Dialogue categories and Frobenius monoids

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Higher topological quantum field theory
and categorical quantum mechanics

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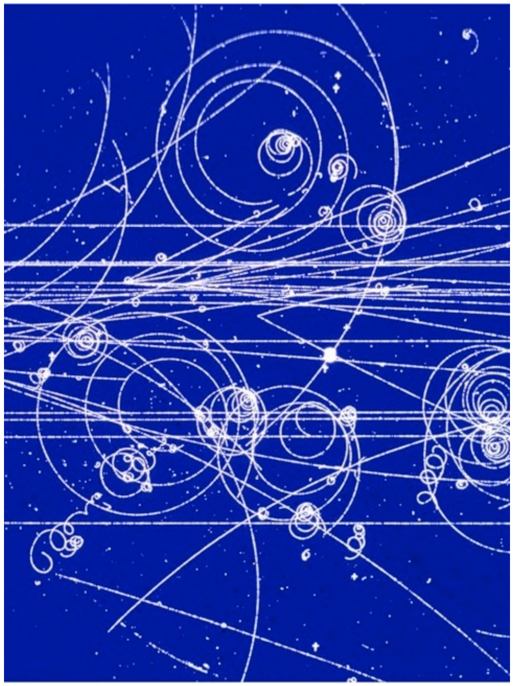
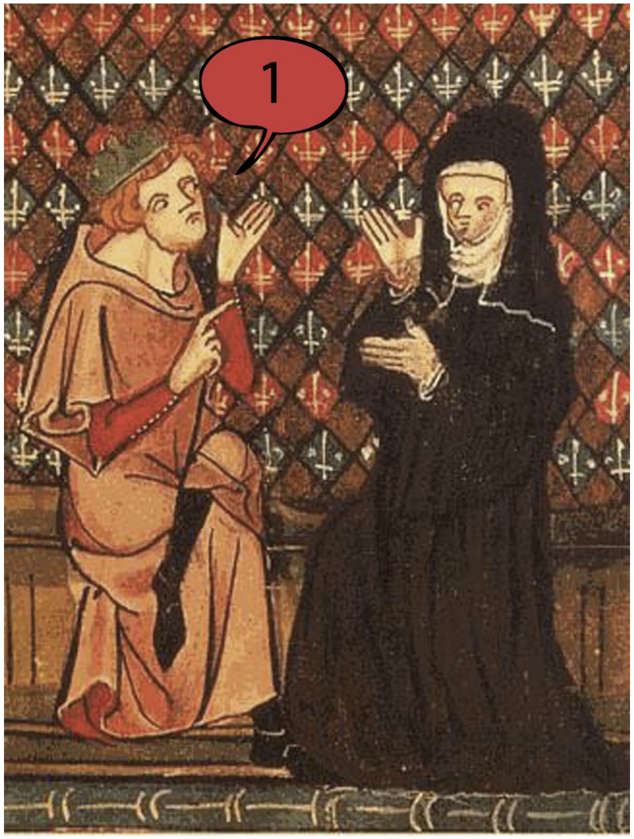
Logic

Physics



Like physics, logic should be the description of **a material event**...

The logical phenomenon



What is the topological structure of a dialogue?

The logical phenomenon

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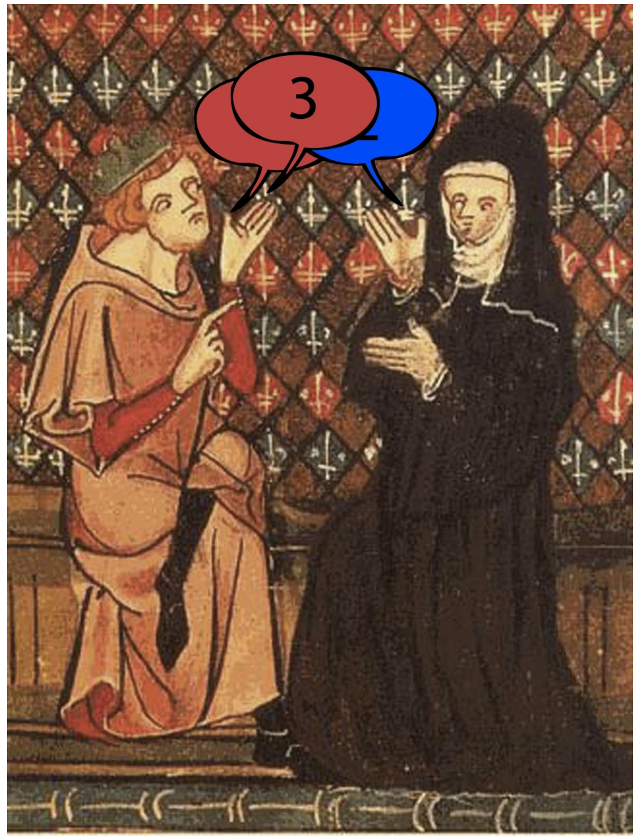
The logical phenomenon



What is the topological structure of a dialogue?

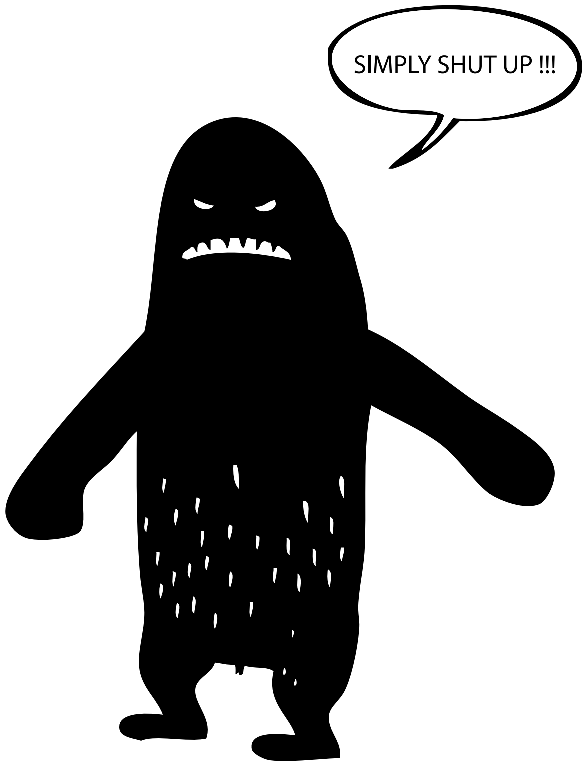
The basic symmetry of logic

The discourse of reason is **symmetric** between Player and Opponent

Claim: this symmetry is the foundation of logic

Next question: can we reconstruct logic from this basic symmetry?

The microcosm principle



No contradiction (thus no formal logic) can emerge in a tyranny...

A microcosm principle in algebra
[Baez & Dolan 1997]

The definition of a **monoid**

$$M \quad \times \quad M \qquad \longrightarrow \qquad M$$

requires the ability to define a **cartesian product** of sets

$$A \quad , \quad B \qquad \mapsto \qquad A \times B$$

Structure at dimension 0 requires structure at dimension 1

A microcosm principle in algebra [**Baez** & **Dolan** **1997**]

The definition of a **cartesian** category

$$\mathscr{C} \quad \times \quad \mathscr{C} \qquad \longrightarrow \quad \mathscr{C}$$

requires the ability to define a **cartesian product** of categories

$$\mathscr{A} \quad , \quad \mathscr{B} \qquad \mapsto \qquad \mathscr{A} \times \mathscr{B}$$

Structure at dimension 1 requires structure at dimension 2

A similar microcosm principle in logic

The definition of a cartesian **closed** category

$$\mathscr{C}^{op} \times \mathscr{C} \longrightarrow \mathscr{C}$$

requires the ability to define the **opposite** of a category

$$\mathscr{A} \mapsto \mathscr{A}^{op}$$

Hence, the “implication” at level 1 requires a “negation” at level 2

An automorphism in <u>Cat</u>

The 2-functor

$$op \quad : \quad \underline{Cat} \quad \longrightarrow \quad \underline{Cat}^{\,op(2)}$$

transports every natural transformation

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & \mathscr{D} \arrow[Rightarrow, from=U, to=D, "\theta"] \end{tikzcd}$$

to a natural transformation in the opposite direction:

$$\begin{tikzcd} \mathscr{C}^{\,op} \arrow[r, bend left=50, "F^{\,op}"{name=U}] \arrow[r, bend right=50, "G^{\,op}"'{name=D}] & \mathscr{D}^{\,op} \arrow[Rightarrow, from=D, to=U, "\theta^{\,op}"'] \end{tikzcd}$$

$\longrightarrow$ requires a braiding on $\mathscr{V}$ in the case of $\mathscr{V}$-enriched categories

Chiralities

A symmetrized account of categories

From categories to chiralities

A slightly bizarre idea emerges in order to reflect the symmetry of logic:

decorrelate the category $\mathscr{C}$ from its opposite category $\mathscr{C}^{op}$

So, let us define a **chirality** as a pair of categories $(\mathscr{A}, \mathscr{B})$ such that

$$\mathscr{A} \cong \mathscr{C} \qquad\qquad \mathscr{B} \cong \mathscr{C}^{op}$$

for some category $\mathscr{C}$.

Here $\cong$ means **equivalence** of category

Chirality

More formally:

Definition:

A chirality is a pair of categories $(\mathscr{A}, \mathscr{B})$ equipped with an equivalence:

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "{}^*(-)"] \arrow[r, phantom, "\mathit{equivalence}"] & \mathscr{B}^{op} \arrow[l, bend left, "(-)^*"] \end{tikzcd}$$

Chirality homomorphisms

Definition. A chirality homomorphism

$$(\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is a pair of functors

$$F_\bullet \quad : \quad \mathscr{A}_1 \quad \longrightarrow \quad \mathscr{A}_2 \qquad\qquad\qquad F_\circ \quad : \quad \mathscr{B}_1 \quad \longrightarrow \quad \mathscr{B}_2$$

equipped with a natural isomorphism

$$\begin{tikzcd}
\mathscr{A}_1 \arrow[r, "F_\bullet"] \arrow[d, "{}^*(-)"'] \arrow[dr, phantom, "\widetilde{F}\ \Swarrow"] & \mathscr{A}_2 \arrow[d, "{}^*(-)"] \\
\mathscr{B}_1^{op} \arrow[r, "F_\circ^{op}"'] & \mathscr{B}_2^{op}
\end{tikzcd}$$

Chirality transformations

Definition. A chirality transformation

$$\theta \quad : \quad F \quad \Rightarrow \quad G \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is a pair of natural transformations

$$\begin{tikzcd} \mathscr{A}_1 \arrow[r, bend left, "F_\bullet"] \arrow[r, bend right, "G_\bullet"'] \arrow[r, phantom, "\Downarrow \theta_\bullet"] & \mathscr{A}_2 \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathscr{B}_1 \arrow[r, bend left, "F_\circ"] \arrow[r, bend right, "G_\circ"'] \arrow[r, phantom, "\Uparrow \theta_\circ"] & \mathscr{B}_2 \end{tikzcd}$$

Chirality transformations

satisfying the equality

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathscr{A}_1 \arrow[r, bend left, "F_\bullet"] \arrow[r, bend right, "G_\bullet"'] \arrow[r, phantom, "\Downarrow \theta_\bullet"] \arrow[d, "{}^*(-)"'] & \mathscr{A}_2 \arrow[d, "{}^*(-)"] \\
\mathscr{B}_1^{op} \arrow[r, bend right, "G_\circ^{op}"'] \arrow[ur, phantom, "\widetilde{G}"] & \mathscr{B}_2^{op}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathscr{A}_1 \arrow[r, bend left, "F_\bullet"] \arrow[d, "{}^*(-)"'] & \mathscr{A}_2 \arrow[d, "{}^*(-)"] \\
\mathscr{B}_1^{op} \arrow[r, bend left, "F_\circ^{op}"] \arrow[r, bend right, "G_\circ^{op}"'] \arrow[r, phantom, "\Downarrow \theta_\circ^{op}"] \arrow[ur, phantom, "\widetilde{F}"] & \mathscr{B}_2^{op}
\end{tikzcd}
$$

A technical justification of symmetrization

Let Chir denote the 2-category with

- chiralities as objects
- chirality homomorphism as 1-dimensional cells
- chirality transformations as 2-dimensional cells

Proposition. The 2-category Chir is biequivalent to the 2-category Cat.

Cartesian closed chiralities

A symmetrized account of cartesian closed categories

Cartesian chiralities

Definition. A cartesian chirality is a chirality

- whose category $\mathscr{A}$ has finite products noted

$$a_1 \wedge a_2 \qquad\qquad \mathbf{true}$$

- whose category $\mathscr{B}$ has finite sums noted

$$b_1 \vee b_2 \qquad\qquad \mathbf{false}$$

Cartesian closed chiralities

Definition. A cartesian closed chirality is a cartesian chirality

$$(\mathscr{A}, \wedge, \mathbf{true}) \qquad\qquad (\mathscr{B}, \vee, \mathbf{false})$$

equipped with a pseudo-action

$$\vee \quad : \quad \mathscr{B} \quad \times \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{A}$$

and a bijection

$$\mathscr{A}(a_1 \wedge a_2, a_3) \quad \cong \quad \mathscr{A}(a_1, a_2^* \vee a_3)$$

natural in a_1, a_2 and a_3.

Once symmetrized, the definition of a ccc becomes purely algebraic

Dictionary

The pseudo-action

$$\vee \quad : \quad \mathscr{B} \quad \times \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{A}$$

reflects the functor

$$\Rightarrow \quad : \quad \mathscr{C}^{\,op} \quad \times \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

The isomorphisms defining the pseudo-action

$$(b_1 \vee b_2) \vee a \quad \cong \quad b_1 \vee (b_2 \vee a) \qquad\qquad \mathbf{false} \vee a \quad \cong \quad a$$

reflect the familiar isomorphisms

$$(x_1 \times x_2) \Rightarrow y \quad \cong \quad x_1 \Rightarrow (x_2 \Rightarrow y) \qquad\qquad 1 \Rightarrow x \quad \cong \quad x$$

Dictionary continued

The isomorphism

$$\mathscr{A}(a_1 \wedge a_2, a_3) \quad \cong \quad \mathscr{A}(a_2, a_1^* \vee a_3)$$

reflects the familiar isomorphism

$$\mathscr{A}(x \times y, z) \quad \cong \quad \mathscr{A}(y, x \Rightarrow z)$$

Note that the isomorphism

$$(a_1)^* \vee a_2 \qquad \cong \qquad a_1 \Rightarrow a_2$$

deserves the name of **classical decomposition** of the implication...
although we are in a cartesian closed category!

Dictionary continued

So, what distinguishes classical logic from intuitionistic logic...
are not the connectives themselves, but their algebraic structure.

Typically, the disjunction $\vee$ is:

- a pseudo-action in the case of cartesian closed chiralities,
- a cotensor product $\parr$ in the case of linear logic,
- a tensor product $\otimes$ in the case of pivotal categories.

