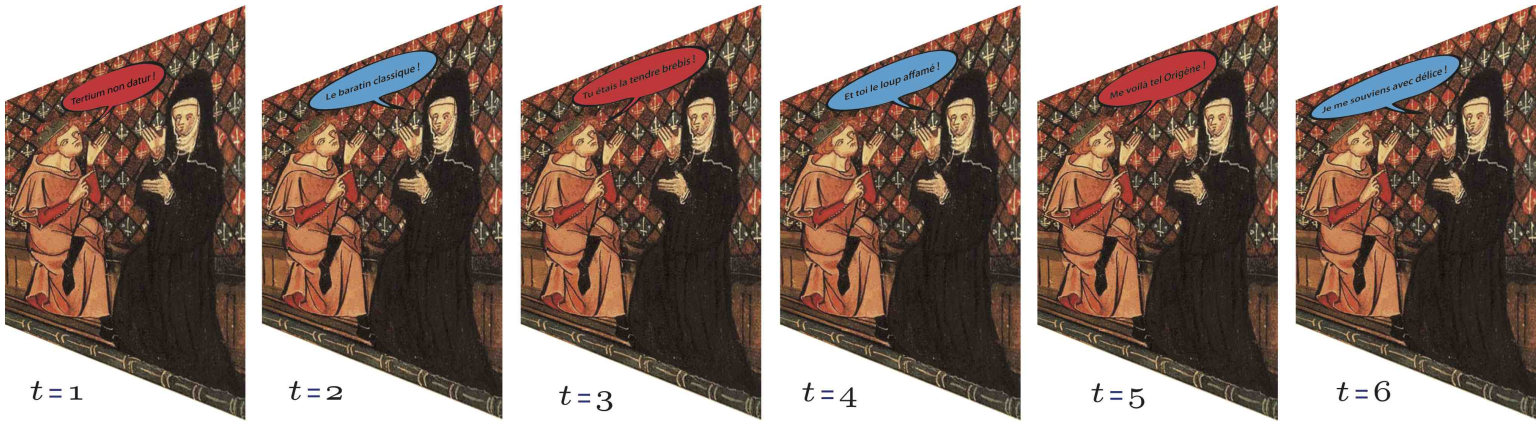
Understanding logic in space and time

What are the principles at work in a dialogue game?

Understanding logic in space and time



What are the principles at work in a dialogue game?

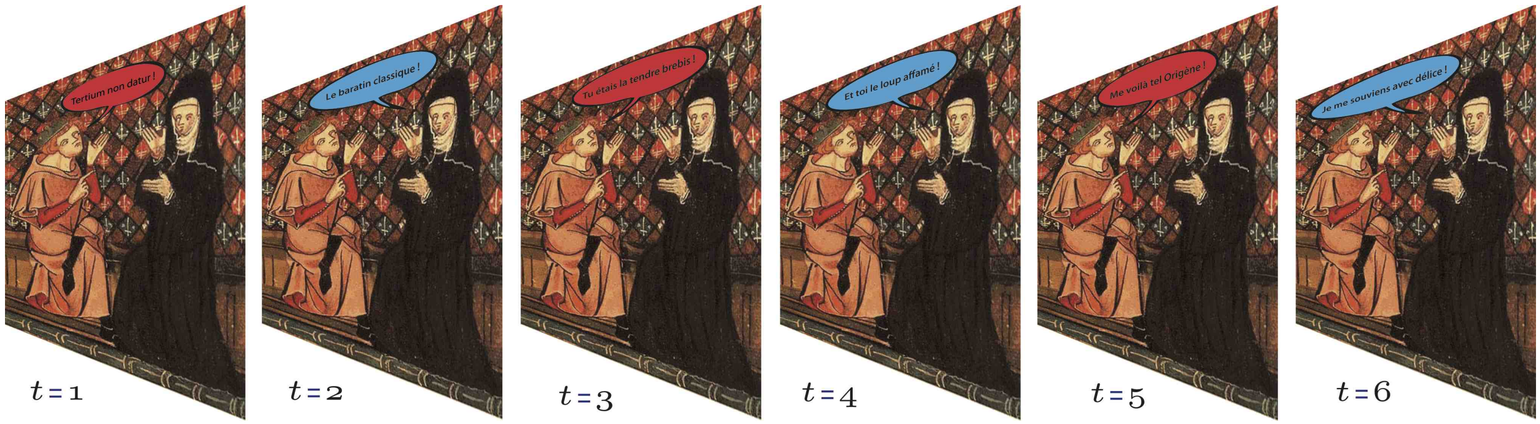


Purpose of this talk:

Understand how different proofs and programs may be

- combined together in space
- synchronized together in time

in the rich and modular ecosystem provided by **game semantics.**



Purpose of this talk:

Understand how different proofs and programs may be

- combined together in space
- synchronized together in time

in the rich and modular ecosystem provided by **differential linear logic.**

Linear logic

Seen through the lens of game semantics

Starting point: game semantics

Every proof of formula A initiates a dialogue where

Proponent tries to convince Opponent

Opponent tries to refute Proponent

An interactive approach to logic and programming languages

The formal proof of the drinker's formula

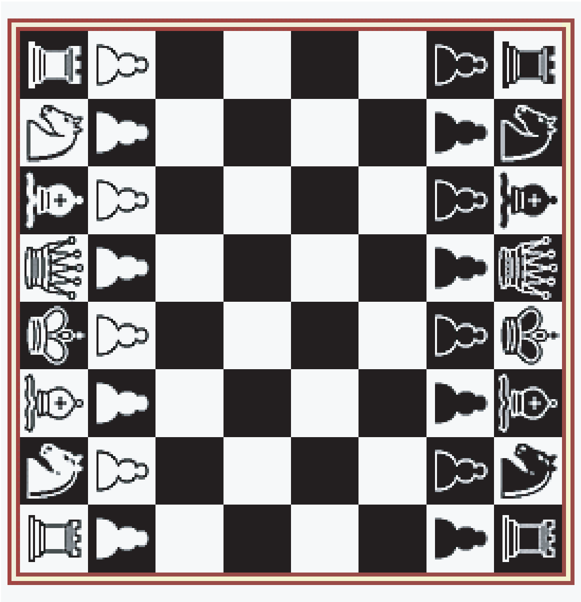
$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A(x_0) \vdash A(x_0)}\ \text{Axiom}}{A(x_0) \vdash A(x_0), \forall x.A(x)}\ \text{Right Weakening}}{\vdash A(x_0), A(x_0) \Rightarrow \forall x.A(x)}\ \text{Right} \Rightarrow}{\vdash A(x_0), \exists y.\{A(y) \Rightarrow \forall x.A(x)\}}\ \text{Right } \exists}{\vdash \forall x.A(x), \exists y.\{A(y) \Rightarrow \forall x.A(x)\}}\ \text{Right } \forall}{A(y_0) \vdash \forall x.A(x), \exists y.\{A(y) \Rightarrow \forall x.A(x)\}}\ \text{Left Weakening}}{\vdash A(y_0) \Rightarrow \forall x.A(x), \exists y.\{A(y) \Rightarrow \forall x.A(x)\}}\ \text{Right} \Rightarrow}{\vdash \exists y.\{A(y) \Rightarrow \forall x.A(x)\}, \exists y.\{A(y) \Rightarrow \forall x.A(x)\}}\ \text{Right } \exists}{\vdash \exists y.\{A(y) \Rightarrow \forall x.A(x)\}}\ \text{Contraction}
$$

Duality

Proponent
Program

plays the game

A



Opponent
Environment

plays the game

$\neg A$

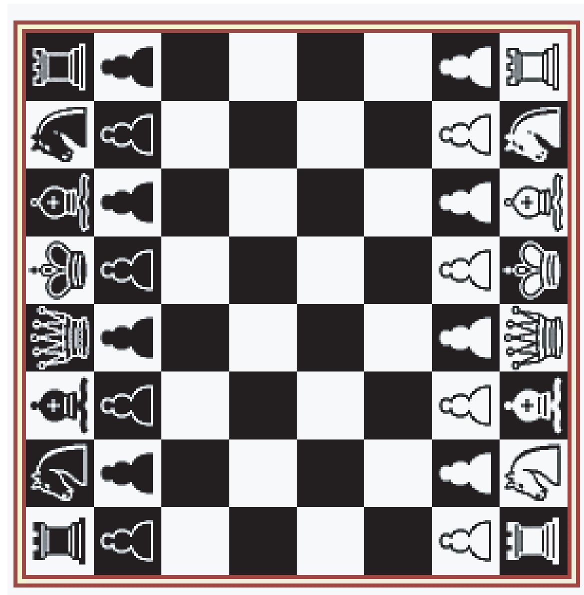
Negation permutes the rôles of Proponent and Opponent

Duality

Opponent
Environment

plays the game

$\neg A$



Proponent
Program

plays the game

A

Negation permutes the rôles of Opponent and Proponent

Sum

$\oplus$

Proponent selects the board which will be played

Sum

$\oplus$

A form of constructive disjunction

Product

&

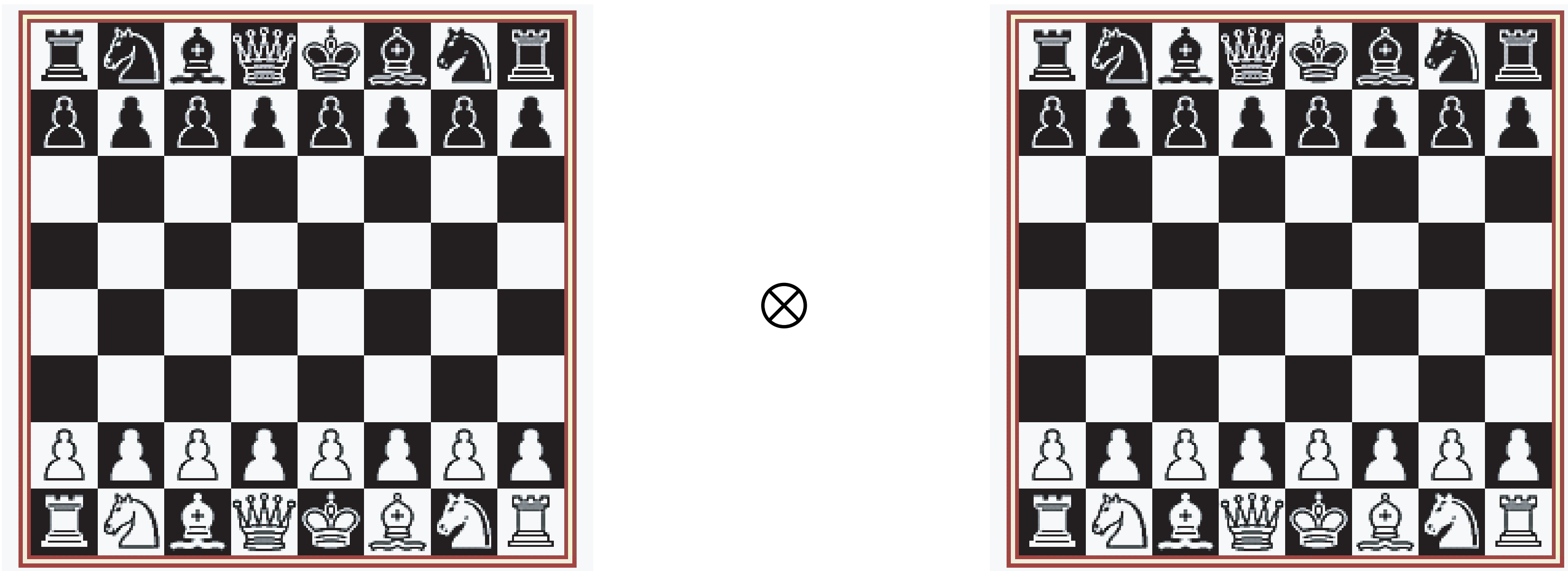
Opponent selects the board which will be played