Tensorial logic

A primitive logic of tensor and negation

Purpose of tensorial logic

To provide a clear type-theoretic foundation to game semantics

Propositions as types $\Leftrightarrow$ Propositions as games

based on the idea that

$$\boxed{\textbf{game semantics is a diagrammatic syntax of negation}}$$

Double negation monad

Captures the difference between addition as a **function**

$$\mathtt{nat} \times \mathtt{nat} \quad \Rightarrow \quad \mathtt{nat}$$

and addition as a **sequential algorithm**

$$(\mathtt{nat} \Rightarrow \bot) \Rightarrow \bot \quad \times \quad (\mathtt{nat} \Rightarrow \bot) \Rightarrow \bot \quad \times \quad (\mathtt{nat} \Rightarrow \bot) \quad \Rightarrow \quad \bot$$

This enables to distinguish the **left-to-right** implementation

$$\mathtt{lradd} \quad = \quad \lambda\varphi.\,\lambda\psi.\,\lambda k.\,\varphi\,(\,\lambda x.\,\psi\,(\,\lambda y.\,k\,(x+y))\,)$$

from the **right-to-left** implementation

$$\mathtt{rladd} \quad = \quad \lambda\varphi.\,\lambda\psi.\,\lambda k.\,\psi\,(\,\lambda y.\,\varphi\,(\,\lambda x.\,k\,(x+y))\,)$$

The left-to-right addition

$\neg\neg$ `nat`	$\times$	$\neg\neg$ `nat`	$\Rightarrow$	$\neg\neg$ `nat`
				`question`
`question`				
12				
		`question`		
		5		
				17

$$\texttt{lradd} \quad = \quad \lambda\varphi.\,\lambda\psi.\,\lambda k.\,\varphi\,(\,\lambda x.\,\psi\,(\,\lambda y.\,k\,(x+y))\,)$$

The right-to-left addition

$\neg\neg$ nat $\times$ $\neg\neg$ nat $\Rightarrow$ $\neg\neg$ nat

question

question

5

question

12

17

$$\texttt{rladd} \;=\; \lambda\varphi.\,\lambda\psi.\,\lambda k.\,\psi\,(\lambda y.\,\varphi\,(\lambda x.\,k\,(x+y)))$$

Tensorial logic

tensorial logic	=	a logic of tensor and negation
	=	linear logic without $A \cong \neg\neg A$
	=	the syntax of tensorial negation
	=	the syntax of dialogue games

Tensorial logic

- Every sequent of the logic is of the form:

$$\boxed{A_1 , \cdots , A_n \quad \vdash \quad B}$$

- Main rules of the logic:

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad\qquad \frac{\Gamma , A , B , \Delta \vdash C}{\Gamma , A \otimes B , \Delta \vdash C}$$

$$\frac{\Gamma , A \vdash \bot}{\Gamma \vdash \neg A} \qquad\qquad \frac{\Gamma \vdash A}{\Gamma , \neg A \vdash \bot}$$

The primitive kernel of logic

The left-to-right scheduler

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A} \quad \dfrac{}{B \vdash B}}{A , B \vdash A \otimes B}\;\text{Right} \otimes}{B , \neg (A \otimes B) , A \vdash}\;\text{Left} \neg}{\neg (A \otimes B) , A \vdash \neg B}\;\text{Right} \neg}{A , \neg\neg B , \neg (A \otimes B) \vdash}\;\text{Left} \neg}{\neg\neg B , \neg (A \otimes B) \vdash \neg A}\;\text{Right} \neg}{\neg (A \otimes B) , \neg\neg A , \neg\neg B \vdash}\;\text{Left} \neg}{\neg\neg A , \neg\neg B \vdash \neg\neg (A \otimes B)}\;\text{Right} \neg}{\neg\neg A \otimes \neg\neg B \vdash \neg\neg (A \otimes B)}\;\text{Left} \otimes
$$

$$
\texttt{lrsched} \quad = \quad \lambda \varphi . \lambda \psi . \lambda k . \varphi (\lambda x . \psi (\lambda y . k (x, y)))
$$

The left-to-right scheduler

$\neg\neg A$	$\times$	$\neg\neg B$	$\Rightarrow$	$\neg\neg A \otimes B$
				`question`
`question`				
`answer`				
		`question`		
		`answer`		
				`answer`

$$\texttt{lrsched} \quad = \quad \lambda\varphi.\,\lambda\psi.\,\lambda k.\,\varphi\,(\,\lambda x.\,\psi\,(\,\lambda y.\,k\,(x, y))\,)$$

The right-to-left scheduler

$$
\begin{array}{c l}
\dfrac{}{A \vdash A} \quad \dfrac{}{B \vdash B} & \\
\hline
A , B \vdash A \otimes B & \text{Right } \otimes \\
\hline
A , B , \neg (A \otimes B) \vdash & \text{Left } \neg \\
\hline
B , \neg (A \otimes B) \vdash \neg A & \text{Right } \neg \\
\hline
B , \neg (A \otimes B) , \neg\neg A \vdash & \text{Left } \neg \\
\hline
\neg (A \otimes B) , \neg\neg A \vdash \neg B & \text{Right } \neg \\
\hline
\neg (A \otimes B) , \neg\neg A , \neg\neg B \vdash & \text{Left } \neg \\
\hline
\neg\neg A , \neg\neg B \vdash \neg\neg (A \otimes B) & \text{Right } \neg \\
\hline
\neg\neg A \otimes \neg\neg B \vdash \neg\neg (A \otimes B) & \text{Left } \otimes
\end{array}
$$

$$\texttt{rlsched} \quad = \quad \lambda\varphi.\, \lambda\psi.\, \lambda k.\, \psi\,(\,\lambda y.\, \varphi\,(\,\lambda x.\, k\,(x, y))\,)$$

The right-to-left scheduler

$\neg\neg A$	$\times$	$\neg\neg B$	$\Rightarrow$	$\neg\neg A \otimes B$
				question
		question		
		answer		
question				
answer				
				answer

$$\texttt{rlsched} \quad = \quad \lambda\varphi.\,\lambda\psi.\,\lambda k.\,\psi\,(\,\lambda y.\,\varphi\,(\,\lambda x.\,k\,(x,y))\,)$$

Dialogue categories

A functorial bridge between proofs and knots

Dialogue categories

A **monoidal category** with a **left duality**

A natural bijection between the set of maps

$$A \otimes B \quad \longrightarrow \quad \perp$$

and the set of maps

$$B \quad \longrightarrow \quad A \multimap \perp$$

A familiar situation in tensorial algebra

Dialogue categories

A **monoidal category** with a **right duality**

A natural bijection between the set of maps

$$A \otimes B \quad \longrightarrow \quad \perp$$

and the set of maps

$$A \quad \longrightarrow \quad \perp \multimapinv B$$

A familiar situation in tensorial algebra

Dialogue categories

Definition. A dialogue category is a monoidal category $\mathscr{C}$ equipped with

▷ an object $\perp$

▷ two natural bijections

$$\begin{array}{ccccc} \varphi_{A,B} & : & \mathscr{C}(A \otimes B, \perp) & \longrightarrow & \mathscr{C}(B, A \multimap \perp) \\ \psi_{A,B} & : & \mathscr{C}(A \otimes B, \perp) & \longrightarrow & \mathscr{C}(A, \perp \multimapinv B) \end{array}$$

Pivotal dialogue categories

A dialogue category equipped with a family of bijections

$$wheel_{A,B} \quad : \quad \mathscr{C}(A \otimes B, \perp) \quad \longrightarrow \quad \mathscr{C}(B \otimes A, \perp)$$

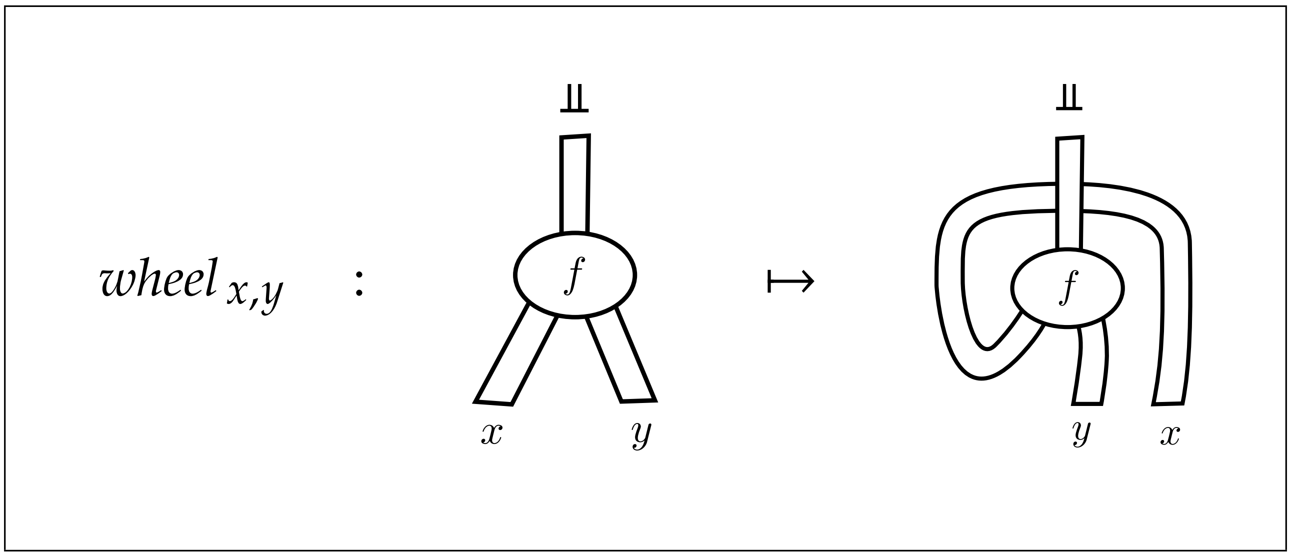
natural in A and B making the diagram

$$\begin{tikzcd}
\mathscr{C}((B \otimes C) \otimes A, \perp) \arrow[rr, "associativity"] & & \mathscr{C}(A \otimes (C \otimes B), \perp) \arrow[d, "wheel_{B,C\otimes A}"] \\
\mathscr{C}(A \otimes (B \otimes C)) \arrow[u, "wheel_{A,B\otimes C}"] \arrow[d, "associativity"] & & \mathscr{C}((C \otimes A) \otimes B, \perp) \\
\mathscr{C}((A \otimes B) \otimes C, \perp) \arrow[rr, "wheel_{A\otimes B,C}"] & & \mathscr{C}(C \otimes (A \otimes B), \perp) \arrow[u, "associativity"]
\end{tikzcd}$$

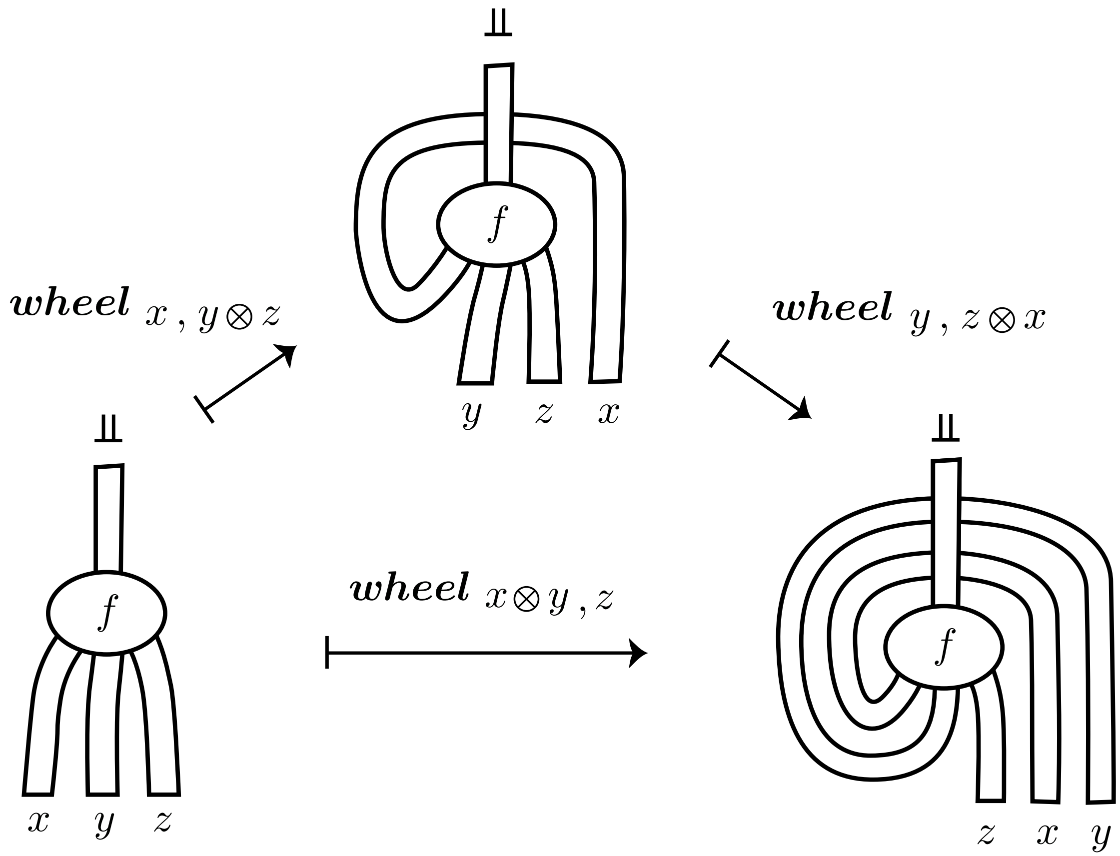
commutes.

Pivotal dialogue categories

The wheel should be understood diagrammatically as:



The coherence diagram



An equivalent formulation

A dialogue category equipped with a natural isomorphism

$$turn_A \quad : \quad A \multimap \perp \quad \longrightarrow \quad \perp \multimapinv A$$

making the diagram below commute:

$$\begin{tikzcd}
 & \perp & \\
(\perp \multimapinv A) \otimes A \arrow[ur, "eval"] & & B \otimes (B \multimap \perp) \arrow[ul, "eval"'] \\
(A \multimap \perp) \otimes A \arrow[u, "turn_A"] & & B \otimes (\perp \multimapinv B) \arrow[u, "turn_B^{-1}"'] \\
B \otimes ((A \otimes B) \multimap \perp) \otimes A \arrow[u, "eval"] \arrow[rr, "turn_{A \otimes B}"] & & B \otimes (\perp \multimapinv (A \otimes B)) \otimes A \arrow[u, "eval"']
\end{tikzcd}$$

Another equivalent formulation

Definition. A pivotal structure is a monoidal natural transformation

$$\tau_A \quad : \quad A \quad \longrightarrow \quad (A \multimap \bot) \multimap \bot$$

such that the composite

$$A \multimap \bot \quad \xrightarrow{\eta_{A\multimap\bot}} \quad \bot \circ\!\!- ((A \multimap \bot) \multimap \bot) \quad \xrightarrow{\tau_A} \quad \bot \circ\!\!- A$$

is an isomorphism for every object A. Hence, the diagram below commutes

$$\begin{tikzcd}
& A \otimes B \arrow[dl, "\tau_A \otimes \tau_B"'] \arrow[dr, "\tau_{A\otimes B}"] & \\
(A \multimap \bot) \multimap \bot \otimes (B \multimap \bot) \multimap \bot \arrow[rr, "m_{A,B}"] & & ((A \otimes B) \multimap \bot) \multimap \bot
\end{tikzcd}$$

and

$$\tau_I = m_I \quad : \quad I \quad \longrightarrow \quad (I \multimap \bot) \multimap \bot$$

The free dialogue category

The objects of the category **free-dialogue**$(\mathscr{C})$ are the **formulas** of tensorial logic:

$$A, B \quad ::= \quad X \quad | \quad A \otimes B \quad | \quad A \multimap \bot \quad | \quad \bot \multimapinv A \quad | \quad 1$$

where X is an object of the category $\mathscr{C}$.

The morphisms are the **proofs** of the logic modulo equality.

A proof-as-tangle theorem

Every category $\mathscr{C}$ of atomic formulas induces a functor $[-]$ such that

$$\begin{tikzcd} \textbf{free-dialogue}(\mathscr{C}) \arrow[rr, "{[-]}"] & & \textbf{free-ribbon}(\mathscr{C}_{\perp}) \\ & \mathscr{C} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

where $\mathscr{C}_{\perp}$ is the category $\mathscr{C}$ extended with an object $\perp$.

Theorem. The functor $[-]$ is faithful.

$\longrightarrow$ a topological foundation for game semantics

An illustration

Imagine that we want to check that the diagram

$$\begin{tikzcd}
\bot \multimapinv (\bot \multimapinv x) \arrow[rr, "\bot \multimapinv turn_{x}"] & & \bot \multimapinv (x \multimap \bot) \\
(\bot \multimapinv x) \multimap \bot \arrow[u, "turn_{\bot \multimapinv x}"] & & \bot \multimapinv (x \multimap \bot) \arrow[u, "twist_{\multimapinv (x \multimap \bot)}"'] \\
& x \arrow[ul, "\eta'"'] \arrow[ur, "\eta"] &
\end{tikzcd}$$

commutes in every balanced dialogue category.