Product

&

A form of constructive conjunction

Tensor product



The two games are played in parallel
Opponent is allowed to switch board but not Player

Tensor product

$\otimes$

A form of classical conjunction

Parallel product

$\parr$

The two games are played in parallel

Player is allowed to switch board but not Opponent

Parallel product

⅋

A form of classical disjunction

The law of excluded middle

Karpov **Korchnoi**

⅋

Player wins by playing Karpov against Korchnoi

The exponential modality

 $\otimes$  $\otimes$  $\otimes \cdots$

Opponent opens as many copies as necessary to beat Proponent but is not allowed to open an infinite number of copies

Hence, the modality is $\begin{cases} \textbf{coinductive} \text{ from the point of view of Player,} \\ \textbf{inductive} \text{ from the point of view of Opponent.} \end{cases}$

A beautiful isomorphism of linear logic

For every pair of formulas A and B of linear logic

$$\boxed{\;!A \otimes !B \quad \cong \quad !(A \,\&\, B)\;}$$

reminiscent of the isomorphism

$$\boxed{\;\wp A \times \wp B \quad \cong \quad \wp(A + B)\;}$$

This isomorphism is the origin for the name of **exponential** modality

Template games

Categorical combinatorics of synchronization

The category of polarities

We introduce the category

$$\pm_{\mathrm{game}}$$

freely generated by the graph

$$\begin{tikzcd} \langle \ominus \rangle \arrow[r, shift right, "P"'] & \langle \oplus \rangle \arrow[l, shift right, "O"'] \end{tikzcd}$$

the category $\pm_{\mathrm{game}}$ will play a fundamental role in the talk

Template games

First idea:

Define a **game** as a category A equipped with a functor

$$\begin{tikzcd} A \arrow[d, "\lambda_A"] \\ \pm_{\mathrm{game}} \end{tikzcd}$$

to the category $\pm_{\mathrm{game}}$ freely generated by the graph

$$\begin{tikzcd} \langle\ominus\rangle \arrow[r, shift right, "P"'] & \langle\oplus\rangle \arrow[l, shift right, "O"'] \end{tikzcd}$$

Inspired by the notion of **coloring** in graph theory

Positions and trajectories

It is convenient to use the following terminology

$$\begin{array}{ccc} \text{objects} & \leftrightarrow & \text{positions} \\ \text{morphisms} & \leftrightarrow & \text{trajectories} \end{array}$$

and to see the category A as an **unlabelled** transition system.

The polarity functor

The polarity functor

$$\lambda_A \quad : \quad A \longrightarrow \pm_{\text{game}}$$

assigns a polarity $\oplus$ or $\ominus$ to every position of the game A.

Definition. A position $a \in A$ is called

Player	when its polarity	$\lambda_A(a) = \oplus$	is positive
Opponent	when its polarity	$\lambda_A(a) = \ominus$	is negative

Opponent moves

Definition. An **Opponent move**

$$m \quad : \quad a^{\oplus} \longrightarrow b^{\ominus}$$

is a trajectory of the game A transported to the edge

$$O \quad : \quad \langle \oplus \rangle \longrightarrow \langle \ominus \rangle$$

of the template category $\pm_{\mathrm{game}}$.

Player moves

Definition. A **Player move**

$$m \quad : \quad a^{\ominus} \longrightarrow b^{\oplus}$$

is a trajectory of the game A transported to the edge

$$P \quad : \quad \langle\ominus\rangle \longrightarrow \langle\oplus\rangle$$

of the template category $\pm_{\mathrm{game}}$.

Silent trajectories

Definition. A **silent move**

$$m \quad : \quad a \longrightarrow b$$

is a trajectory of the game A transported to an identity morphism

$$id_{\langle \oplus \rangle} \quad : \quad \langle \oplus \rangle \longrightarrow \langle \oplus \rangle$$

$$id_{\langle \ominus \rangle} \quad : \quad \langle \ominus \rangle \longrightarrow \langle \ominus \rangle$$

of the template category $\pm_{\mathrm{game}}$.

The template of strategies

Categorical combinatorics of synchronization

The template of strategies

In order to describe the strategies between two games

$$\sigma \quad : \quad A \longrightarrow\!\!\!\!\!\!\!\!+\;\;\;\; B$$

we introduce the **template of strategies**

$$\text{\char"2A5F}_{\mathrm{strat}}$$

defined as the category freely generated by the graph

$$\begin{tikzcd} \langle \ominus, \ominus \rangle \arrow[r, shift right, "O_s"'] & \langle \oplus, \ominus \rangle \arrow[l, shift right, "P_s"'] \arrow[r, shift right, "P_t"'] & \langle \oplus, \oplus \rangle \arrow[l, shift right, "O_t"'] \end{tikzcd}$$

The template of strategies

Each of the four labels

$$O_s \qquad P_s \qquad O_t \qquad P_t$$

describes a specific kind of Opponent and Player move

O_s	:	Opponent move	played at	the source game
P_s	:	Player move	played at	the source game
O_t	:	Opponent move	played at	the target game
P_t	:	Player move	played at	the target game

which may appear on the interactive trajectory played by a strategy

$$\sigma \quad : \quad A \longrightarrow\!\!\!\!\!\!\!+\;\; B.$$

The template of strategies

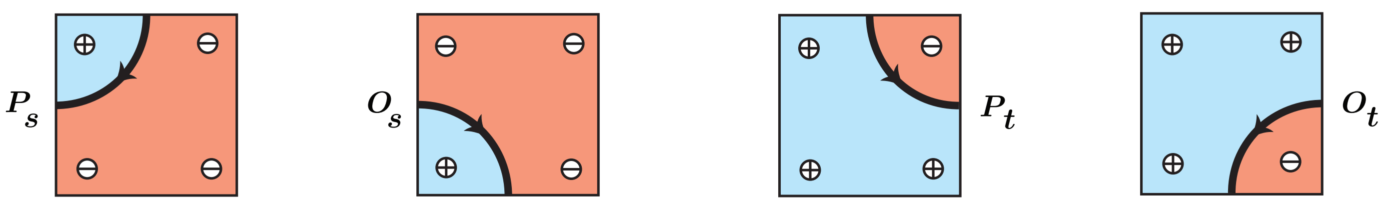
The four generators

$$
\begin{tikzcd}
\langle \ominus, \ominus \rangle \arrow[r, shift right, "O_s"'] & \langle \oplus, \ominus \rangle \arrow[l, shift right, "P_s"'] \arrow[r, shift right, "P_t"'] & \langle \oplus, \oplus \rangle \arrow[l, shift right, "O_t"']
\end{tikzcd}
$$

of the category

$$\pitchfork\mathrm{strat}$$

may be depicted as follows:

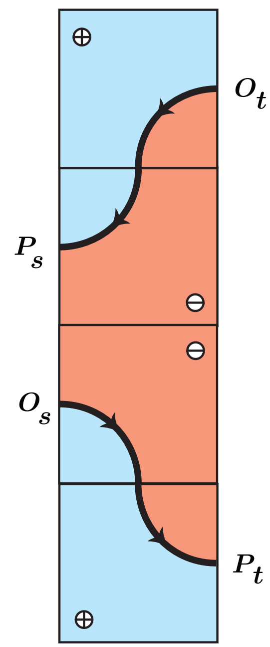


The template of strategies

In that graphical notation, the sequence

$$O_t \cdot P_s \cdot O_s \cdot P_t$$

is depicted as



The template of strategies

The category $\pitchfork_{\mathrm{strat}}$ comes equipped with a span of functors

$$\pitchfork_{\mathrm{game}} \xleftarrow{s=(1)} \pitchfork_{\mathrm{strat}} \xrightarrow{t=(2)} \pitchfork_{\mathrm{game}}$$

defined as the projection $s = (1)$ on the first component:

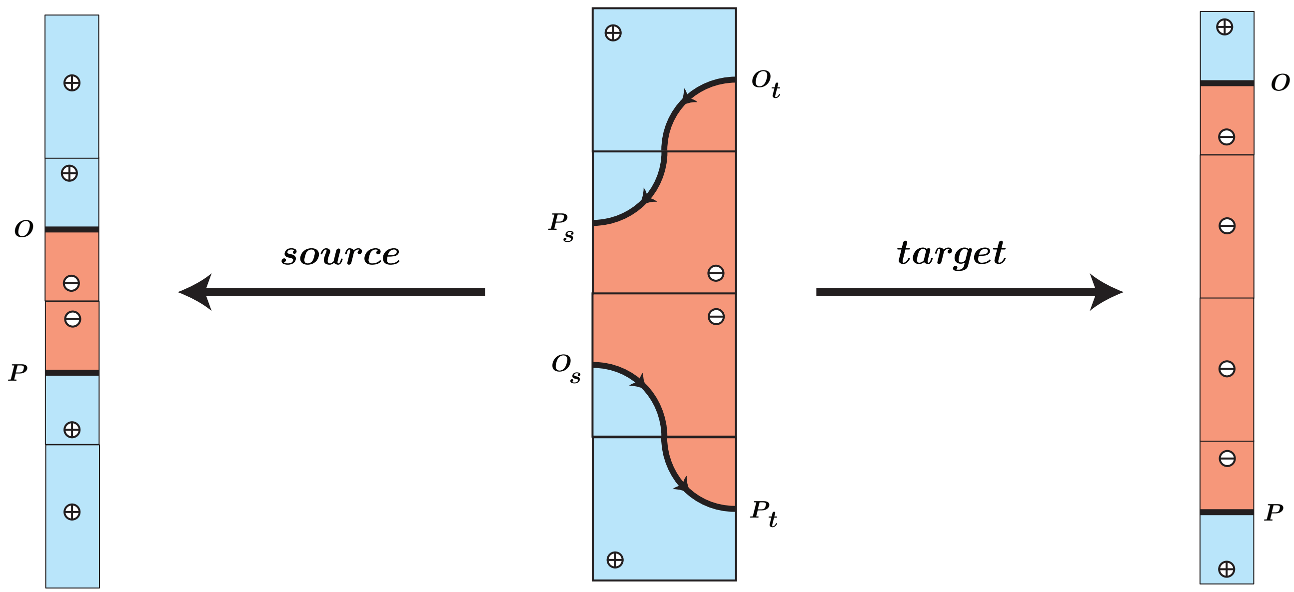
$$\begin{array}{rcl} \langle\ominus,\ominus\rangle & \mapsto & \langle\ominus\rangle \\ \langle\oplus,\ominus\rangle\,,\,\langle\oplus,\oplus\rangle & \mapsto & \langle\oplus\rangle \end{array} \qquad \begin{array}{c} O_s \mapsto P \qquad P_s \mapsto O \\ O_t\,,\,P_t \mapsto id_{\langle\oplus\rangle} \end{array}$$

and as the projection $t = (2)$ on the second component:

$$\begin{array}{rcl} \langle\oplus,\oplus\rangle & \mapsto & \langle\oplus\rangle \\ \langle\ominus,\ominus\rangle\,,\,\langle\oplus,\ominus\rangle & \mapsto & \langle\ominus\rangle \end{array} \qquad \begin{array}{c} O_t \mapsto O \qquad P_t \mapsto P \\ O_s\,,\,P_s \mapsto id_{\langle\ominus\rangle} \end{array}$$

The template of strategies

The two functors s and t are illustrated below:



Strategies between games

Second idea:

Define a **strategy** between two games

$$\sigma \quad : \quad A \longrightarrow\!\!\!\!\!\!\!\!\!+\quad B$$

as a **span of functors**

$$A \xleftarrow{\quad s \quad} S \xrightarrow{\quad t \quad} B$$

together with a **scheduling functor**

$$S \xrightarrow{\quad \lambda_\sigma \quad} \pitchfork_{\mathrm{strat}}$$

Strategies between games

making the diagram below commute

$$\begin{tikzcd} A \arrow[d, "\lambda_A"'] & S \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "\lambda_\sigma"] & B \arrow[d, "\lambda_B"] \\ \pm\mathrm{game} & \pm\mathrm{strat} \arrow[l, "s"'] \arrow[r, "t"] & \pm\mathrm{game} \end{tikzcd}$$

Key idea:

Every trajectory $s \in S$ induces a pair of trajectories $s_A \in A$ and $s_B \in B$.