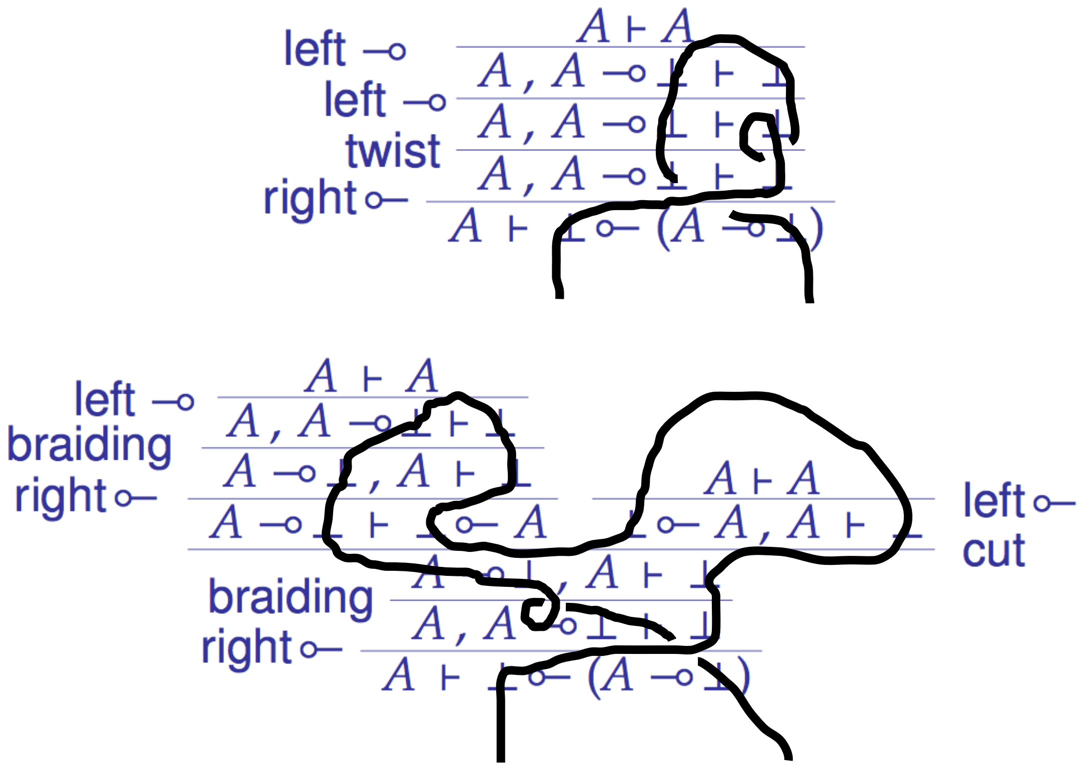
An illustration

Equivalently, we want to check that the two derivation trees are equal:

$$\text{right}\multimapinv\;\dfrac{\text{twist}\;\dfrac{\text{left}\multimap\;\dfrac{\text{left}\multimap\;\dfrac{A \vdash A}{A\,,\, A \multimap \bot \vdash \bot}}{A\,,\, A \multimap \bot \vdash \bot}}{A\,,\, A \multimap \bot \vdash \bot}}{A \vdash \bot \multimapinv (A \multimap \bot)}$$

$$\text{right}\multimapinv\;\dfrac{\text{braiding}\;\dfrac{\dfrac{\text{right}\multimapinv\;\dfrac{\text{braiding}\;\dfrac{\text{left}\multimap\;\dfrac{A \vdash A}{A\,,\, A \multimap \bot \vdash \bot}}{A \multimap \bot\,,\, A \vdash \bot}}{A \multimap \bot \vdash \bot \multimapinv A} \qquad \dfrac{A \vdash A}{\bot \multimapinv A\,,\, A \vdash \bot}\;\text{left}\multimapinv}{A \multimap \bot\,,\, A \vdash \bot}\;\text{cut}}{A\,,\, A \multimap \bot \vdash \bot}}{A \vdash \bot \multimapinv (A \multimap \bot)}$$

An illustration



equality of proofs $\Longleftrightarrow$ equality of tangles

Game semantics in string diagrams

Main theorem

The objects of the free **symmetric** dialogue category are **dialogue games** constructed by the grammar

$$A, B \quad ::= \quad X \quad | \quad A \otimes B \quad | \quad \neg A \quad | \quad 1$$

where X is an object of the category $\mathscr{C}$.

The morphisms are **total** and **innocent strategies** on dialogue games.

As we will see: proofs become 3-dimensional variants of knots...

An algebraic presentation of dialogue categories

Negation defines a pair of **adjoint functors**

$$\begin{tikzcd} \mathscr{C} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathscr{C}^{op} \arrow[ll, bend left, "R"] \end{tikzcd}$$

witnessed by the series of bijection:

$$\mathscr{C}(A, \neg B) \quad \cong \quad \mathscr{C}(B, \neg A) \quad \cong \quad \mathscr{C}^{op}(\neg A, B)$$

An algebraic presentation of dialogue chiralities

The algebraic presentation starts by the pair of **adjoint functors**

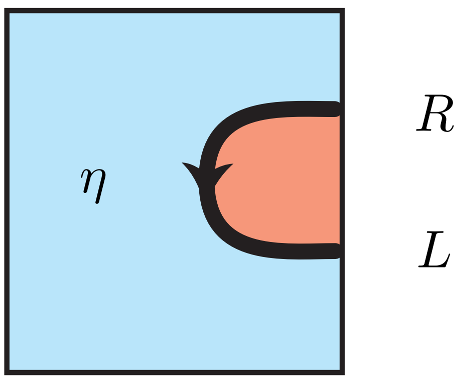
$$\begin{tikzcd} \mathscr{A} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathscr{B} \arrow[ll, bend left, "R"] \end{tikzcd}$$

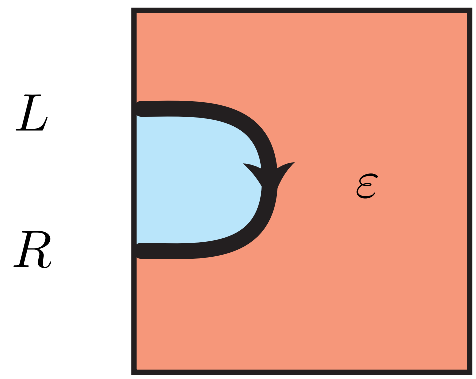
between the two components $\mathscr{A}$ and $\mathscr{B}$ of the dialogue chirality.

The 2-dimensional topology of adjunctions

The **unit** and **counit** of the adjunction $L \dashv R$ are depicted as

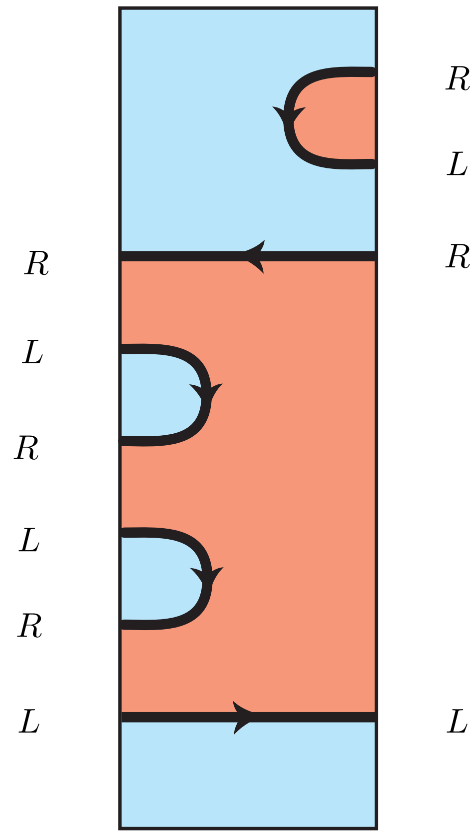
$\eta : Id \longrightarrow R \circ L$ $\varepsilon : L \circ R \longrightarrow Id$





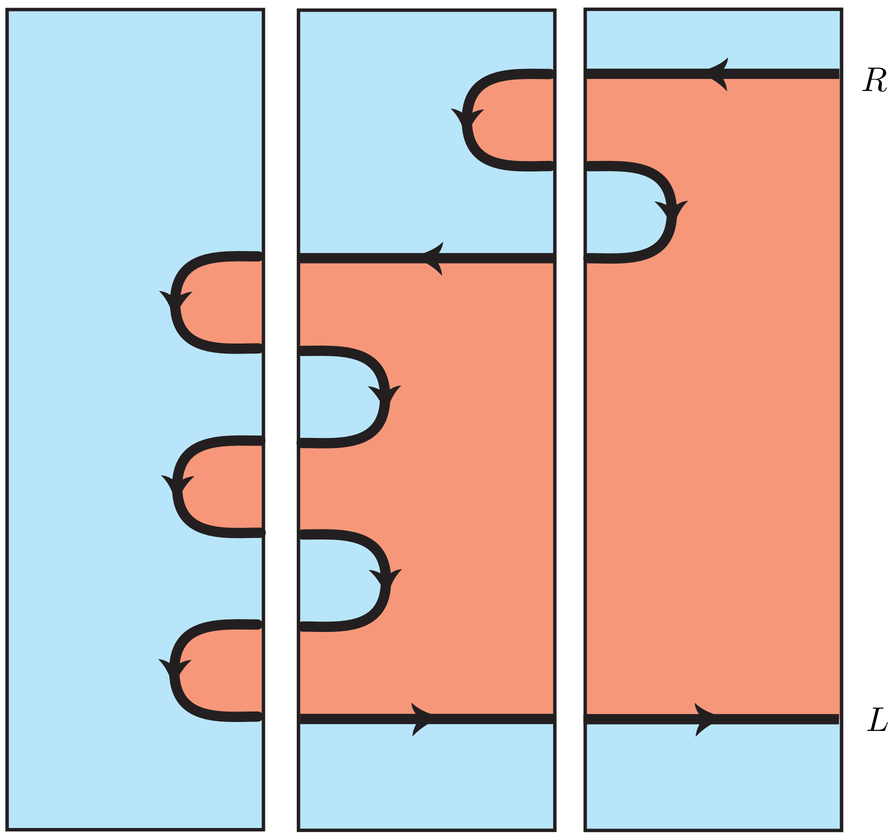
Opponent move = functor R Proponent move = functor L

A typical proof

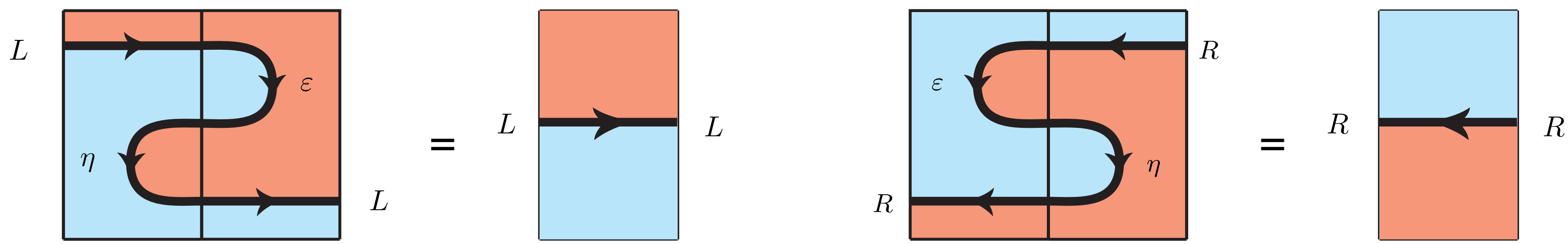


Reveals the algebraic nature of game semantics

A purely diagrammatic cut elimination



The 2-dimensional dynamics of adjunctions



Recovers the usual way to compose strategies in game semantics

When a tensor meets a negation...

The continuation monad is strong

$$\boxed{(\neg\neg A) \otimes B \longrightarrow \neg\neg(A \otimes B)}$$

As Gordon explained, this is the starting point of algebraic effects

Tensor vs. negation

Proofs are generated by a **parametric strength**

$$\kappa_X \quad : \quad \neg\,(X \otimes \neg\,A) \otimes B \;\longrightarrow\; \neg\,(X \otimes \neg\,(A \otimes B))$$

which generalizes the usual notion of **strong monad** :

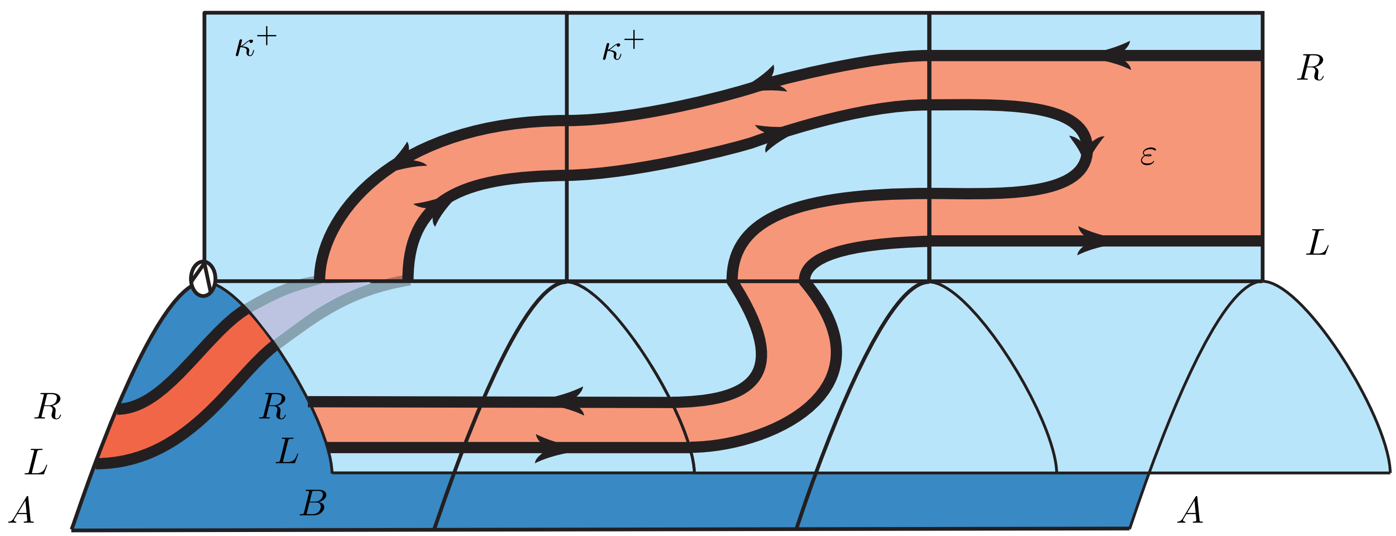
$$\kappa \quad : \quad \neg\neg\,A \otimes B \;\longrightarrow\; \neg\neg\,(A \otimes B)$$

Proofs as 3-dimensional string diagrams

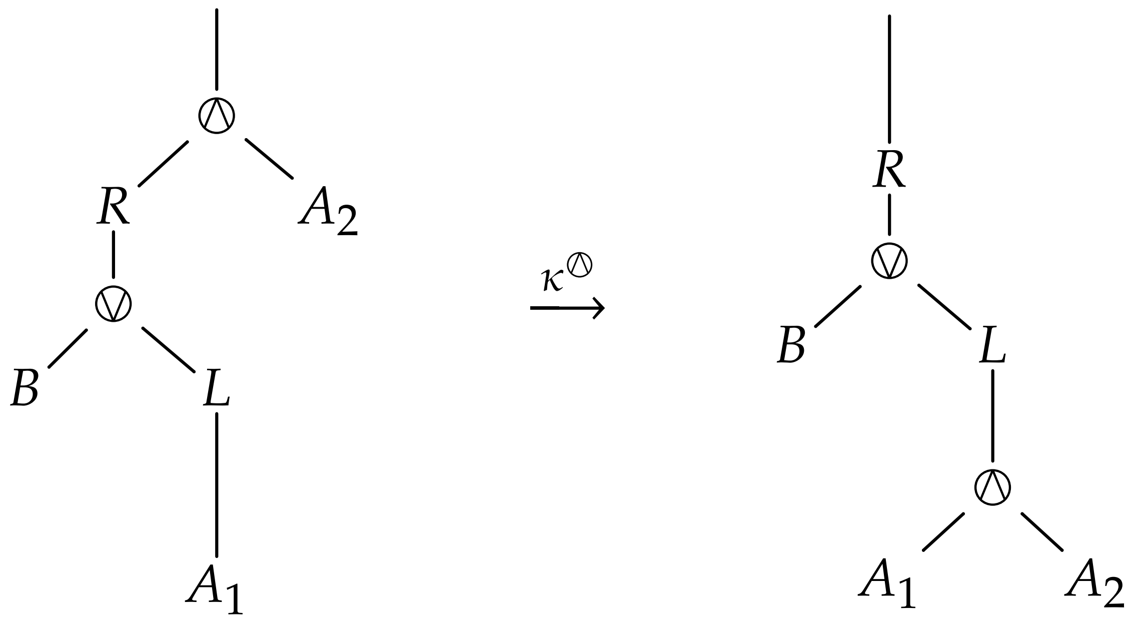
The left-to-right proof of the sequent

$$\neg\neg A \otimes \neg\neg B \quad \vdash \quad \neg\neg(A \otimes B)$$

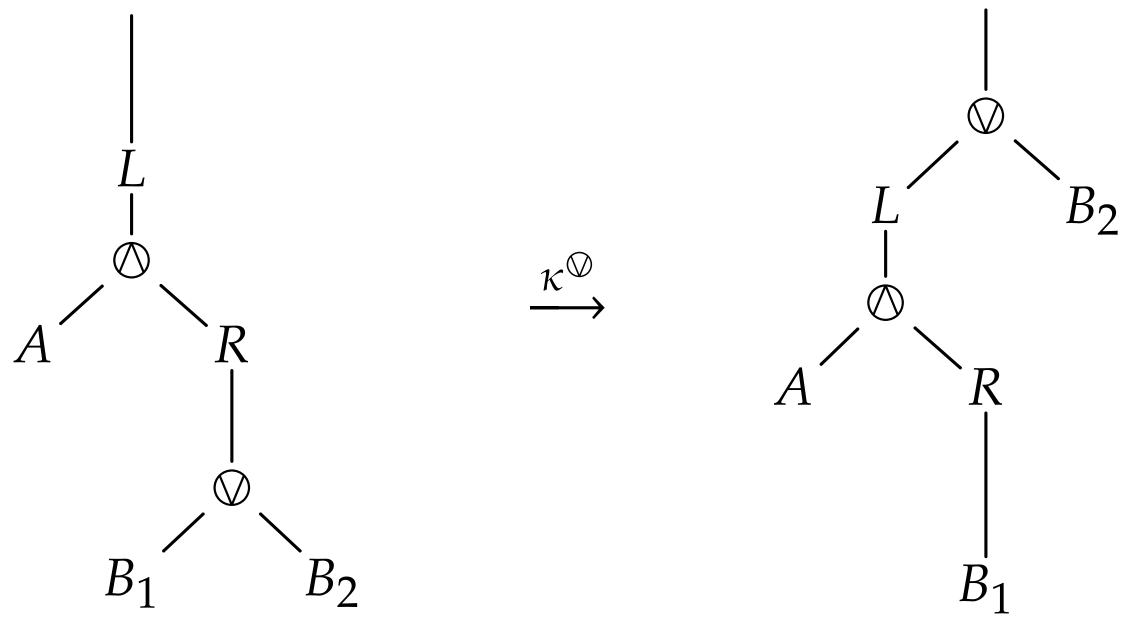
is depicted as



Tensor vs. negation : conjunctive strength



Tensor vs. negation : disjunctive strength



A factorization theorem

The four proofs $\eta, \epsilon, \kappa^{\oslash}$ and $\kappa^{\obslash}$ generate every proof of the logic. Moreover, every such proof