The functor λ_σ describes how s_A and s_B are scheduled together by σ.

Support of a strategy

Terminology. The category S defining the span

$$A \xleftarrow{\ s\ } S \xrightarrow{\ t\ } B$$

is called the **support** of the strategy

$$\sigma \quad : \quad A \longrightarrow B$$

Basic intuition:

« the support S contains the trajectories played by σ »

A typical scheduling $B \cdot A \cdot A \cdot B$

A trajectory $s \in S$ of the strategy σ with schedule

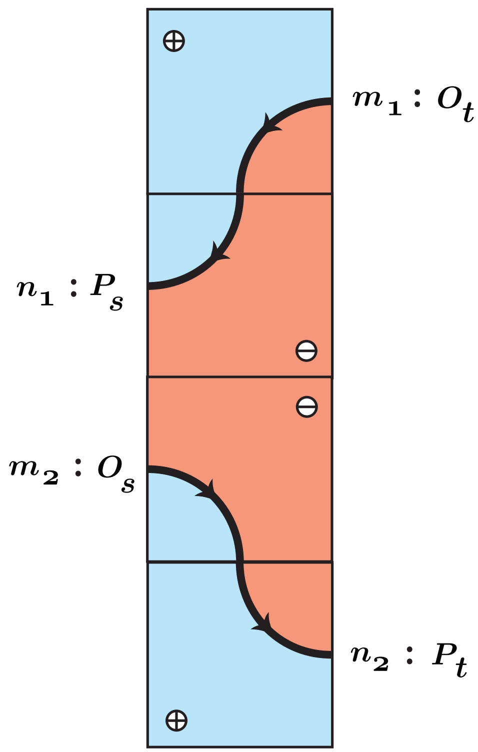
$$\langle \oplus, \oplus \rangle \xrightarrow{O_t} \langle \oplus, \ominus \rangle \xrightarrow{P_s} \langle \ominus, \ominus \rangle \xrightarrow{O_s} \langle \ominus, \oplus \rangle \xrightarrow{P_t} \langle \oplus, \oplus \rangle$$

is traditionally depicted as

	A	$\xrightarrow{\sigma}$	B
first move m_1 of polarity O_t			m_1
second move n_1 of polarity P_s	n_1		
third move m_2 of polarity O_s	m_2		
fourth move n_2 of polarity P_t			n_2

A typical scheduling $B \cdot A \cdot A \cdot B$

Thanks to the approach, one gets the more informative picture:



Simulations

Definition: A **simulation** between strategies

$$\theta \ : \ \sigma \Longrightarrow \tau \ : \ A \relbar\joinrel\mapstochar\joinrel\rightarrow B$$

is a **functor** from the support of σ to the support of τ

$$\theta \ : \ S \longrightarrow T$$

making the three triangles commute

$$\begin{tikzcd}
S \arrow[rr, "\theta"] \arrow[dr, "s"'] & & T \arrow[dl, "s"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[rr, "\theta"] \arrow[dr, "\lambda_\sigma"'] & & T \arrow[dl, "\lambda_\tau"] \\
& \pm_{\mathrm{strat}} &
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[rr, "\theta"] \arrow[dr, "t"'] & & T \arrow[dl, "t"] \\
& B &
\end{tikzcd}$$

The category of strategies and simulations

Suppose given two games A and B.

The category **Games** (A, B) has **strategies** between A and B

$$\sigma, \tau \quad : \quad A \longrightarrow\!\!\!\!\!\!+\;\; B$$

as objects and **simulations** between strategies

$$\theta \;:\; \sigma \Longrightarrow \tau \;:\; A \longrightarrow\!\!\!\!\!\!+\;\; B$$

as morphisms.

The bicategory **Games**

A bicategory of games, strategies and simulations

The bicategory **Games** of games and strategies

At this stage, we want to turn the family of categories

$$\textbf{Games}\,(A, B)$$

into a **bicategory**

$$\textbf{Games}$$

of games and strategies.

The bicategory **Games** of games and strategies

To that purpose, we need to define a composition functor

$$\circ_{A,B,C} \; : \; \textbf{Games}\,(B,C) \times \textbf{Games}\,(A,B) \longrightarrow \textbf{Games}\,(A,C)$$

which composes a pair of strategies

$$\sigma \quad : \quad A \longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\;\; B \qquad\qquad \tau \quad : \quad B \longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\;\; C$$

into a strategy

$$\sigma \;\circ_{A,B,C}\; \tau \quad : \quad A \longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\;\; C$$

Composition of strategies

The construction starts by putting the pair of functorial spans side by side:

$$
\begin{tikzcd}
 & S \arrow[dl, "s"'] \arrow[d, "\lambda_\sigma"'] \arrow[dr, "t"] & & T \arrow[dl, "s"'] \arrow[d, "\lambda_\tau"] \arrow[dr, "t"] & \\
A \arrow[d, "\lambda_A"'] & \rotatebox[origin=c]{180}{$\pitchfork$}\,\mathrm{strat} \arrow[dl, "s"] \arrow[dr, "t"'] & B \arrow[d, "\lambda_B"] & \rotatebox[origin=c]{180}{$\pitchfork$}\,\mathrm{strat} \arrow[dl, "s"] \arrow[dr, "t"'] & C \arrow[d, "\lambda_C"] \\
\rotatebox[origin=c]{180}{$\pitchfork$}\,\mathrm{game} & & \rotatebox[origin=c]{180}{$\pitchfork$}\,\mathrm{game} & & \rotatebox[origin=c]{180}{$\pitchfork$}\,\mathrm{game}
\end{tikzcd}
$$

Fine, but how shall one carry on and perform the composition?

The template of interactions

Third idea:

We define the **template of interactions**

$$ナ_{\mathrm{int}}$$

as the category obtained by the pullback diagram below

$$\begin{tikzcd}
 & & ナ_{\mathrm{int}} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, phantom, "pb"] & & \\
 & ナ_{\mathrm{strat}} \arrow[dl, "s"'] \arrow[dr, "t"'] & & ナ_{\mathrm{strat}} \arrow[dl, "s"] \arrow[dr, "t"] & \\
ナ_{\mathrm{game}} & & ナ_{\mathrm{game}} & & ナ_{\mathrm{game}}
\end{tikzcd}$$

The template of interactions

Somewhat surprisingly, the category

$$\text{大}_{\mathrm{int}}$$

is simple to describe, as the **free category** generated by the graph

$$\langle\ominus,\ominus,\ominus\rangle \underset{O_s}{\overset{P_s}{\rightleftarrows}} \langle\oplus,\ominus,\ominus\rangle \underset{P|O}{\overset{O|P}{\rightleftarrows}} \langle\oplus,\oplus,\ominus\rangle \underset{P_t}{\overset{O_t}{\rightleftarrows}} \langle\oplus,\oplus,\oplus\rangle$$

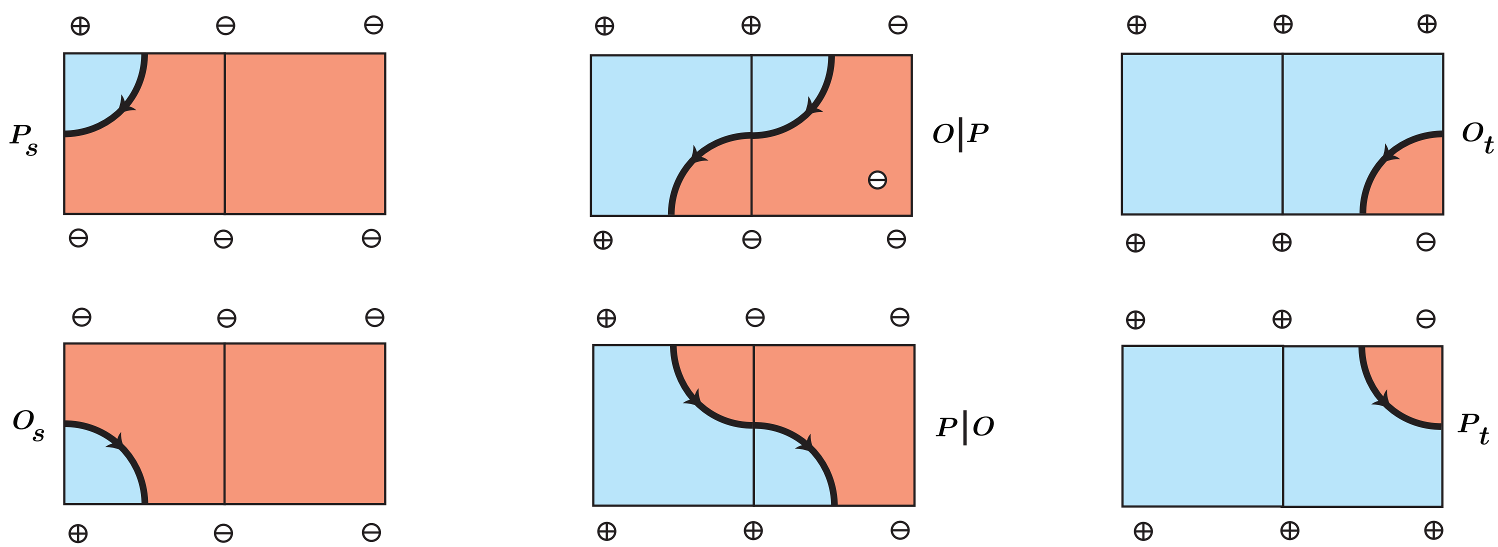
with four states or positions.

The template of interactions

The six generators

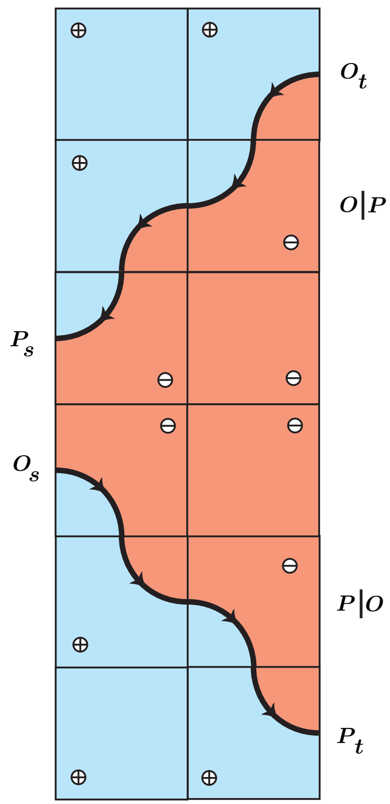
$$\langle\ominus,\ominus,\ominus\rangle \underset{O_s}{\overset{P_s}{\leftrightarrows}} \langle\oplus,\ominus,\ominus\rangle \underset{P|O}{\overset{O|P}{\leftrightarrows}} \langle\oplus,\oplus,\ominus\rangle \underset{P_t}{\overset{O_t}{\leftrightarrows}} \langle\oplus,\oplus,\oplus\rangle$$

may be depicted as follows:



A typical interaction $C \cdot B \cdot A \cdot A \cdot B \cdot C$

This typical sequence of interactions is depicted as follows:



The template of interactions

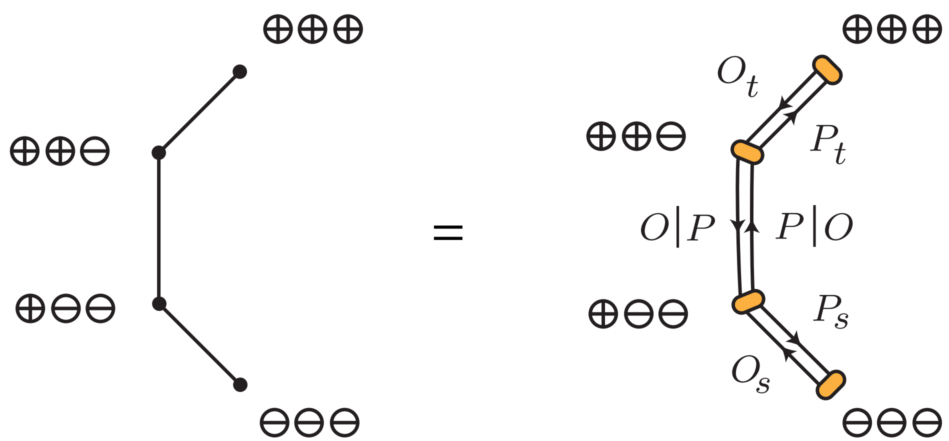
We find illuminating to depict the canonical functor

$$\pm_{\mathrm{int}} \xrightarrow{(1223)} \pm_{\mathrm{strat}} \times \pm_{\mathrm{strat}}$$

induced by the pullback diagram in the following way:

The template of interactions

In order to fully appreciate the diagram, one needs to “fatten” it



in such a way as to recover the template of interactions

$$\langle\ominus,\ominus,\ominus\rangle \;\overset{P_s}{\underset{O_s}{\leftrightarrows}}\; \langle\oplus,\ominus,\ominus\rangle \;\overset{O|P}{\underset{P|O}{\leftrightarrows}}\; \langle\oplus,\oplus,\ominus\rangle \;\overset{O_t}{\underset{P_t}{\leftrightarrows}}\; \langle\oplus,\oplus,\oplus\rangle$$

Key observation

The template $\pitchfork_{\mathrm{int}}$ of interactions comes equipped with a functor

$$hide \quad : \quad \pitchfork_{\mathrm{int}} \longrightarrow \pitchfork_{\mathrm{strat}}$$

which makes the diagram below commute:

$$\begin{tikzcd}
\pitchfork_{\mathrm{strat}} \arrow[d, "(1)"'] & \pitchfork_{\mathrm{int}} \arrow[l, "(12)"'] \arrow[r, "(23)"] \arrow[d, "hide"] & \pitchfork_{\mathrm{strat}} \arrow[d, "(2)"] \\
\pitchfork_{\mathrm{game}} & \pitchfork_{\mathrm{strat}} \arrow[l, "s=(1)"'] \arrow[r, "t=(2)"] & \pitchfork_{\mathrm{game}}
\end{tikzcd}$$

and thus defines a map of span.

Key observation

The functor

$$hide \quad : \quad \pitchfork_{\mathrm{int}} \longrightarrow \pitchfork_{\mathrm{strat}}$$

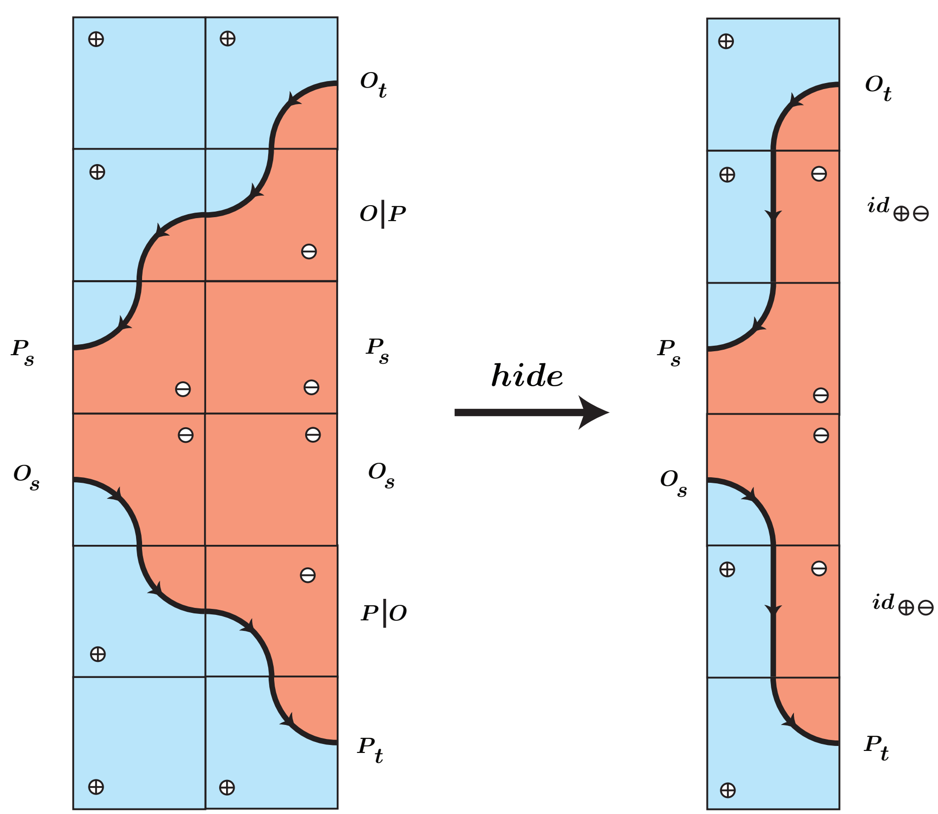
is defined by **projecting** the positions of the interaction category

$$\langle\, \varepsilon_1 \,,\, \varepsilon_2 \,,\, \varepsilon_3 \,\rangle$$

on their first and third components:

$$\begin{array}{rcl@{\qquad}l}
\langle \ominus, \ominus, \ominus \rangle & \mapsto & \langle \ominus, \ominus \rangle & O_s \mapsto O_s \quad P_s \mapsto P_s \\
\langle \oplus, \ominus, \ominus \rangle \,,\, \langle \oplus, \oplus, \ominus \rangle & \mapsto & \langle \oplus, \ominus \rangle & O|P \,,\, P|O \;\mapsto\; id_{\langle \oplus, \ominus \rangle} \\
\langle \oplus, \oplus, \oplus \rangle & \mapsto & \langle \oplus, \oplus \rangle & O_s \mapsto O_s \quad P_s \mapsto P_s
\end{array}$$

Illustration



Composition of strategies

$$\begin{tikzcd}
 & & S \times_B T \arrow[dl, "\pi_1"'] \arrow[d, "\lambda_\sigma \parallel \lambda_\tau" description] \arrow[dr, "\pi_2"] & & \\
 & S \arrow[dl, "s"'] \arrow[d, "\lambda_\sigma"'] & \pm_{\mathrm{int}} \arrow[dl, "\pi_1"] \arrow[dd, "hide" description] \arrow[dr, "\pi_2"'] & T \arrow[d, "\lambda_\tau"] \arrow[dr, "t"] & \\
A \arrow[d, "\lambda_A"'] & \pm_{\mathrm{strat}} \arrow[dl, "s"'] & & \pm_{\mathrm{strat}} \arrow[dr, "t"] & C \arrow[d, "\lambda_C"] \\
\pm_{\mathrm{game}} & & \pm_{\mathrm{strat}} \arrow[ll, "s"'] \arrow[rr, "t"] & & \pm_{\mathrm{game}}
\end{tikzcd}$$

Composition of strategies

This definition of composition implements the slogan that

$$\boxed{\text{composition} \quad = \quad \text{synchronization} \ + \ \text{hiding}}$$

What about identities?

There exists a functor

$$copycat \quad : \quad \pitchfork_{\mathrm{game}} \longrightarrow \pitchfork_{\mathrm{strat}}$$

which makes the diagram commute:

$$\begin{tikzcd}
 & \pitchfork_{\mathrm{game}} \arrow[dl, "id"'] \arrow[dr, "id"] \arrow[dd, "copycat"] & \\
\pitchfork_{\mathrm{game}} & & \pitchfork_{\mathrm{game}} \\
 & \pitchfork_{\mathrm{strat}} \arrow[ul, "s=(1)"] \arrow[ur, "t=(2)"'] &
\end{tikzcd}$$

and thus defines a morphism of spans.

What about identities?

The functor

$$copycat \quad : \quad \pm_{\mathrm{game}} \longrightarrow \pm_{\mathrm{strat}}$$

is defined by **duplicating** the positions of the polarity category

$$\langle \varepsilon \rangle$$

in the following way:

$$\begin{array}{rcl} \langle \ominus \rangle & \mapsto & \langle \ominus, \ominus \rangle \\ \langle \oplus \rangle & \mapsto & \langle \oplus, \oplus \rangle \end{array} \qquad \begin{array}{rcl} O & \mapsto & O_t \cdot P_s \\ P & \mapsto & O_s \cdot P_t \end{array}$$

A synchronous copycat strategy

The functor

$$copycat \quad : \quad ⅄_{\mathrm{game}} \longrightarrow ⅄_{\mathrm{strat}}$$

transports the edge

$$\langle \ominus \rangle \xleftarrow{\ O\ } \langle \oplus \rangle$$

to the trajectory consisting of two moves

$$\langle \ominus, \ominus \rangle \xleftarrow{\ P_s\ } \langle \oplus, \ominus \rangle \xleftarrow{\ O_t\ } \langle \oplus, \oplus \rangle$$

A synchronous copycat strategy

The functor

$$copycat \quad : \quad \pm_{\mathrm{game}} \longrightarrow \pm_{\mathrm{strat}}$$

transports the edge

$$\langle \ominus \rangle \xrightarrow{\;P\;} \langle \oplus \rangle$$

to the trajectory consisting of two moves

$$\langle \ominus, \ominus \rangle \xrightarrow{\;O_s\;} \langle \oplus, \ominus \rangle \xrightarrow{\;P_t\;} \langle \oplus, \oplus \rangle$$

The identity strategy

Given a game A, the copycat strategy

$$\mathbf{cc}_A \quad : \quad A \longrightarrow A$$

is defined as the functional span

$$A \xleftarrow{\ \textit{identity}\ } A \xrightarrow{\ \textit{identity}\ } A$$

together with the scheduling functor

$$\lambda_{\mathbf{cc}_A} \;=\; A \xrightarrow{\ \lambda_A\ } \pitchfork_{\mathrm{game}} \xrightarrow{\ \textit{copycat}\ } \pitchfork_{\mathrm{strat}}$$

Identity strategy

$$
\begin{tikzcd}
 & A \arrow[ddl, "identity"'] \arrow[d, "\lambda_A"] \arrow[ddr, "identity"] & \\
 & \text{よ}\mathrm{game} \arrow[dd, "copycat"] \arrow[ddl, "id"'] \arrow[ddr, "id"] & \\
A \arrow[d, "\lambda_A"'] & & A \arrow[d, "\lambda_A"] \\
\text{よ}\mathrm{game} & \text{よ}\mathrm{strat} \arrow[l, "s"'] \arrow[r, "t"] & \text{よ}\mathrm{game}
\end{tikzcd}
$$

Discovery of an unexpected structure

Key observation: the categories

$$\pitchfork[0] = \pitchfork_{\mathrm{game}} \qquad \pitchfork[1] = \pitchfork_{\mathrm{strat}} \qquad \pitchfork[2] = \pitchfork_{\mathrm{int}}$$

and the span of functors

$$\pitchfork[0] \xleftarrow{\;s\;} \pitchfork[1] \xrightarrow{\;t\;} \pitchfork[0]$$

define an **internal category** in $\mathbf{Cat}$ with composition and identity

$$\pitchfork[2] \xrightarrow{\;hide\;} \pitchfork[1] \qquad\qquad \pitchfork[0] \xrightarrow{\;copycat\;} \pitchfork[1]$$

As an immediate consequence...