$$X \xrightarrow{\epsilon} \xrightarrow{\kappa^{\oslash}} \xrightarrow{\epsilon} \xrightarrow{\epsilon} \xrightarrow{\eta} \xrightarrow{\eta} \xrightarrow{\kappa^{\obslash}} \xrightarrow{\epsilon} \xrightarrow{\eta} \xrightarrow{\epsilon} \xrightarrow{\kappa^{\obslash}} \xrightarrow{\eta} \xrightarrow{\eta} Z$$

factors uniquely as

$$X \overset{\kappa^{\oslash}}{\twoheadrightarrow} \overset{\epsilon}{\twoheadrightarrow} \overset{\eta}{\twoheadrightarrow} \overset{\kappa^{\obslash}}{\twoheadrightarrow} Z$$

This factorization reflects a Player – Opponent view factorization

Axiom and cut links

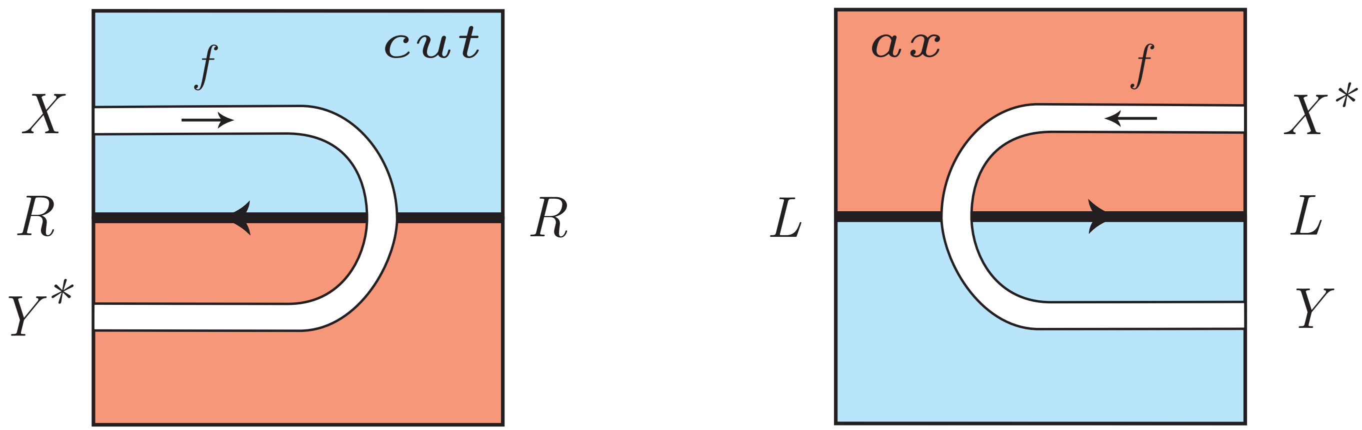
The basic building blocks of linear logic

Axiom and cut links

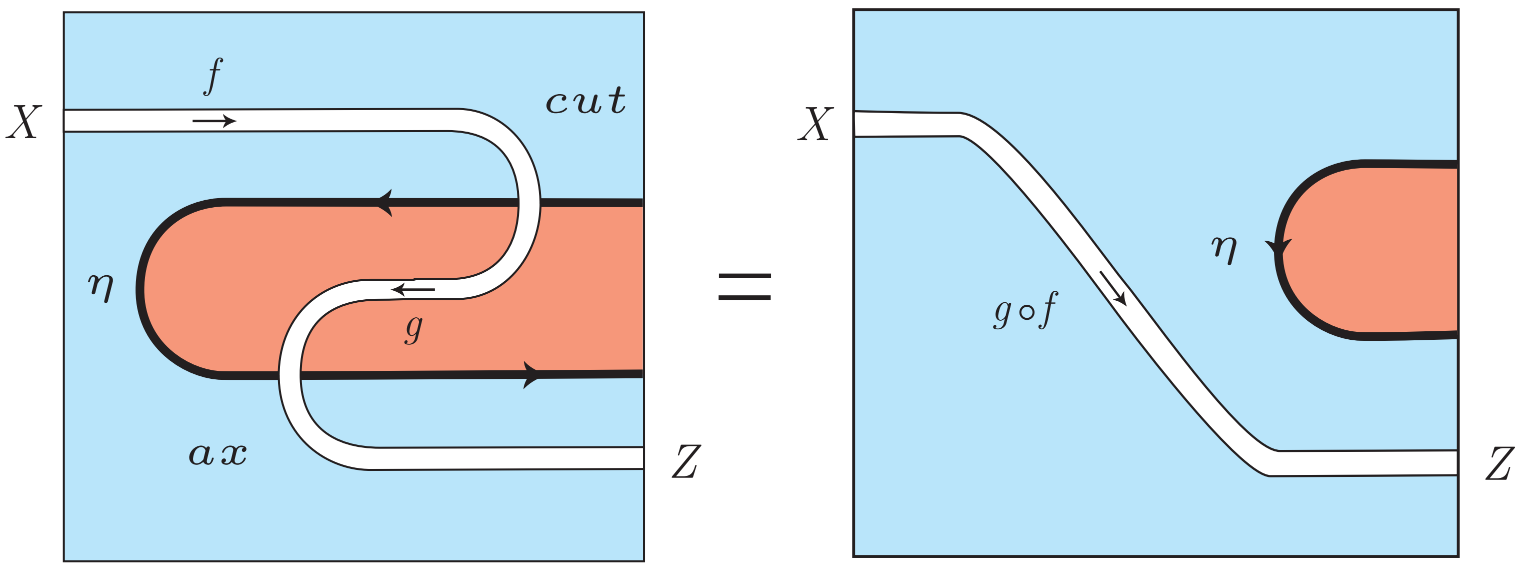
Every map

$$f \quad : \quad X \quad \longrightarrow \quad Y$$

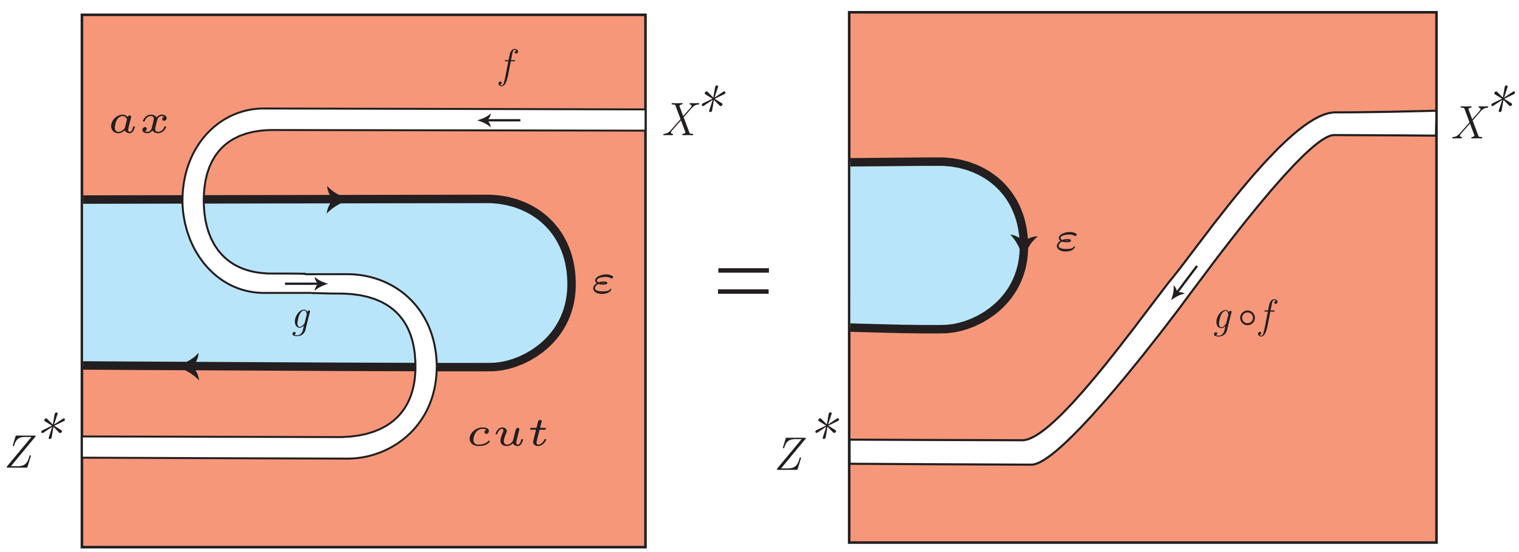
between atoms in the category $\mathscr{C}$ induces an axiom and a cut combinator:



Equalities between axiom and cut links



Equalities between axiom and cut links



Dialogue chiralities

A symmetric account of dialogue categories

Dialogue chiralities

A **dialogue chirality** is a pair of monoidal categories

$$(\mathscr{A}, \owedge, \mathrm{true}) \qquad\qquad (\mathscr{B}, \ovee, \mathrm{false})$$

with a monoidal equivalence

$$\begin{tikzcd} \mathscr{A} \arrow[rr, bend left, "(-)^*"] \arrow[rr, phantom, "\substack{\textit{monoidal}\\ \textit{equivalence}}"] & & \mathscr{B}^{op(0,1)} \arrow[ll, bend left, "(-)^*"] \end{tikzcd}$$

together with an adjunction

$$\begin{tikzcd} \mathscr{A} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathscr{B} \arrow[ll, bend left, "R"] \end{tikzcd}$$

Dialogue chiralities

and two natural bijections

$$\chi^{L}_{m,a,b} \quad : \quad \langle\, m \owedge a \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, a \,|\, m^{*} \ovee b \,\rangle$$

$$\chi^{R}_{m,a,b} \quad : \quad \langle\, a \owedge m \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, a \,|\, b \ovee m^{*} \,\rangle$$

where the evaluation bracket

$$\langle\, - \,|\, - \,\rangle \quad : \quad \mathscr{A}^{\,op} \times \mathscr{B} \quad \longrightarrow \quad Set$$

is defined as

$$\langle\, a \,|\, b \,\rangle \quad := \quad \mathscr{A}(\, a \,,\, Rb \,)$$

Dialogue chiralities

These are required to make the diagrams commute:

$$\begin{tikzcd}
\langle (m \owedge n) \owedge a \,|\, b \rangle \arrow[rr, "\chi^L_{m \owedge n}"] \arrow[d] \arrow[drr, phantom, "{[1]}"] & & \langle a \,|\, (m \owedge n)^* \ovee b \rangle \\
\langle m \owedge (n \owedge a) \,|\, b \rangle \arrow[r, "\chi^L_m"] & \langle n \owedge a \,|\, m^* \ovee b \rangle \arrow[r, "\chi^L_n"] & \langle a \,|\, n^* \ovee (m^* \ovee b) \rangle \arrow[u]
\end{tikzcd}$$

Dialogue chiralities

These are required to make the diagrams commute:

$$
\begin{tikzcd}
\langle\, a \owedge (m \owedge n) \,|\, b \,\rangle \arrow[rr, "\chi^R_{m \owedge n}"] \arrow[d] \arrow[drr, phantom, "{[2]}"] & & \langle\, a \,|\, b \ovee (m \owedge n)^* \,\rangle \\
\langle\, (a \owedge m) \owedge n \,|\, b \,\rangle \arrow[r, "\chi^R_n"] & \langle\, a \owedge m \,|\, b \ovee n^* \,\rangle \arrow[r, "\chi^R_m"] & \langle\, a \,|\, (b \ovee n^*) \ovee m^* \,\rangle \arrow[u]
\end{tikzcd}
$$

Dialogue chiralities

These are required to make the diagrams commute:

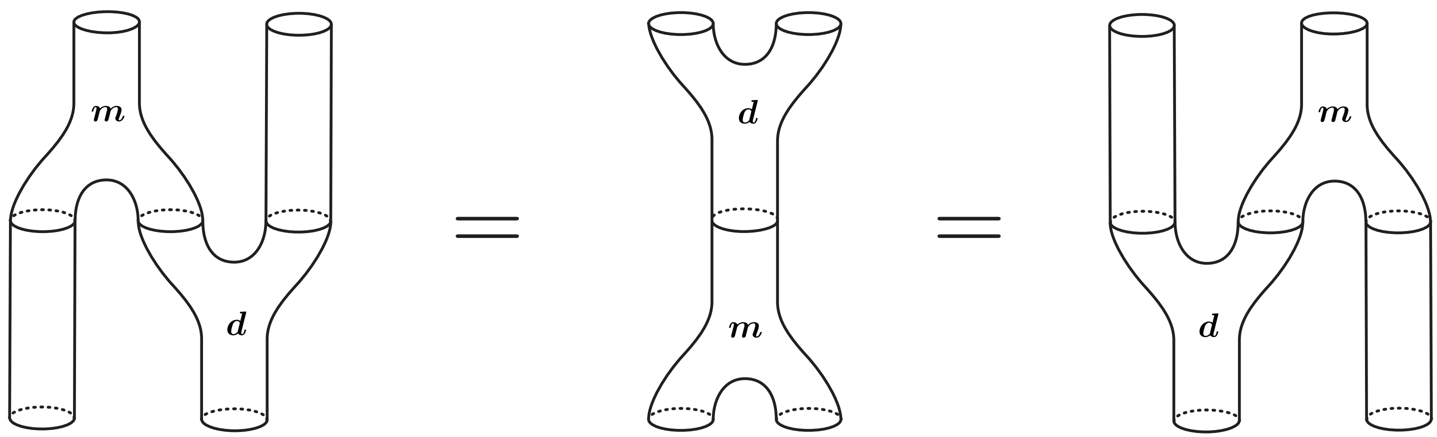
$$
\begin{tikzcd}
\langle (m \owedge a) \owedge n \,|\, b \rangle \arrow[r, "\chi_n^R"] \arrow[d, dash] & \langle m \owedge a \,|\, b \ovee n^* \rangle \arrow[r, "\chi_m^L"] \arrow[d, phantom, "{[3]}"] & \langle a \,|\, m^* \ovee (b \ovee n^*) \rangle \arrow[d, dash] \\
\langle m \owedge (a \owedge n) \,|\, b \rangle \arrow[r, "\chi_m^L"] & \langle a \owedge n \,|\, m^* \ovee b \rangle \arrow[r, "\chi_n^R"] & \langle a \,|\, (m^* \ovee b) \ovee n^* \rangle
\end{tikzcd}
$$

Chiralities as Frobenius monoids

A bialgebraic account of dialogue categories

An observation by Day and Street

A Frobenius monoid F is a monoid and a comonoid satisfying



A surprising relationship with $*$-autonomous categories discovered by Brian Day and Ross Street.