Theorem A. The construction just given defines a **bicategory**

Games

of games, strategies and simulations.

Main technical result of the paper

Theorem B. The bicategory

Games

of games, strategies and simulations is **symmetric monoidal.**

Main technical result of the paper

Theorem C. The bicategory

Games

of games, strategies and simulations is **star-autonomous.**

All these results are based on the same recipe!

One constructs an **internal category** of tensorial schedules

$$\pitchfork^{\otimes}$$

together with a pair of **internal functors**

$$\pitchfork \times \pitchfork \xleftarrow{\quad pick \quad} \pitchfork^{\otimes} \xrightarrow{\quad pince \quad} \pitchfork$$

where *pick* and *pince* are moreover required to be **acute.**

All these results are based on the same recipe!

One constructs an **internal category** of cotensorial schedules

$$\pitchfork^{\parr}$$

together with a pair of **internal functors**

$$\pitchfork \times \pitchfork \xleftarrow{\;\;pick\;\;} \pitchfork^{\parr} \xrightarrow{\;\;pince\;\;} \pitchfork$$

where $pick$ and $pince$ are moreover required to be **acute.**

All these results are based on the same recipe!

One constructs an **internal functor**

$$reverse \quad : \quad \pitchfork^{op} \longrightarrow \pitchfork$$

which reverses the polarity of every position and move

$$\begin{array}{ccc} \oplus & \mapsto & \ominus \\ \ominus & \mapsto & \oplus \end{array} \qquad\qquad \begin{array}{ccc} O & \mapsto & P \\ P & \mapsto & O \end{array}$$

Acute internal functors

Definition An internal functor

$$F \quad : \quad \text{ホ}_1 \longrightarrow \text{ホ}_2$$

is **acute** when the two diagrams

$$\begin{tikzcd}
\text{ホ}_1[2] \arrow[r, "{F[2]}"] \arrow[d, "hide_1"'] & \text{ホ}_2[2] \arrow[d, "hide_2"] \\
\text{ホ}_1[1] \arrow[r, "{F[1]}"'] & \text{ホ}_2[1]
\end{tikzcd}
\qquad
\begin{tikzcd}
\text{ホ}_1[0] \arrow[r, "{F[0]}"] \arrow[d, "copycat_1"'] & \text{ホ}_2[0] \arrow[d, "copycat_2"] \\
\text{ホ}_1[1] \arrow[r, "{F[1]}"'] & \text{ホ}_2[1]
\end{tikzcd}$$

are pullback diagrams.

The backward action

Every acute internal functor $F : \pm_1 \to \pm_2$ induces a homomorphism

$$F^{\triangleleft} \quad : \quad \textbf{Games}(\pm_2) \longrightarrow \textbf{Games}(\pm_1)$$

defined by **pullback** on games and strategies:

$$\begin{tikzcd}
A \arrow[d, "\lambda_A"'] & & S \arrow[ll, "s"'] \arrow[rr, "t"] \arrow[d, "\lambda_\sigma"] & & B \arrow[d, "\lambda_B"] & \\
\pm_2[0] & & \pm_2[1] \arrow[ll, "s"'] \arrow[rr, "t"] & & \pm_2[0] & \\
& \pm_1[0] \arrow[ul, "{F[0]}"'] & & \pm_1[1] \arrow[ll, "s"'] \arrow[rr, "t"] \arrow[ul, "{F[1]}"'] & & \pm_1[0] \arrow[ul, "{F[0]}"']
\end{tikzcd}$$

The forward action

Every acute internal functor $F : \pitchfork_1 \to \pitchfork_2$ induces a homomorphism

$$F^{\triangleright} \quad : \quad \textbf{Games}(\pitchfork_1) \longrightarrow \textbf{Games}(\pitchfork_2)$$

defined by **postcomposition** on games and strategies:

$$\begin{tikzcd}
A \arrow[d, "\lambda_A"'] & S \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "\lambda_\sigma"] & B \arrow[d, "\lambda_B"] \\
\pitchfork_1[0] \arrow[d, "F[0]"'] & \pitchfork_1[1] \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "F[1]"] & \pitchfork_1[0] \arrow[d, "F[0]"] \\
\pitchfork_2[0] & \pitchfork_2[1] \arrow[l, "s"'] \arrow[r, "t"] & \pitchfork_2[0]
\end{tikzcd}$$

The template of tensorial schedules

The recipe for the tensor product

The template of tensorial schedules

We consider the category

$$\pm^{\otimes}_{\mathrm{game}}$$

freely generated by the graph

$$\begin{tikzcd}
\langle \ominus, \oplus \rangle \arrow[r, shift right, "P_l"'] & \langle \oplus, \oplus \rangle \arrow[l, shift right, "O_l"'] \arrow[r, shift left, "O_r"] & \langle \oplus, \ominus \rangle \arrow[l, shift left, "P_r"]
\end{tikzcd}$$

The template of tensorial schedules

Idea: The three positions

$$\langle \ominus, \oplus \rangle \qquad \langle \oplus, \oplus \rangle \qquad \langle \oplus, \ominus \rangle$$

represent the three polarities

$$\langle \varepsilon_1, \varepsilon_2 \rangle$$

possibly reached by a position $a_1 \otimes a_2$ in the game

$$A_1 \otimes A_2$$

obtained by tensoring the games A_1 and A_2.

The template of tensorial schedules

The category

$$\pitchfork^{\otimes}_{\mathrm{strat}}$$

is freely generated by the graph

$$\begin{tikzcd}
 & & \langle \oplus, \oplus, \oplus, \oplus \rangle \arrow[dl, shift right, "O_{l,t}"'] \arrow[dr, shift left, "O_{r,t}"] & & \\
 & \langle \oplus, \oplus, \ominus, \oplus \rangle \arrow[ur, shift right, "P_{l,t}"'] \arrow[dl, shift right, "P_{l,s}"'] & & \langle \oplus, \oplus, \oplus, \ominus \rangle \arrow[ul, shift left, "P_{r,t}"] \arrow[dr, shift left, "P_{r,s}"] & \\
\langle \ominus, \oplus, \ominus, \oplus \rangle \arrow[ur, shift right, "O_{l,s}"'] & & & & \langle \oplus, \ominus, \oplus, \ominus \rangle \arrow[ul, shift left, "O_{r,s}"]
\end{tikzcd}$$

The template of tensorial schedules

The five positions of the category

$$\langle\ominus,\oplus,\ominus,\oplus\rangle \quad \langle\oplus,\oplus,\ominus,\oplus\rangle \quad \langle\oplus,\oplus,\oplus,\oplus\rangle \quad \langle\oplus,\oplus,\oplus,\ominus\rangle \quad \langle\oplus,\ominus,\oplus,\ominus\rangle$$

describe the five possible sequences of polarities

$$\langle\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4\rangle$$

reached by a position of the games A_1, A_2, A_3, A_4 in a trajectory of

$$\sigma \quad : \quad A_1 \otimes A_2 \longrightarrow A_3 \otimes A_4$$

Key observation

Theorem. The categories

$$\text{ナ}^{\otimes}[0] = \text{ナ}^{\otimes}_{\text{game}} \qquad\qquad \text{ナ}^{\otimes}[1] = \text{ナ}^{\otimes}_{\text{sched}}$$

and the span of functors

$$\text{ナ}^{\otimes}_{\text{game}} \xleftarrow{\quad s \quad} \text{ナ}^{\otimes}_{\text{strat}} \xrightarrow{\quad t \quad} \text{ナ}^{\otimes}_{\text{game}}$$

define an **internal category** $\text{ナ}^{\otimes}$ in the category $\mathbf{Cat}$.

A pair of internal functors

The internal category

$$\pitchfork^{\otimes}$$

comes equipped with a pair of **internal functors**

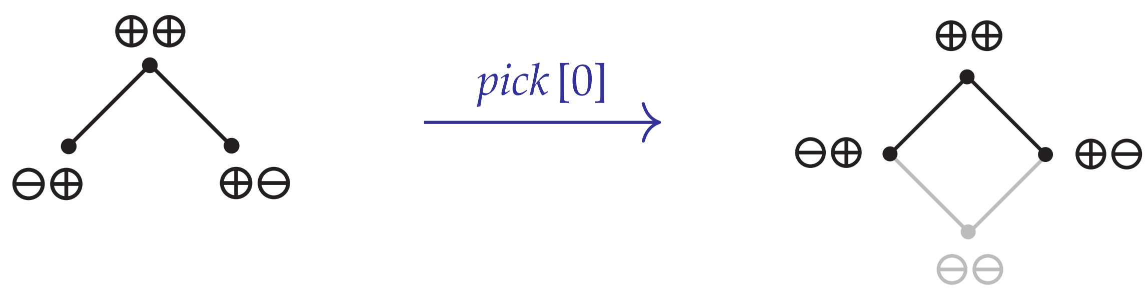
$$\pitchfork \times \pitchfork \xleftarrow{\ pick\ } \pitchfork^{\otimes} \xrightarrow{\ pince\ } \pitchfork$$

The pick functor

The internal functor

$$pick \quad : \quad \pm^{\otimes} \longrightarrow \pm \times \pm$$

is defined at dimension 0 by the functor:

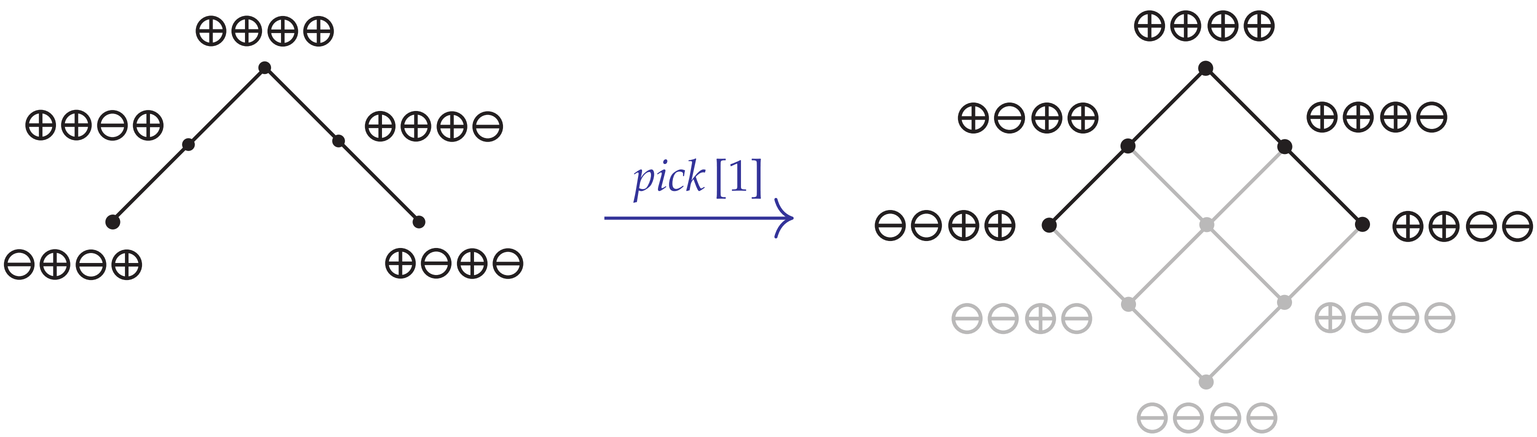


The pick functor

The internal functor

$$pick \quad : \quad \pm^{\otimes} \longrightarrow \pm \times \pm$$

is defined at dimension 1 by the functor:



The pince functor

The internal functor

$$\mathit{pince} \quad : \quad \rotatebox[origin=c]{180}{$\pitchfork$}^{\otimes} \longrightarrow \rotatebox[origin=c]{180}{$\pitchfork$}$$

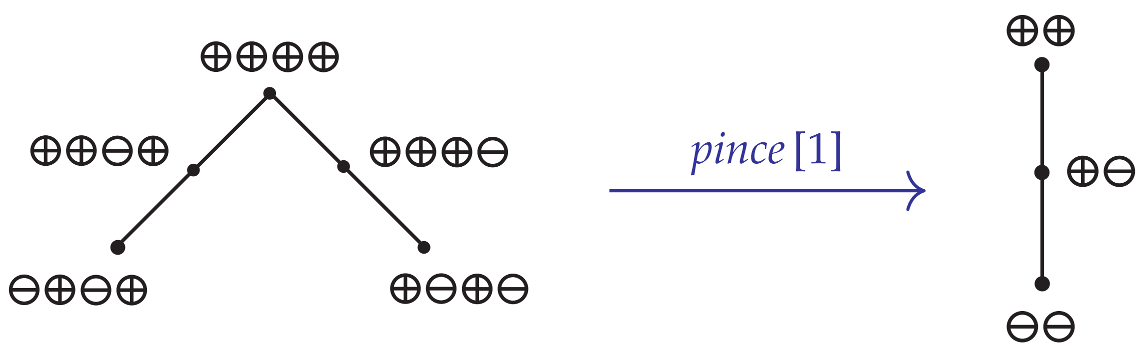
is defined at dimension 0 by the functor:

The pince functor

The internal functor

$$pince \quad : \quad \rotatebox{180}{$\pitchfork$}^{\otimes} \longrightarrow \rotatebox{180}{$\pitchfork$}$$

is defined at dimension 1 by the functor:



The tensor product of template games

The tensor product $A \otimes B$ of two template games

$$A \xrightarrow{\quad\lambda_A\quad} \pm_{\mathrm{game}} \qquad\qquad B \xrightarrow{\quad\lambda_B\quad} \pm_{\mathrm{game}}$$

is computed by pullback along *pick* followed by composition with *pince*:

$$\begin{tikzcd}
A \times B \arrow[d, "\lambda_A \times \lambda_B"'] \arrow[dr, phantom, "pullback"] & & A \otimes B \arrow[ll] \arrow[d] \arrow[dr, bend left, "\lambda_{A\otimes B}"] & \\
\pm_{\mathrm{game}} \times \pm_{\mathrm{game}} & & \pm^{\otimes}_{\mathrm{game}} \arrow[ll, "pick"] \arrow[r, "pince"'] & \pm_{\mathrm{game}}
\end{tikzcd}$$

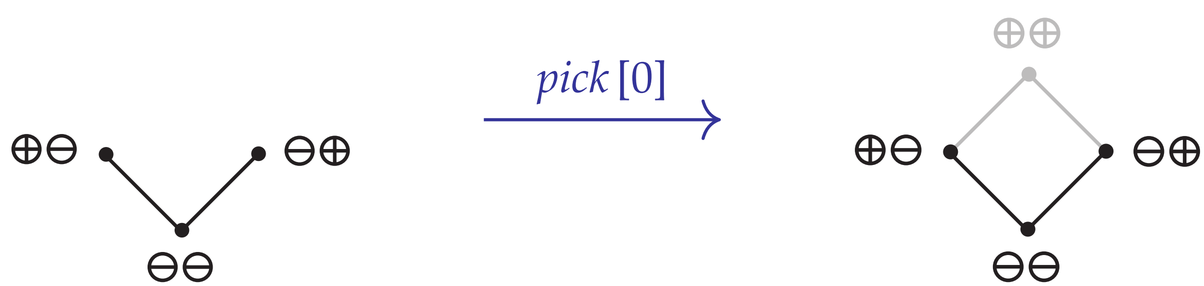
A categorical version of Milner's idea of **synchronization algebra.**

The pick functor

The internal functor

$$pick \quad : \quad \pitchfork^{\rotatebox[origin=c]{180}{\&}} \longrightarrow \pitchfork \times \pitchfork$$

is defined at dimension 0 by the functor:

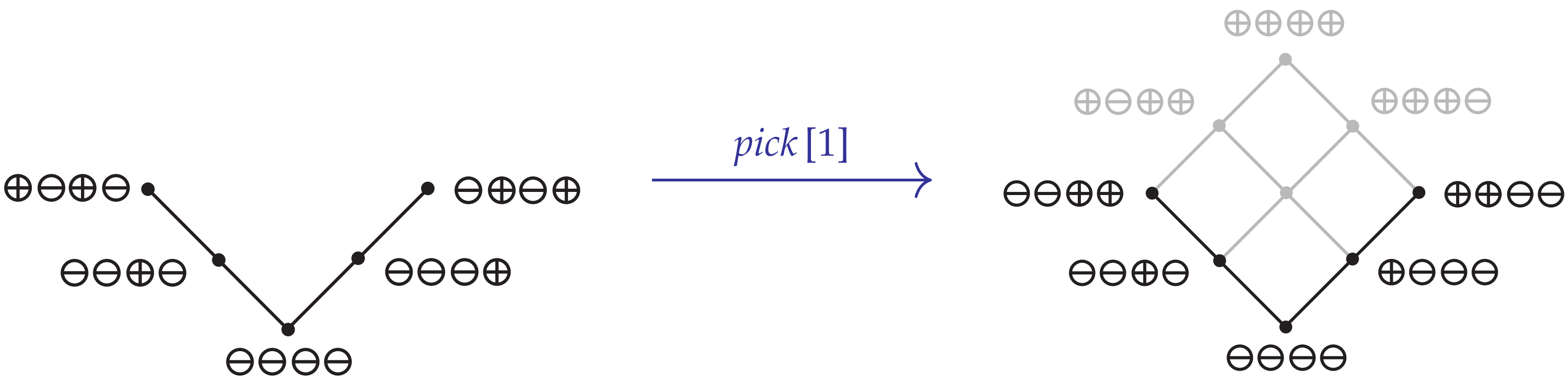


The pick functor

The internal functor

$$pick \quad : \quad \pm^{\invamp} \longrightarrow \pm \times \pm$$

is defined at dimension 1 by the functor:

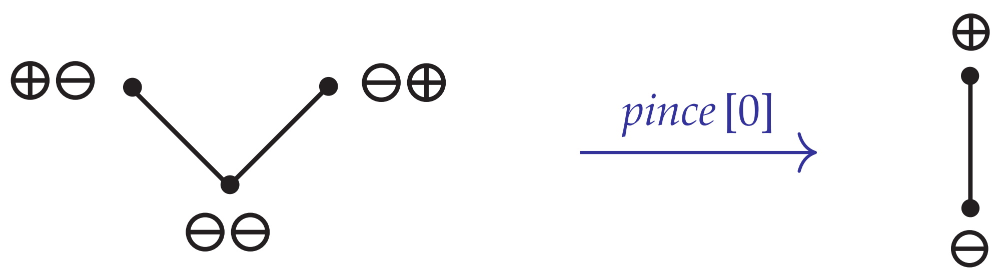


The pince functor

The internal functor

$$\mathit{pince} \quad : \quad \pitchfork^{\wp} \longrightarrow \pitchfork$$

is defined at dimension 0 by the functor:

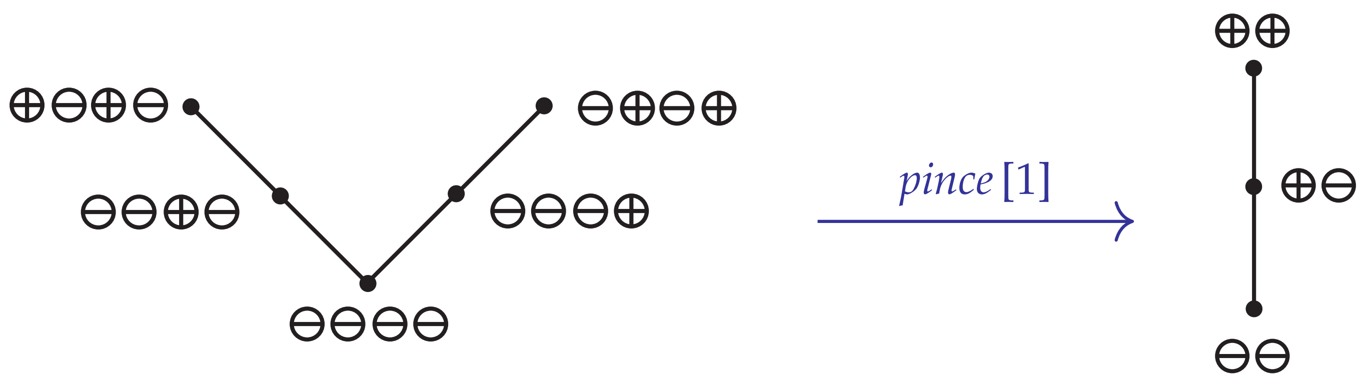


The pince functor

The internal functor

$$pince \quad : \quad \rotatebox[origin=c]{180}{$\pitchfork$}^{\parr} \longrightarrow \rotatebox[origin=c]{180}{$\pitchfork$}$$

is defined at dimension 1 by the functor:



The cotensor product of template games

The cotensor product $A \parr B$ of two template games

$$A \xrightarrow{\quad \lambda_A \quad} \pitchfork_{\mathrm{game}} \qquad\qquad B \xrightarrow{\quad \lambda_B \quad} \pitchfork_{\mathrm{game}}$$

is computed by pullback along *pick* followed by composition with *pince*:

$$\begin{tikzcd}
A \times B \arrow[d, "\lambda_A \times \lambda_B"'] & & A \parr B \arrow[ll] \arrow[d] \arrow[dr, bend left, "\lambda_{A \parr B}"] & \\
\pitchfork_{\mathrm{game}} \times \pitchfork_{\mathrm{game}} \arrow[urr, phantom, "\mathit{pullback}"] & & \pitchfork^{\parr} \arrow[ll, "\mathit{pick}"] \arrow[r, "\mathit{pince}"'] & \pitchfork_{\mathrm{game}}
\end{tikzcd}$$