A symmetric presentation of Frobenius algebras

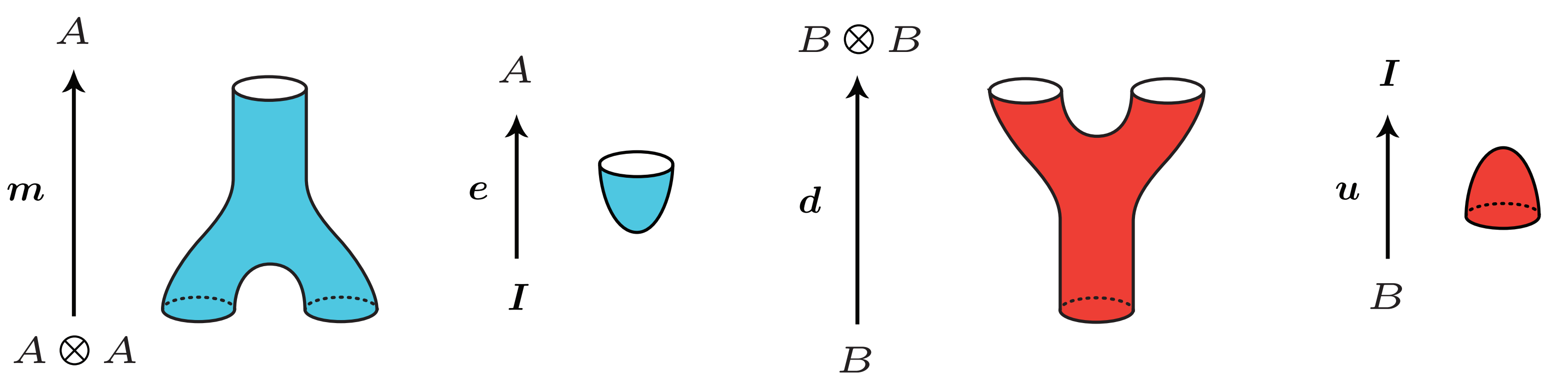
Key idea. Separate the monoid part

$$m : A \otimes A \longrightarrow A \qquad\qquad e : A \otimes A \longrightarrow A$$

from the comonoid part

$$m : B \longrightarrow B \otimes B \qquad\qquad d : B \longrightarrow I$$

in a Frobenius algebra:

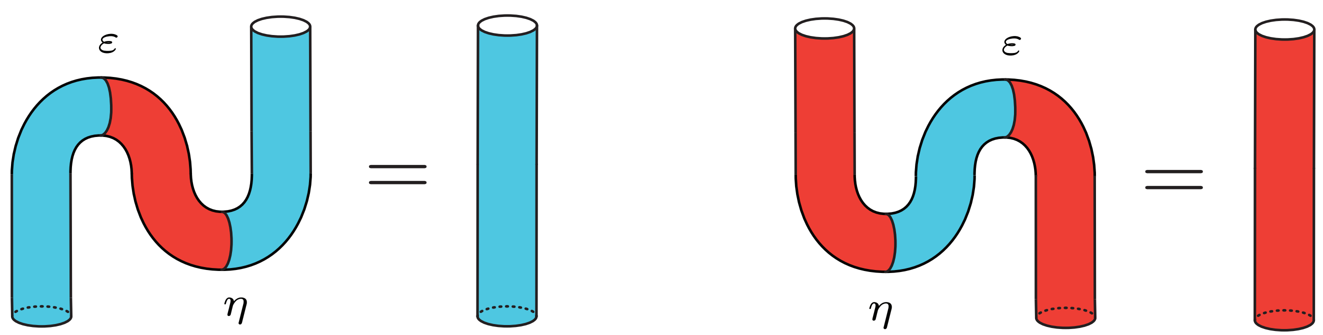


A symmetric presentation of Frobenius algebras

Then, relate A and B by a dual pair

$$\eta \ : \ I \longrightarrow B \otimes A \qquad\qquad \varepsilon \ : \ A \otimes B \longrightarrow I$$

in the sense that:

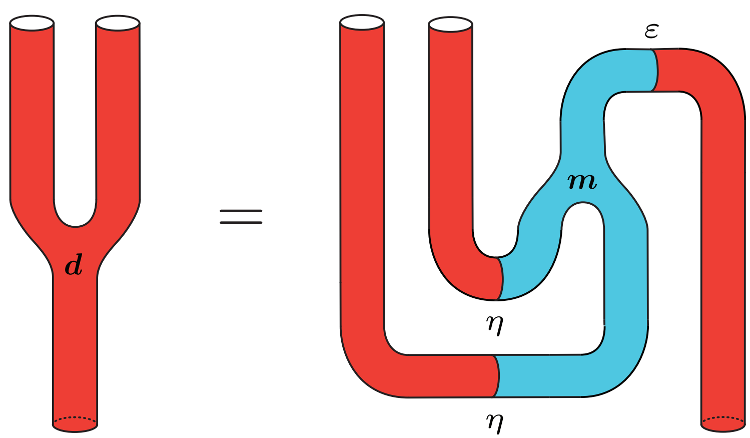


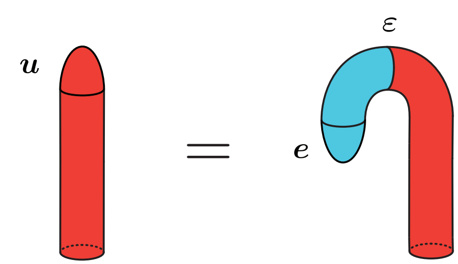
A symmetric presentation of Frobenius algebras

Require moreover that the dual pair

$$(A, m, e) \quad \dashv \quad (B, d, u)$$

relates the algebra structure to the coalgebra structure, in the sense that:



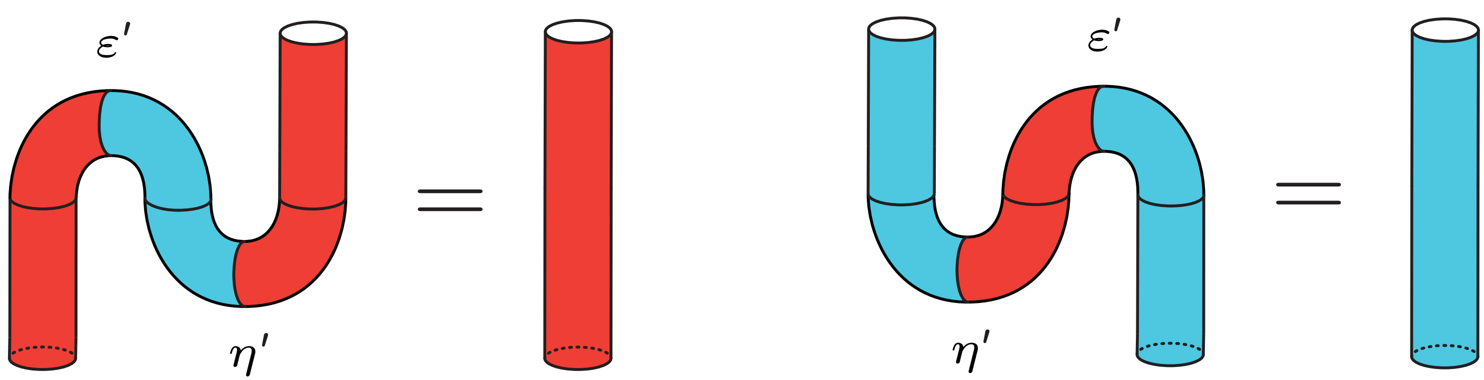


Symmetrically

Relate B and A by a dual pair

$$\eta' \;:\; I \longrightarrow B \otimes A \qquad\qquad \varepsilon' \;:\; A \otimes B \longrightarrow I$$

this meaning that the equations below hold:

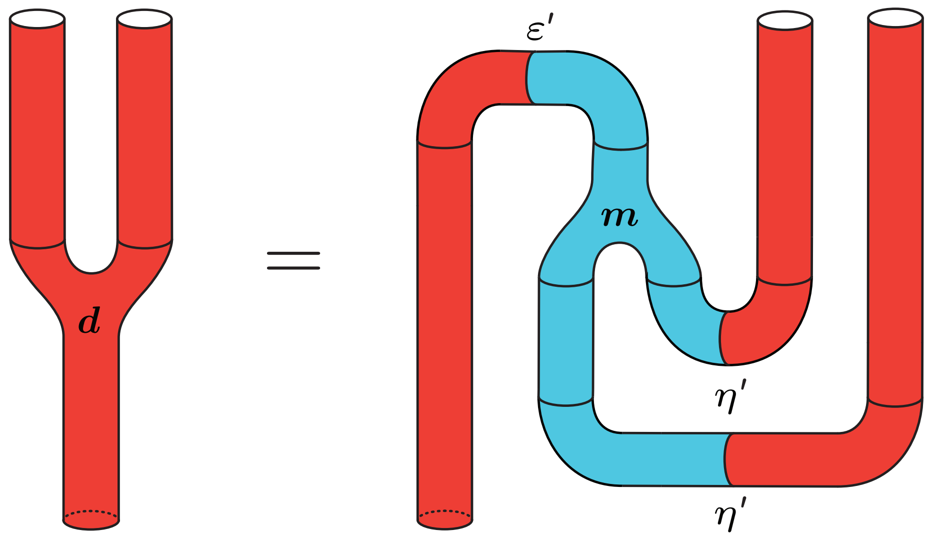


Symmetrically

and ask that the dual pair

$$A \dashv B$$

relates the coalgebra structure to the algebra structure, in the sense that:



An alternative formulation

Key observation:

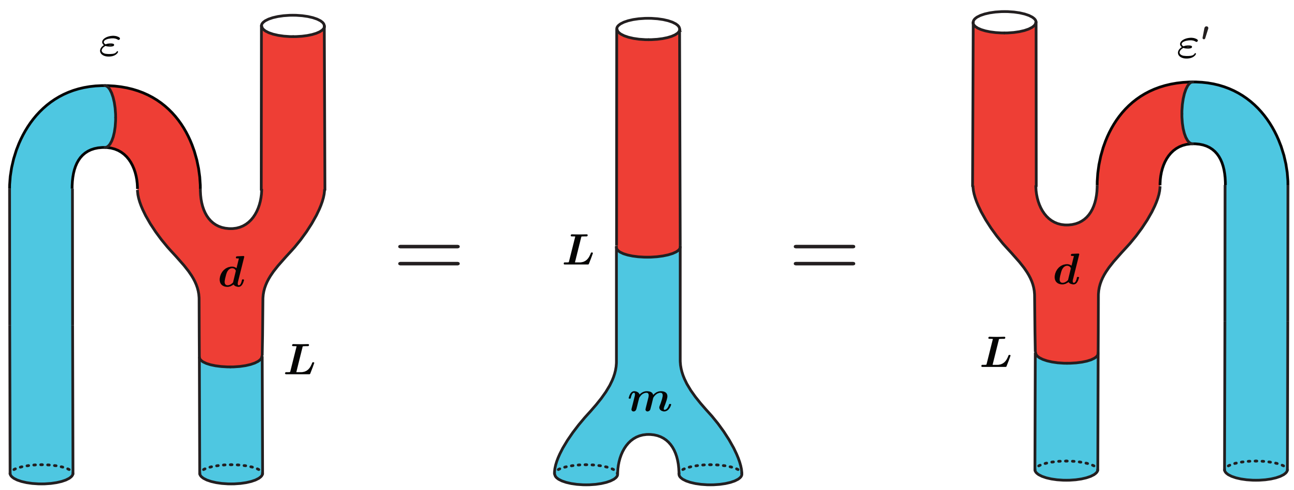
A Frobenius monoid is the same thing as such a pair (A, B) equipped with

$$\begin{tikzcd} A \arrow[r, bend left, "L"] \arrow[r, phantom, "isomorphism"] & B \arrow[l, bend left, "R"] \end{tikzcd}$$

between the underlying spaces A and B and...

Frobenius monoids

... satisfying the two equalities below:



Reminiscent of currification in the λ-calculus...

Not far from the connection, but...

Idea: the « self-duality » of Frobenius monoids

$$\begin{tikzcd} A \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\textit{isomorphism}"] & & B \arrow[ll, bend left, "R"] \end{tikzcd}$$

is replaced by an **adjunction** in dialogue chiralities:

$$\begin{tikzcd} \mathscr{A} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathscr{B} \arrow[ll, bend left, "R"] \end{tikzcd}$$

Key objection: the category $\mathscr{B} \cong \mathscr{A}^{op}$ is not dual to the category $\mathscr{A}$.

Categorical bimodules

A bimodule

$$M \quad : \quad \mathscr{A} \quad \longrightarrow\!\!\!\!\!\!\!\!\!\!+\quad \mathscr{B}$$

between categories $\mathscr{A}$ and $\mathscr{B}$ is defined as a functor

$$M \quad : \quad \mathscr{A}^{op} \times \mathscr{B} \quad \longrightarrow \quad \textbf{Set}$$

Composition of two bimodules

$$\mathscr{A} \xrightarrow[\quad+\quad]{M} \mathscr{B} \xrightarrow[\quad+\quad]{N} \mathscr{C}$$

is defined by the coend formula:

$$M \circledast N \quad : \quad (a,c) \quad \mapsto \quad \int^{b\in\mathscr{B}} M(a,b)\times N(b,c)$$

The coend formula

The coend

$$\int^{b\in\mathscr{B}} M(a,b)\times N(b,c)$$

is defined as the sum

$$\coprod_{b\in ob(\mathscr{B})} M(a,b)\times N(b,c)$$

modulo the equation

$$(x, h\cdot y) \quad \sim \quad (x\cdot h, y)$$

for every triple

$$x\in M(a,b) \qquad h : b \longrightarrow b' \qquad y\in N(b',c)$$

A well-known 2-categorical miracle

Fact. Every category $\mathscr{C}$ comes with a biexact pairing

$$\mathscr{C} \quad \dashv \quad \mathscr{C}^{\,op}$$

defined as the bimodule

$$\mathrm{hom} \quad : \quad (x,y) \quad \mapsto \quad \mathscr{A}(x,y) \quad : \quad \mathscr{C}^{\,op} \times \mathscr{C} \quad \longrightarrow \quad \mathbf{Set}$$

in the bicategory **BiMod** of categorical bimodules.

The opposite category $\mathscr{C}^{\,op}$ becomes <u>dual</u> to the category $\mathscr{C}$

Biexact pairing

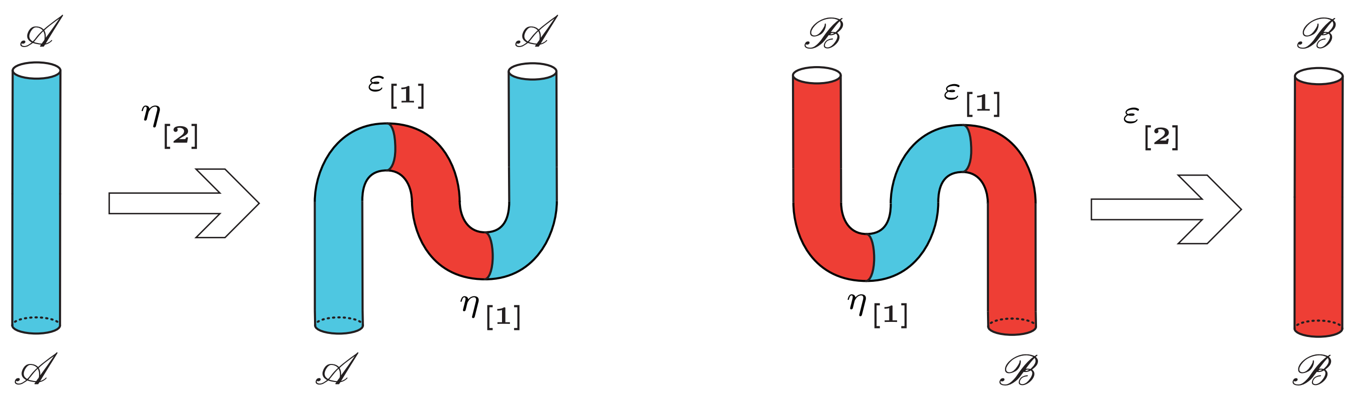
Definition. A biexact pairing

$$\mathscr{A} \dashv \mathscr{B}$$

in a monoidal bicategory is a pair of 1-dimensional cells

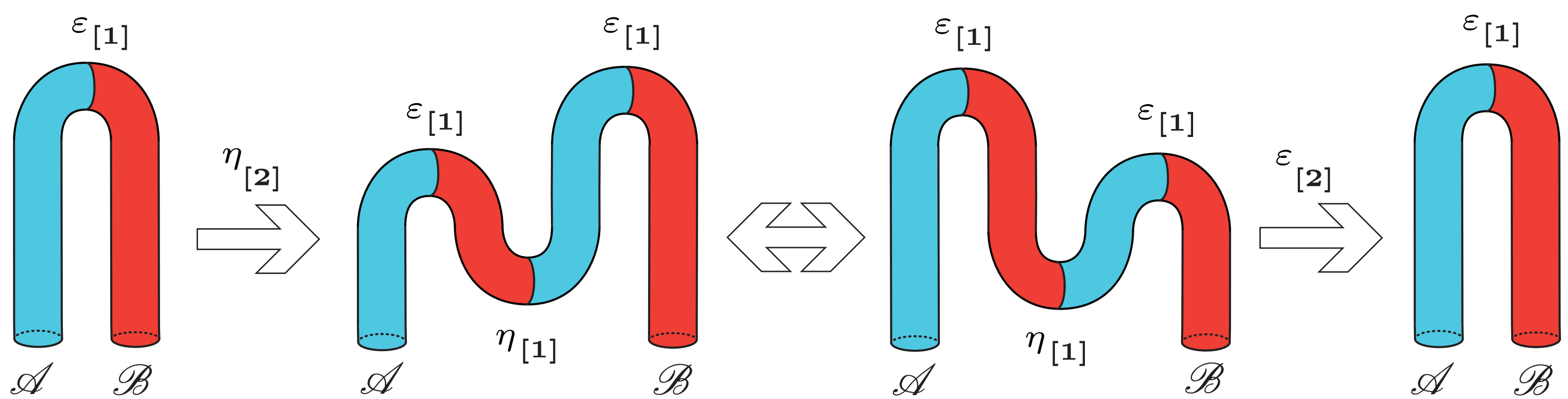
$$\eta_{[1]} : \mathscr{A} \otimes \mathscr{B} \longrightarrow I \qquad\qquad \varepsilon_{[1]} : I \longrightarrow \mathscr{B} \otimes \mathscr{A}$$

together with a pair of invertible 2-dimensional cells



Biexact pairing

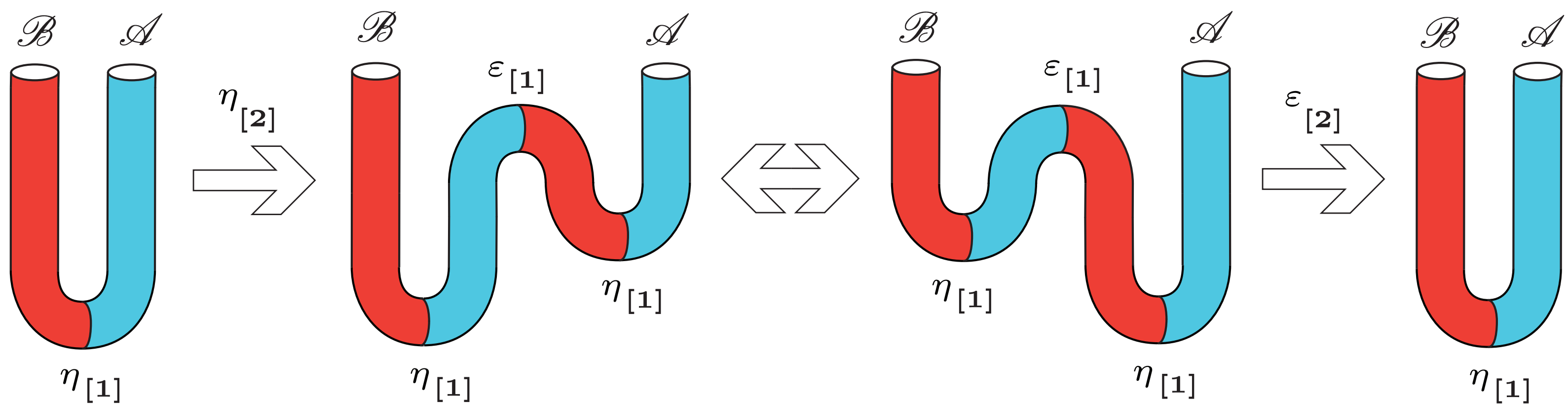
such that the composite 2-dimensional cell



coincides with the identity on the 1-dimensional cell $\varepsilon_{[1]}$,

Biexact pairing

and symmetrically, such that the composite 2-dimensional cell



coincides with the identity on the 1-dimensional cell $\eta_{[1]}$.

Amphimonoid

In any symmetric monoidal bicategory like **BiMod**...

Definition. An amphimonoid is a pseudomonoid

$$(\mathscr{A}, \owedge, \mathrm{true})$$

and a pseudocomonoid

$$(\mathscr{B}, \ovee, \mathrm{false})$$

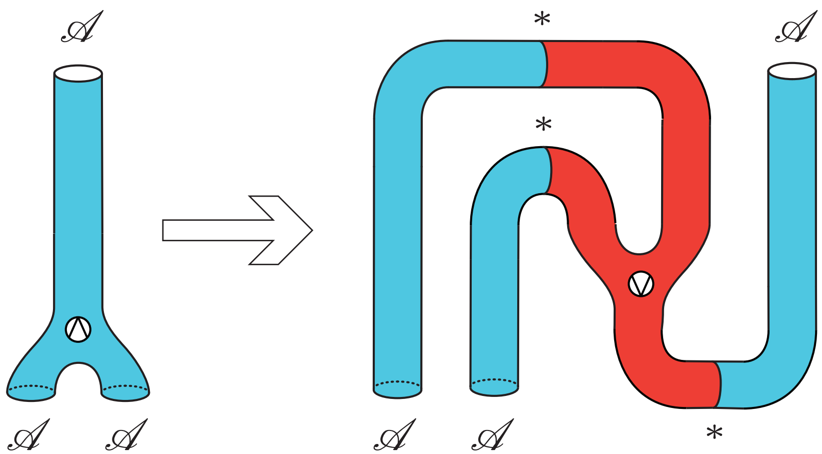
equipped with a biexact pairing

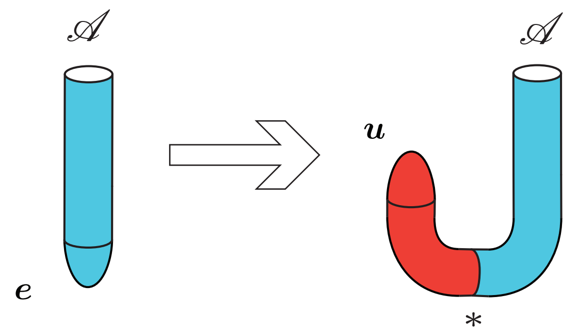
$$\mathscr{A} \dashv \mathscr{B}$$

Bialgebraic counterpart to the notion of chirality

Amphimonoid

together with a pair of invertible 2-dimensional cells





defining a pseudomonoid equivalence.

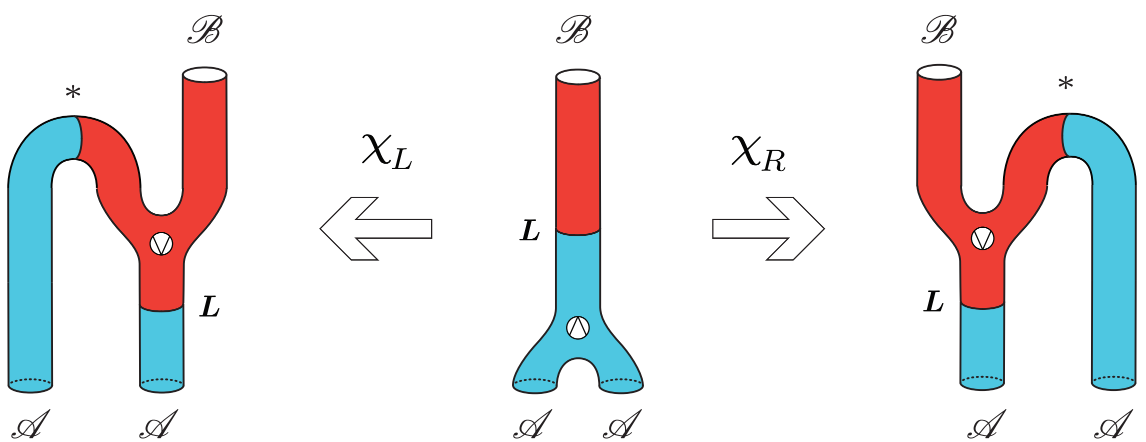
Bialgebraic counterpart to the notion of monoidal chirality

Frobenius amphimonoid

Definition. An amphimonoid together with an adjunction

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, bend left, "R"] \end{tikzcd}$$

and two invertible 2-dimensional cells:



Bialgebraic counterpart to the notion of dialogue chirality

Frobenius amphimonoid

The 1-dimensional cell

$$L \quad : \quad \mathscr{A} \quad \to \quad \mathscr{B}$$

may be understood as defining a bracket

$$\langle\, a \,|\, b \,\rangle$$

between the objects $\mathscr{A}$ and $\mathscr{B}$ of the bicategory $\mathscr{V}$.

Each side of the equation implements currification:

$$\chi_L : \langle\, a_1 \owedge a_2 \,|\, b \,\rangle \Rightarrow \langle\, a_2 \,|\, a_1^* \ovee b \,\rangle \qquad \chi_R : \langle\, a_1 \owedge a_2 \,|\, b \,\rangle \Rightarrow \langle\, a_1 \,|\, b \ovee a_2^* \,\rangle$$

Frobenius amphimonoid

These are required to make the diagrams commute:

$$
\begin{tikzcd}
\langle (m \owedge n) \owedge a \,|\, b \rangle \arrow[rr, "\chi^L_{m \owedge n}"] \arrow[d] \arrow[drr, phantom, "{[1]}"] & & \langle a \,|\, (m \owedge n)^* \ovee b \rangle \\
\langle m \owedge (n \owedge a) \,|\, b \rangle \arrow[r, "\chi^L_m"] & \langle n \owedge a \,|\, m^* \ovee b \rangle \arrow[r, "\chi^L_n"] & \langle a \,|\, n^* \ovee (m^* \ovee b) \rangle \arrow[u]
\end{tikzcd}
$$

Frobenius amphimonoid

These are required to make the diagrams commute:

$$
\begin{tikzcd}
\langle\, a \owedge (m \owedge n) \,|\, b \,\rangle \arrow[rr, "\chi^R_{m \owedge n}"] \arrow[d] \arrow[drr, phantom, "{[2]}"] & & \langle\, a \,|\, b \ovee (m \owedge n)^* \,\rangle \\
\langle\, (a \owedge m) \owedge n \,|\, b \,\rangle \arrow[r, "\chi^R_n"] & \langle\, a \owedge m \,|\, b \ovee n^* \,\rangle \arrow[r, "\chi^R_m"] & \langle\, a \,|\, (b \ovee n^*) \ovee m^* \,\rangle \arrow[u]
\end{tikzcd}
$$

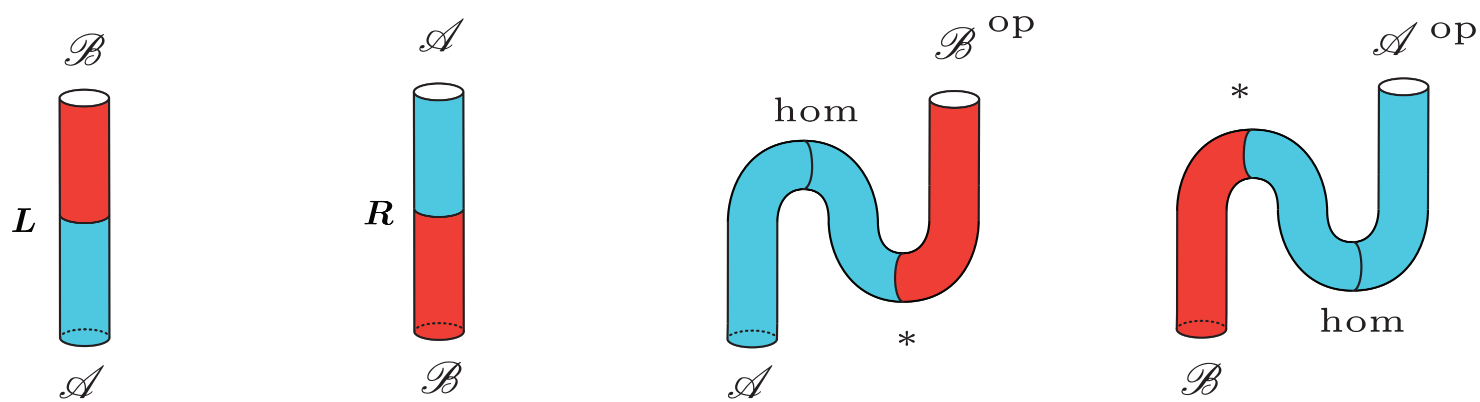
Frobenius amphimonoid

These are required to make the diagrams commute:

$$
\begin{tikzcd}
\langle (m \owedge a) \owedge n \,|\, b \rangle \arrow[r, "\chi_n^R"] \arrow[dd, no head] & \langle m \owedge a \,|\, b \ovee n^* \rangle \arrow[r, "\chi_m^L"] \arrow[dd, phantom, "{[3]}"] & \langle a \,|\, m^* \ovee (b \ovee n^*) \rangle \arrow[dd, no head] \\
 & & \\
\langle m \owedge (a \owedge n) \,|\, b \rangle \arrow[r, "\chi_m^L"] & \langle a \owedge n \,|\, m^* \ovee b \rangle \arrow[r, "\chi_n^R"] & \langle a \,|\, (m^* \ovee b) \ovee n^* \rangle
\end{tikzcd}
$$

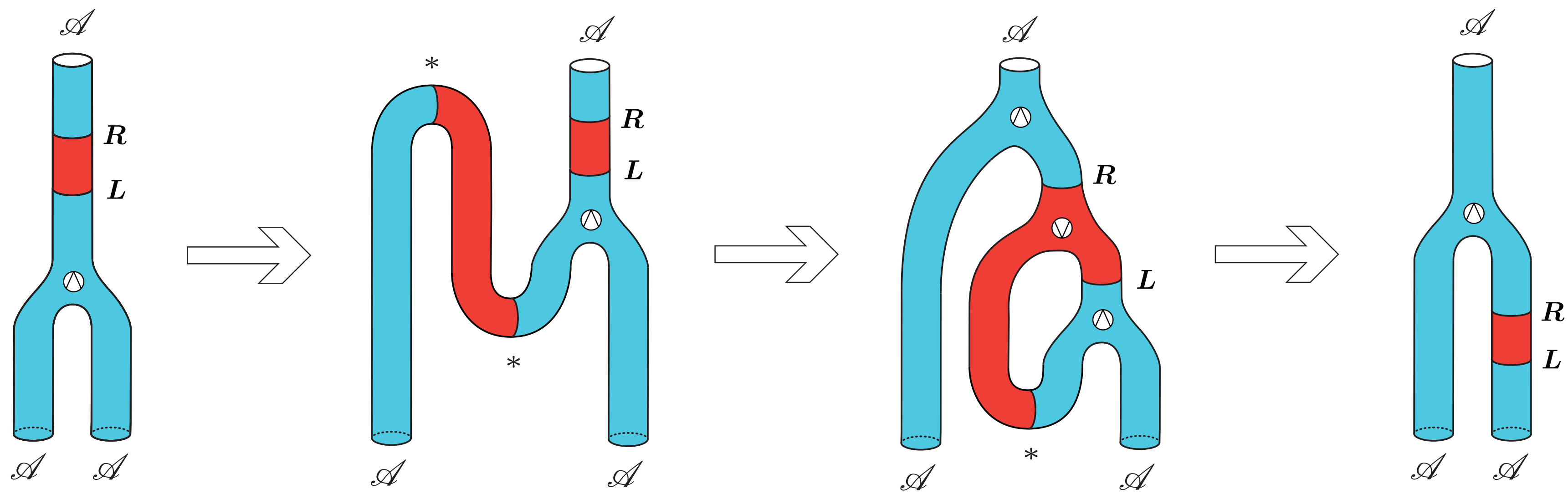
Correspondence theorem

Theorem. A pivotal chirality is the same thing as a Frobenius amphimonoid in the bicategory **BiMod** whose 1-dimensional cells



are representable, that is, induced by functors.