The distributivity law of linear logic

A game semantics of linear logic

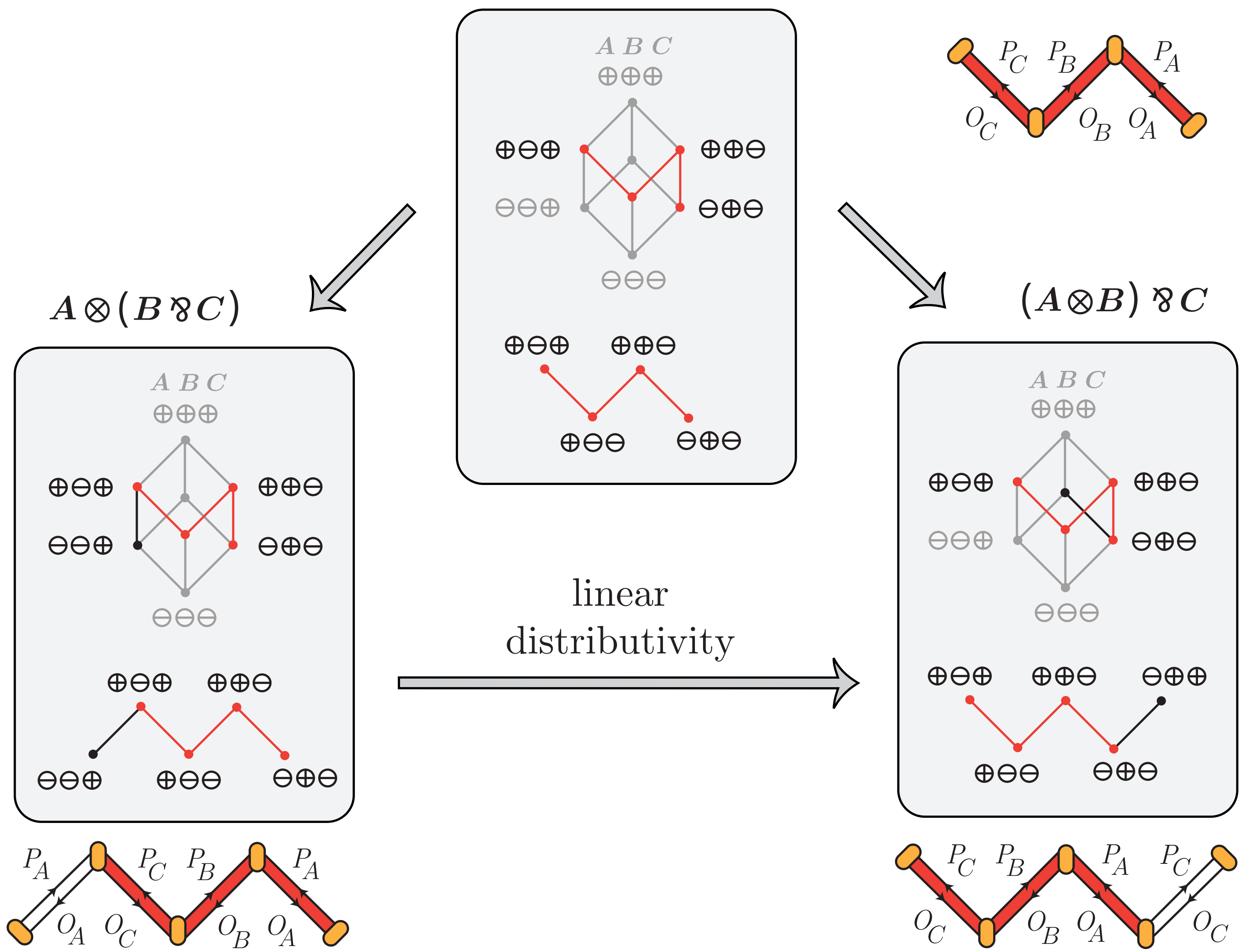
The distributivity law of linear logic

The main ingredient of linear logic

$$\kappa_{A,B,C} \quad : \quad A \otimes (B \parr C) \longrightarrow (A \otimes B) \parr C$$

cannot be interpreted in traditional game semantics.

When one interprets it in template games, here is what one gets...



A B C
$A \otimes (B \parr C)$
$(A \otimes B) \parr C$
linear
distributivity
P_A P_B P_C O_A O_B O_C

A homotopy structure on functorial spans

A homotopy model of differential linear logic

The natural model structure on $\mathbf{Cat}$

We distinguish three classes of functors

$$F \quad : \quad \mathscr{A} \longrightarrow \mathscr{B}$$

between small categories:

- ▷ the class $\mathscr{C}$ of **monos on objects**
- ▷ the class $\mathscr{F}$ of **isofibrations**
- ▷ the class $\mathscr{W}$ of **categorical equivalences**

Theorem [Joyal]

The category $\mathbf{Cat}$ of small categories and functors equipped with

$\mathscr{C}$: cofibrations $\quad$ $\mathscr{F}$: fibrations $\quad$ $\mathscr{W}$: weak equivalences

defines a Quillen model structure.

The Seely equivalence

The usual Seely isomorphism of linear logic

$$!(A \& B) \quad \cong \quad !A \otimes !B$$

is replaced in the 2-category $\mathbf{Cat}$ by a **categorical equivalence**

$$\mathbf{Sym}(A + B) \xrightarrow{\quad deshuffle \quad} \mathbf{Sym}\,A \times \mathbf{Sym}\,B$$

which happens to be an **isofibration** and thus in $\mathscr{F} \cap \mathscr{W}$.

The categorical equivalence in the converse direction

$$\mathbf{Sym}\,A \times \mathbf{Sym}\,B \xrightarrow{\quad concat \quad} \mathbf{Sym}(A + B)$$

happens to be a **mono on object** and thus in $\mathscr{C} \cap \mathscr{W}$.

In the case of distributors

Every functor between small categories

$$F \quad : \quad A \longrightarrow B$$

induces an adjoint pair $L_F \dashv R_F$ of distributors

$$L_F \;:\; A \longrightarrow\!\!\!\!\!\!+\;\; B \qquad\qquad R_F \;:\; B \longrightarrow\!\!\!\!\!\!+\;\; A$$

in the bicategory $\mathbf{Dist}$, where the distributors are defined as

$$L_F\,(b,a) \;=\; B\,(Fb,a) \quad : \quad B^{op} \times A \longrightarrow \mathbf{Set}$$

$$R_F\,(a,b) \;=\; B\,(a,Fb) \quad : \quad A^{op} \times B \longrightarrow \mathbf{Set}$$

In the case of functorial spans

Similarly, every functor between small categories

$$F \quad : \quad A \longrightarrow B$$

induces an adjoint pair $L_F \dashv R_F$ of categorical spans

$$L_F \ : \ A \longrightarrow\!\!\!\!\!\!\!\!\!\!+\quad B \qquad\qquad R_F \ : \ B \longrightarrow\!\!\!\!\!\!\!\!\!\!+\quad A$$

in the bicategory $\mathbf{Span}$, where the spans L_F and R_F are defined as

$$L_F \ = \quad A \xleftarrow{id} A \xrightarrow{F} B$$

$$R_F \ = \quad B \xleftarrow{F} B \xrightarrow{id} A$$

Same recipe for contractions and co-contractions

This enables one to deduce from the monoid structure in $\mathbf{Cat}$

$$\begin{aligned} \otimes_A &: \mathsf{Sym}\,A \times \mathsf{Sym}\,A \longrightarrow \mathsf{Sym}\,A \\ I_A &: \mathbf{1} \longrightarrow \mathsf{Sym}\,A \end{aligned}$$

the comonoid structure in $\mathbf{Dist}$ of the exponential modality

$$\begin{aligned} d_A = R_{\otimes_A} &: \mathsf{Sym}\,A \;\mathrel{\longrightarrow\!\!\!\!\!\!\!\!\mid\;\;\;} \mathsf{Sym}\,A \otimes \mathsf{Sym}\,A \\ e_A = R_{I_A} &: \mathsf{Sym}\,A \;\mathrel{\longrightarrow\!\!\!\!\!\!\!\!\mid\;\;\;} \mathbf{1} \end{aligned}$$

as well as its monoid structure coming from the differential structure:

$$\begin{aligned} m_A = L_{\otimes_A} &: \mathsf{Sym}\,A \otimes \mathsf{Sym}\,A \;\mathrel{\longrightarrow\!\!\!\!\!\!\!\!\mid\;\;\;} \mathsf{Sym}\,A \\ u_A = L_{I_A} &: \mathbf{1} \;\mathrel{\longrightarrow\!\!\!\!\!\!\!\!\mid\;\;\;} \mathsf{Sym}\,A \end{aligned}$$

In the case of distributors

Every natural transformation in **Cat**

$$\begin{tikzcd} A \arrow[rr, bend left, "F"] \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\Downarrow\varphi"] & & B \end{tikzcd}$$

is transported to a pair of 2-cells in **Dist**

$$\begin{tikzcd} A \arrow[rr, bend left, "L_F", "\shortmid"{marking}] \arrow[rr, bend right, "L_G"', "\shortmid"{marking}] \arrow[rr, phantom, "\Downarrow L_\varphi"] & & B \end{tikzcd} \qquad \begin{tikzcd} B \arrow[rr, bend left, "R_F", "\shortmid"{marking}] \arrow[rr, bend right, "R_G"', "\shortmid"{marking}] \arrow[rr, phantom, "\Uparrow R_\varphi"] & & A \end{tikzcd}$$

Commutativity up to an invertible 2-cell

The multiplication in $\mathbf{Cat}$ is commutative up to an isomorphism

$$\begin{tikzcd} \textbf{Sym}\,A \times \textbf{Sym}\,A \arrow[rr, "(21)"] \arrow[dr, "\otimes_A"'] & \arrow[d, phantom, "\overset{\gamma}{\Longrightarrow}"] & \textbf{Sym}\,A \times \textbf{Sym}\,A \arrow[dl, "\otimes_A"] \\ & \textbf{Sym}A & \end{tikzcd}$$

Hence, the comultiplication in $\mathbf{Dist}$ is commutative up to an isomorphism

$$\begin{tikzcd} & \textbf{Sym}\,A \arrow[dl, "d_A"'] \arrow[dr, "d_A"] \arrow[d, phantom, "\overset{\gamma}{\Longrightarrow}"] & \\ \textbf{Sym}\,A \otimes \textbf{Sym}\,A \arrow[rr, "(21)"'] & & \textbf{Sym}\,A \otimes \textbf{Sym}\,A \end{tikzcd}$$