Tensorial strength formulated in cobordism



$$a_1 \owedge RL(a_2) \quad \vdash \quad RL(a_1 \owedge a_2)$$

$$\mathscr{A}(RL(a_1 \owedge a_2), a) \quad \longrightarrow \quad \mathscr{A}(a_1 \owedge RL(a_2), a)$$

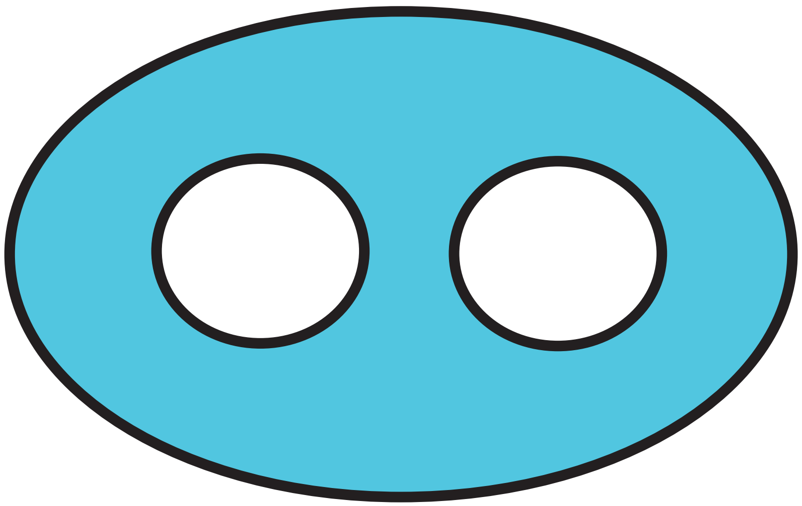
Connection with topology

Idea: interpret tensorial logic in topological field theory with defects.

- ▷ Formulas as 1+1 topological field theories with defects
- ▷ Tensorial proofs as 2+1 topological field theories with defects

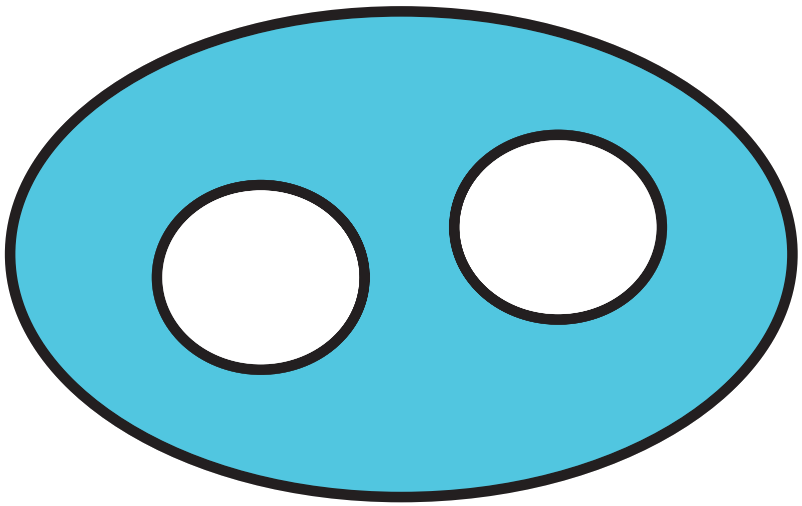
- ▷ a coherence theorem including the microcosm?
- ▷ what about dialogue 2-categories and 3-categories?