An apparent obstruction

In contrast to what happens with $\mathbf{Dist}$, a natural transformation in $\mathbf{Cat}$

$$\begin{tikzcd} A \arrow[rr, bend left, "F"{name=U}] \arrow[rr, bend right, "G"'{name=D}] & & B \arrow[from=U, to=D, Rightarrow, "\varphi"] \end{tikzcd}$$

is **not transported** to a pair of 2-cells in the bicategory $\mathbf{Span(Cat)}$

$$\begin{tikzcd} A \arrow[rr, bend left, "L_F", "\shortmid"{marking}, ""'{name=U}] \arrow[rr, bend right, "L_G"', "\shortmid"{marking}, ""{name=D}] & & B \arrow[from=U, to=D, Rightarrow, "L_\varphi"] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[rr, bend left, "R_F", "\shortmid"{marking}, ""'{name=U}] \arrow[rr, bend right, "R_G"', "\shortmid"{marking}, ""{name=D}] & & A \arrow[from=D, to=U, Rightarrow, "R_\varphi"'] \end{tikzcd}$$

Resolving the obstruction up to homotopy

However, every natural isomorphism in $\mathbf{Cat}$

$$\begin{tikzcd} A \arrow[rr, bend left, "F"] \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\Downarrow\varphi"] & & B \end{tikzcd}$$

is transported to a pair of **cospans of simulations**

$$L_F \xRightarrow{inl} \widetilde{L}_\varphi \xLeftarrow{inr} L_G \qquad\qquad R_F \xRightarrow{inl} \widetilde{R}_\varphi \xLeftarrow{inr} R_G$$

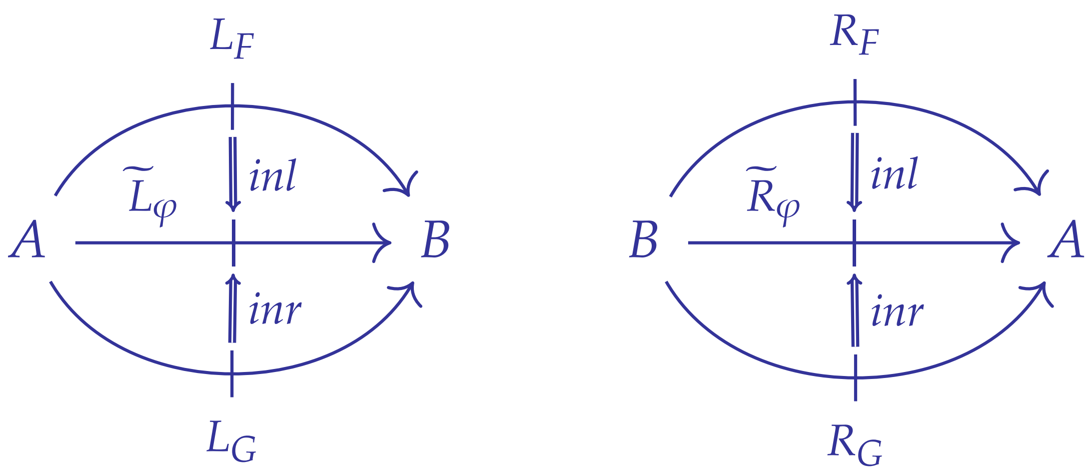
each of them defining a **cospan of 2-cells** in the bicategory $\mathbf{SpanCat}$.

Resolving the obstruction up to homotopy

However, every natural isomorphism in $\mathbf{Cat}$

$$\begin{tikzcd} A \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\varphi"] & B \end{tikzcd}$$

is transported to a pair of **cospans of simulations**



each of them defining a **cospan of 2-cells** in the bicategory $\mathbf{SpanCat}$.

Resolving the obstruction up to homotopy

These **cospans** of 2-cells in $\mathbf{SpanCat}$

$$L_F \xRightarrow{inl} \widetilde{L}_\varphi \xLeftarrow{inr} L_G \qquad\qquad R_F \xRightarrow{inl} \widetilde{R}_\varphi \xLeftarrow{inr} R_G$$

are defined as the following simulations

$$\begin{tikzcd}
 & A \arrow[dl, "id_A"'] \arrow[d, "inl"] \arrow[dr, "F"] & \\
A & Cyl(A) \arrow[l, "proj"'] \arrow[r, "\varphi"] & B \\
 & A \arrow[ul, "id_A"] \arrow[u, "inr"'] \arrow[ur, "G"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[dl, "F"'] \arrow[d, "inl"] \arrow[dr, "id_A"] & \\
B & Cyl(A) \arrow[l, "\varphi"'] \arrow[r, "proj"] & A \\
 & A \arrow[ul, "G"] \arrow[u, "inr"'] \arrow[ur, "id_A"'] &
\end{tikzcd}$$

Resolving the obstruction up to homotopy

Here, $Cyl(A)$ denotes the **cylinder category** defined as

$$Cyl(A) \quad = \quad \mathbb{J} \times A$$

where the **interval category** $\mathbb{J}$ is the category

$$0 \xrightarrow{\ j\ } 1$$

with two objects 0 and 1 and an isomorphism $j : 0 \to 1$ between them.

The category $\mathbb{J}$ comes equipped with three functors

$$\begin{tikzcd} \mathbf{1} \arrow[r, shift left, "0"] \arrow[r, shift right, "1"'] & \mathbb{J} \arrow[r, "p"] & \mathbf{1} \end{tikzcd}$$

Resolving the obstruction up to homotopy

The three functors

$$\begin{tikzcd} A & Cyl(A) = A \times \mathbb{J} \arrow[l, shift right, "inl"'] \arrow[l, shift left, "inr"] & A \arrow[l, "proj"'] \end{tikzcd}$$

are deduced from the three functors

$$\begin{tikzcd} \mathbf{1} \arrow[r, shift left, "0"] \arrow[r, shift right, "1"'] & \mathbb{J} \arrow[r, "p"] & \mathbf{1} \end{tikzcd}$$

in the expected way:

$$inl = 0 \times A \qquad inr = 1 \times A \qquad proj = p \times A.$$

Resolving the obstruction up to homotopy

The two functorial spans

$$\widetilde{L}_\varphi \;:\; A \longrightarrow\!\!\!\!\!\!\!+\;\; B \qquad\qquad \widetilde{R}_\varphi \;:\; B \longrightarrow\!\!\!\!\!\!\!+\;\; A$$

are defined as

$$A \xleftarrow{\;proj\;} Cyl(A) \xrightarrow{\;\varphi\;} B \qquad B \xleftarrow{\;\varphi\;} Cyl(A) \xrightarrow{\;proj\;} A$$

where the functor

$$\varphi \;:\; Cyl(A) \longrightarrow B$$

internalizes the natural isomorphism $\varphi : F \Rightarrow G : A \to B$ and thus satisfies:

$$F = \varphi \circ inl \qquad\qquad G = \varphi \circ inr$$

required for the functors inl and inr to define simulations.

The exponential modality

The exponential modality

The construction of the exponential modality relies on the fact that

Property. The monad

$$\mathbf{Sym} \quad : \quad \mathbf{Cat} \longrightarrow \mathbf{Cat}$$

which associates to every category

$$\mathscr{C} \quad \in \quad \mathbf{Cat}$$

the freely generated symmetric monoidal category

$$\mathbf{Sym}(\mathscr{C}) \quad \in \quad \mathbf{Cat}$$

is a **cartesian monad**.

The exponential modality

From this follows that

Corollary. The monad

$$\mathbf{Sym} \quad : \quad \mathbf{Cat} \longrightarrow \mathbf{Cat}$$

transports the internal category of polarities

$$\pm$$

into an internal category

$$\mathbf{Sym}(\pm)$$

The exponential modality

The objects of

$$\mathbf{Sym}(\pm_{\mathrm{game}})$$

are the finite words

$$\varepsilon_1 \cdots \varepsilon_n$$

on the alphabet with two letters

$$\oplus \qquad \ominus$$

The template of exponential polarities

The category

$$\pm^{!}_{\mathrm{game}}$$

is defined as a the full subcategory of

$$\textbf{Sym}(\pm_{\mathrm{game}})$$

with objects of the form

$$\oplus \cdots \oplus \cdots \oplus$$

containing **only positive polarities**, and objects of the form

$$\oplus \cdots \ominus \cdots \oplus$$

containing **exactly one negative polarity**.

The template of exponential schedules

The internal category

$$\pm^{!}$$

is defined by restricting the internal category

$$\mathbf{Sym}(\pm)$$

to the category of objects $\pm^{!}_{\mathrm{game}}$ using the pullback

$$\begin{tikzcd}
\pm^{!}_{\mathrm{strat}} \arrow[r] \arrow[d] & \mathbf{Sym}(\pm_{\mathrm{strat}}) \arrow[d] \\
\pm^{!}_{\mathrm{game}} \times \pm^{!}_{\mathrm{game}} \arrow[r] & \mathbf{Sym}(\pm_{\mathrm{game}}) \times \mathbf{Sym}(\pm_{\mathrm{game}})
\end{tikzcd}$$

A pair of internal functors

The internal category

$$\pitchfork^{!}$$

comes equipped with a pair of internal functors

$$\mathbf{Sym}(\pitchfork) \xleftarrow{\quad pick \quad} \pitchfork^{!} \xrightarrow{\quad pince \quad} \pitchfork$$

which defines an **exponential modality** of linear logic.

The exponential modality

The exponential of a template game

$$A \xrightarrow{\lambda_A} \pm_{\text{game}}$$

is simply computed by pullback followed by composition:

$$\begin{tikzcd}
\textbf{Sym}(A) \arrow[d, "\textbf{Sym}(\lambda_A)"'] & !A \arrow[l] \arrow[d] \arrow[dr, bend left, "\lambda_{!A}"] & \\
\textbf{Sym}(\pm_{\text{game}}) & \pm^{!}_{\text{game}} \arrow[l, "pick"'] \arrow[r, "pince"] & \pm_{\text{game}}
\end{tikzcd}$$

Main result

Theorem D. The symmetric monoidal category

$$\mathbf{Games}$$

equipped with the exponential modality

$$!$$

defines a bicategorical (homotopy) model of differential linear logic.

Conclusion and perspectives

- games played on **categories** with **synchronous copycats**
- games played on **2-categories** with **asynchronous copycats**
- a number of **different templates** considered already:

$\pitchfork_{\mathrm{alt}}$	alternating games and strategies
$\pitchfork_{\mathrm{asynch}}$	asynchronous games and strategies
$\pitchfork_{\mathrm{span}}$	functorial spans with no scheduling

- a model of **differential linear logic** based on **homotopy theory**
- a model of **concurrent separation logic** based on **cobordisms** and **synchronization on machine states** with Léo Stefanesco.

Short selection of related papers

[1] PAM.
Categorical Combinatorics of Scheduling and Synchronization in Game Semantics.
POPL 2019

[2] PAM.
Template Games and Differential Linear Logic.
LICS 2019

[3] PAM.
Asynchronous Template Games and the Gray Tensor Product of 2-categories
LICS 2021

[4] Clovis Eberhart, Tom Hirschowitz and Alexis Laouar.
Template Games, Simple Games, and Day Convolution.
FSCD 2019

[5] Simon Castellan, Pierre Clairambault and Glynn Winskel.
Thin games with symmetry and concurrent Hyland-Ong games
LMCS 2020

Short selection of related papers

[1] Russ Harmer, Martin Hyland and PAM.
Categorical Combinatorics for Innocent Strategies.
LICS 2007

[2] PAM and Samuel Mimram.
Asynchronous Games: Innocence Without Alternation.
CONCUR 2007

[3] Sylvain Rideau and Glynn Winskel.
Concurrent Strategies.
LICS 2011

[4] PAM and Léo Stefanesco.
An Asynchronous Soundness Theorem for Concurrent Separation Logic.
LICS 2018

[5] PAM and Léo Stefanesco.
Concurrent Separation Logic Meets Template Games.
LICS 2020

Thank you !