The topological nature of proofs



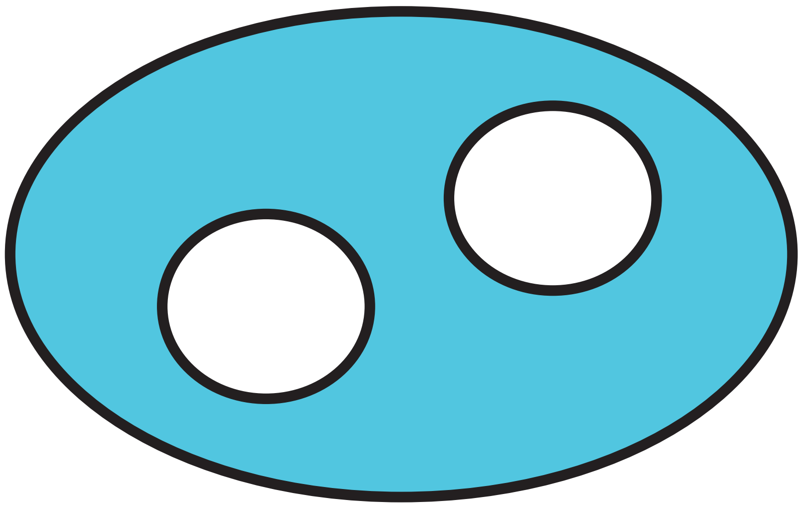
A topological account of exchange

The topological nature of proofs



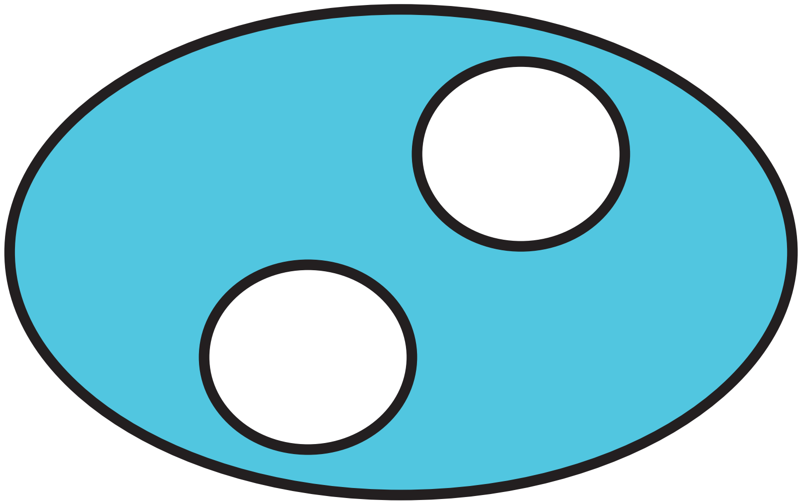
A topological account of exchange

The topological nature of proofs



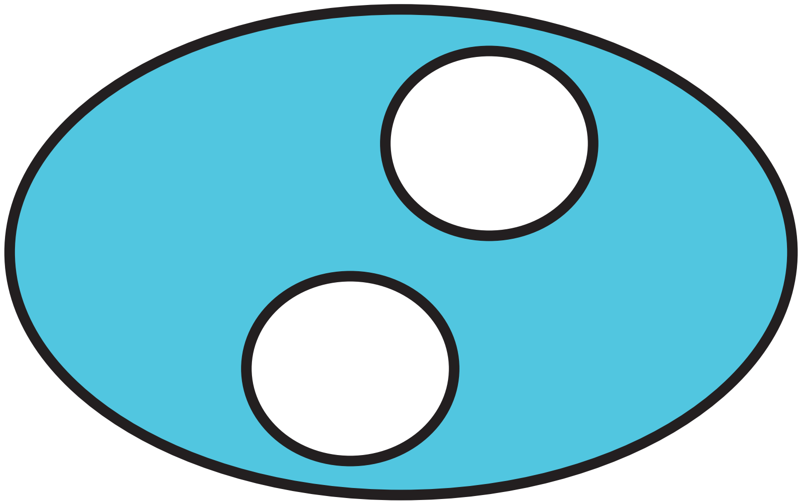
A topological account of exchange

The topological nature of proofs



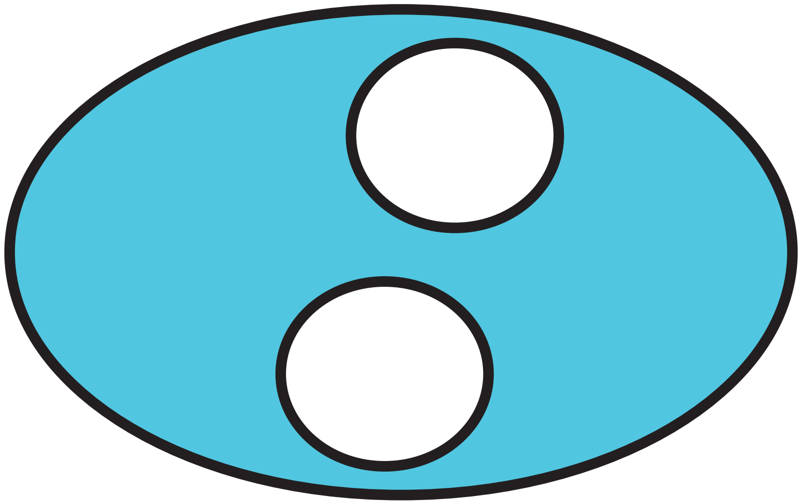
A topological account of exchange

The topological nature of proofs



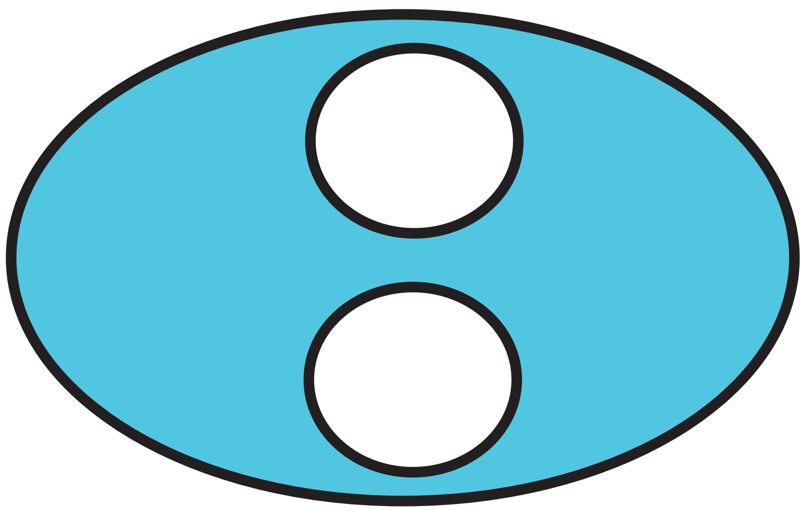
A topological account of exchange

The topological nature of proofs



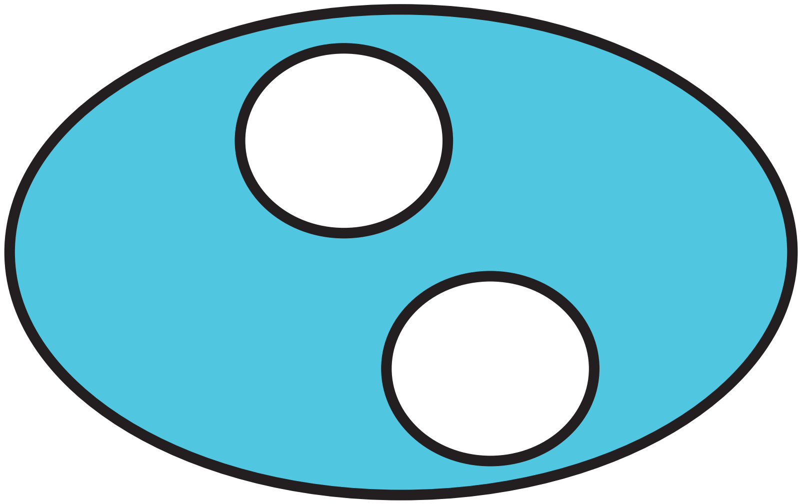
A topological account of exchange

The topological nature of proofs



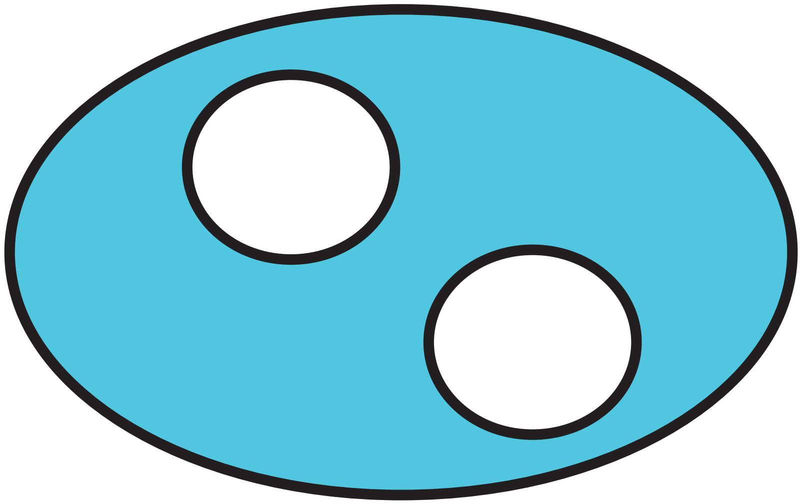
A topological account of exchange

The topological nature of proofs



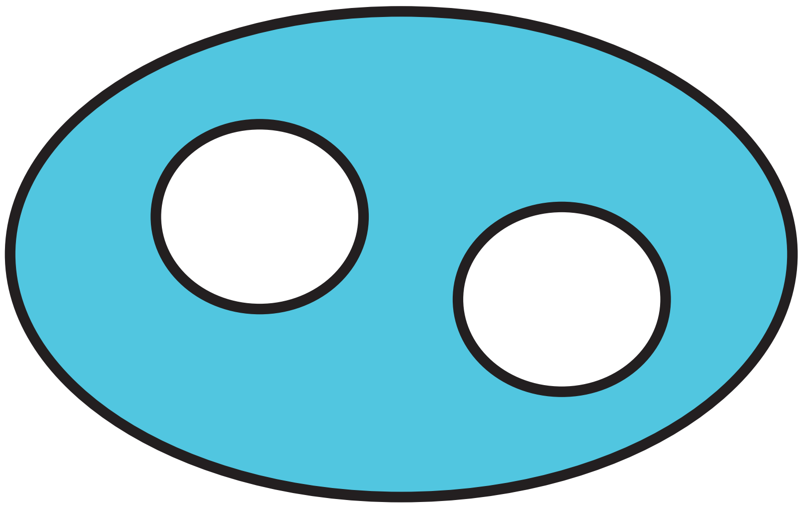
A topological account of exchange

The topological nature of proofs



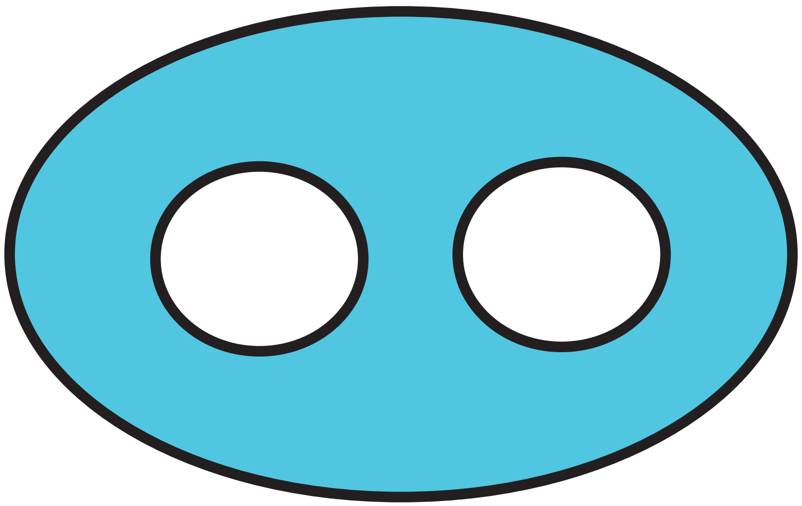
A topological account of exchange

The topological nature of proofs



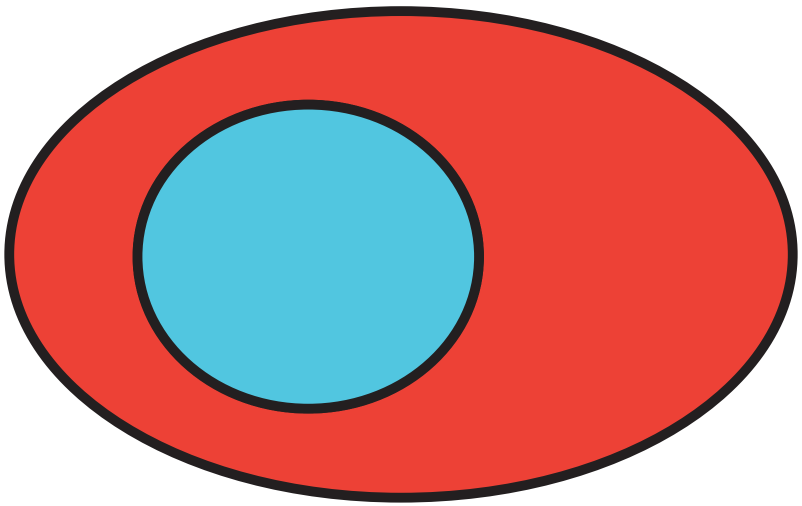
A topological account of exchange

The topological nature of proofs



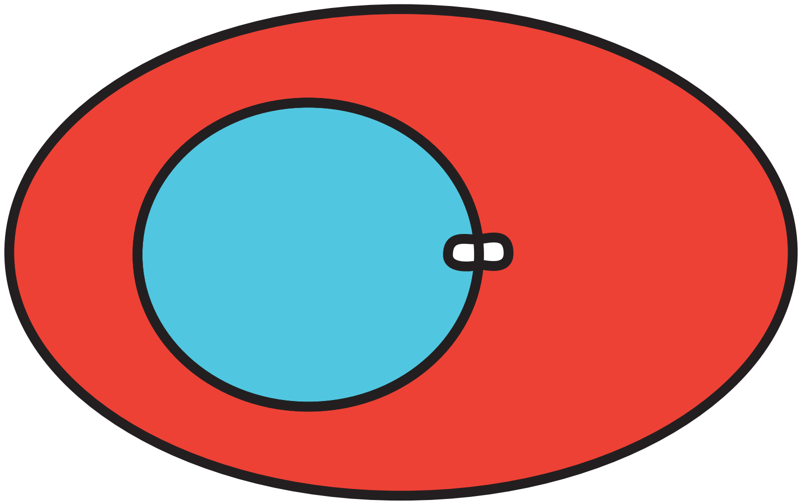
A topological account of exchange

The topological nature of proofs



A topological account of modus ponens

The topological nature of proofs



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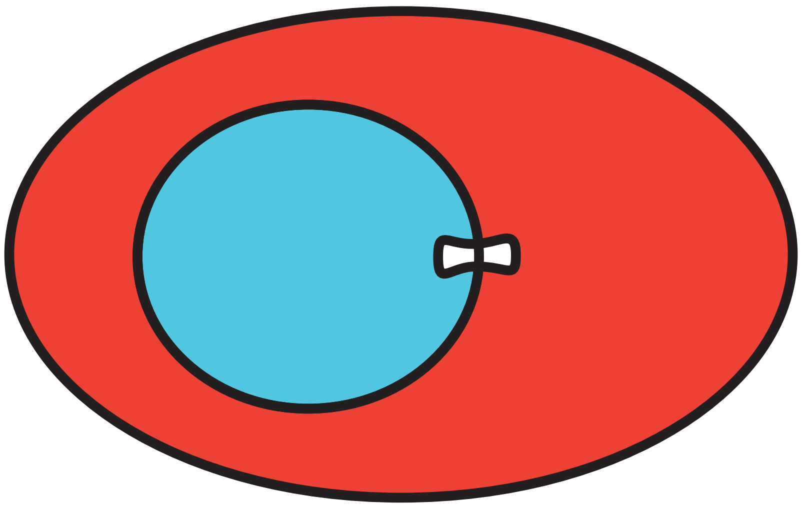

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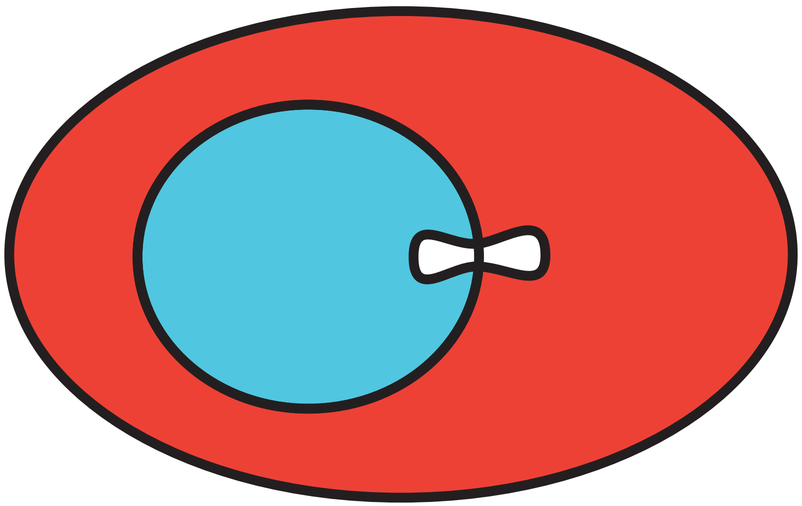

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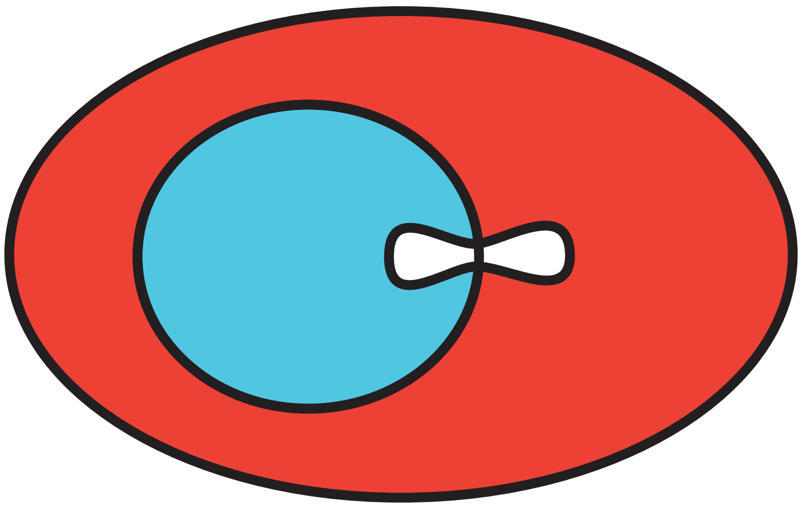

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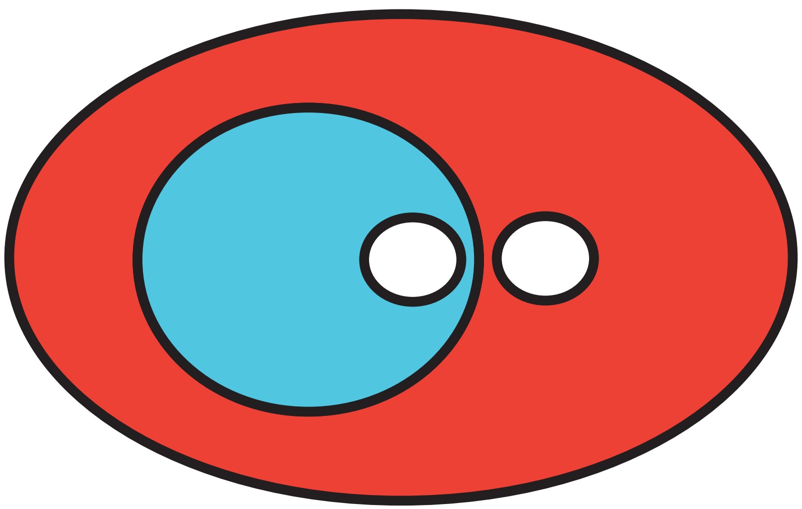

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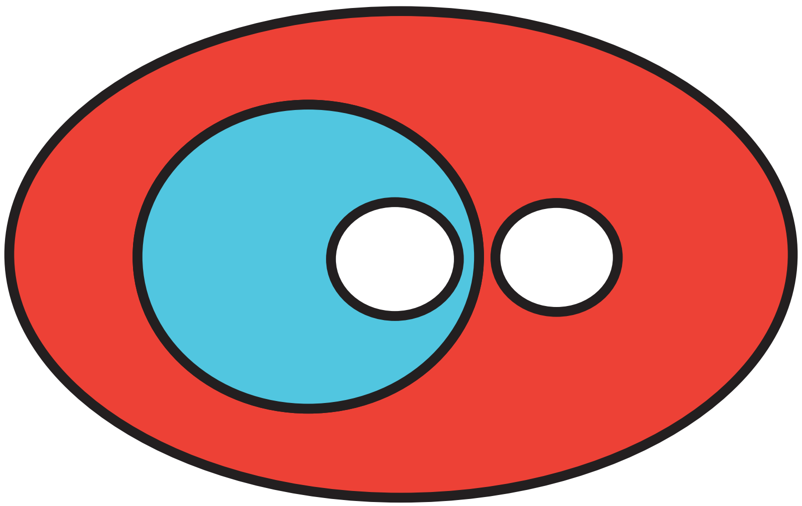

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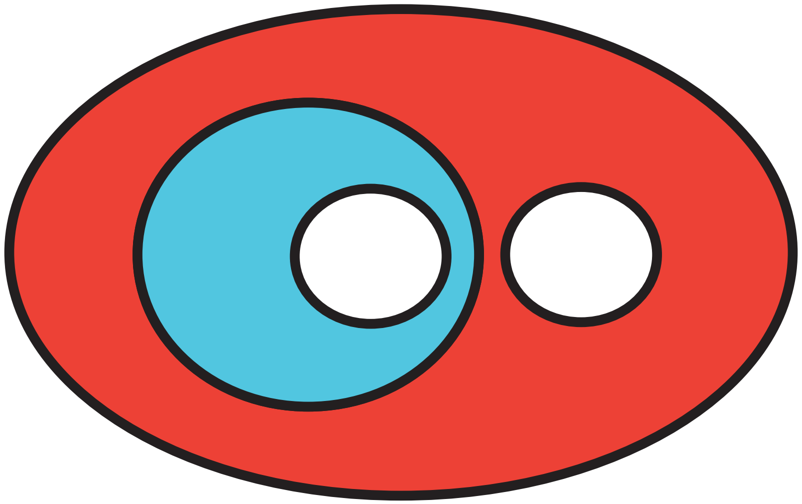

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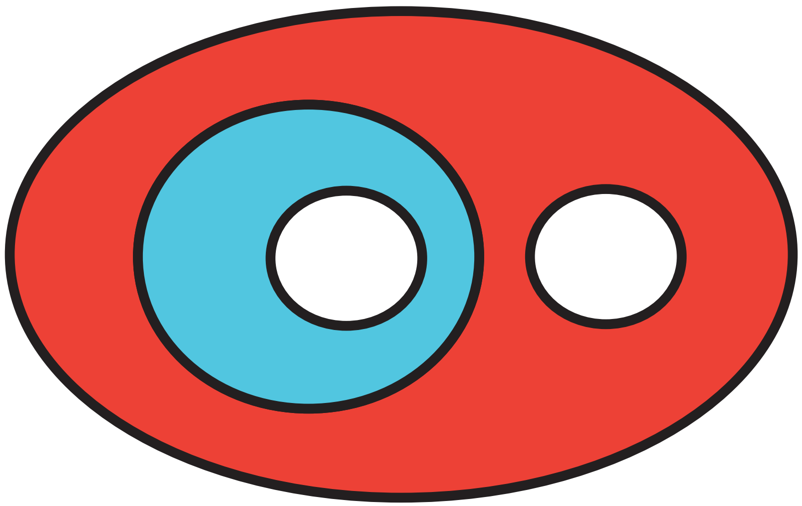

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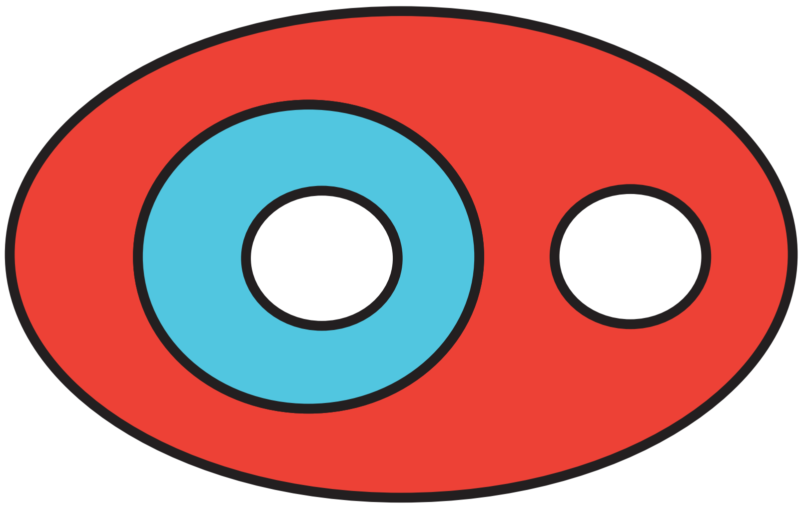

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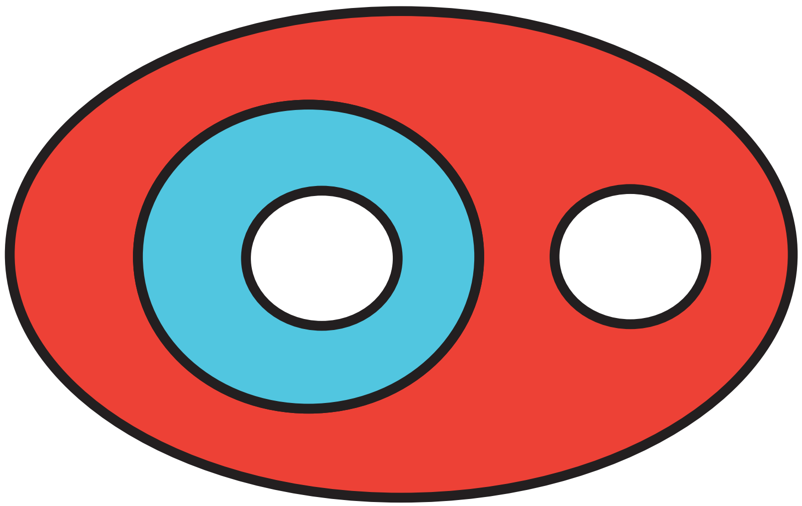

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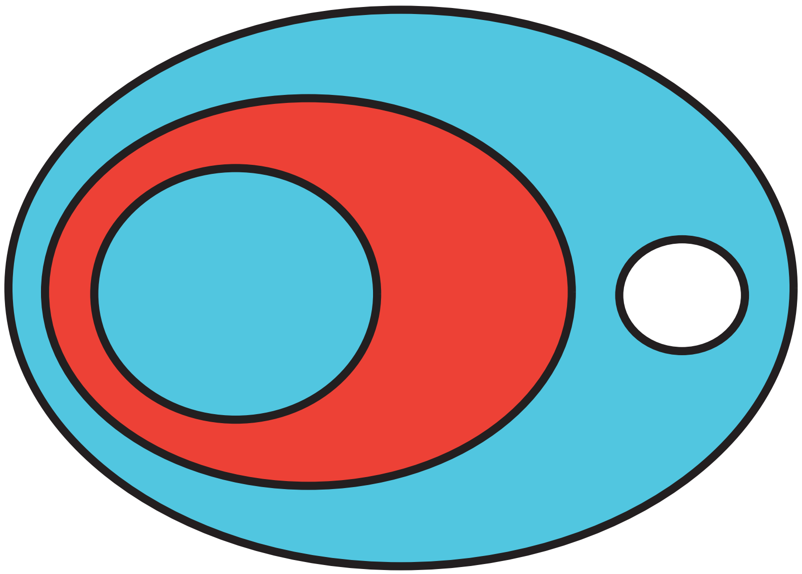

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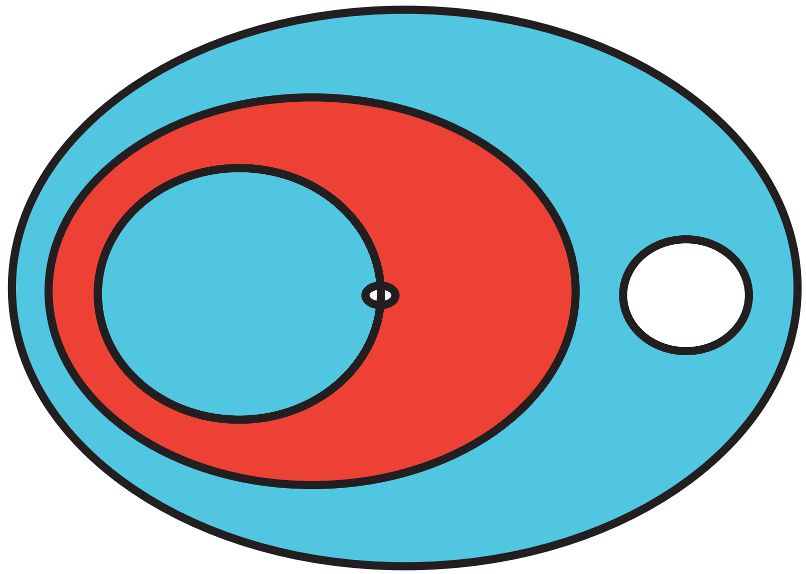

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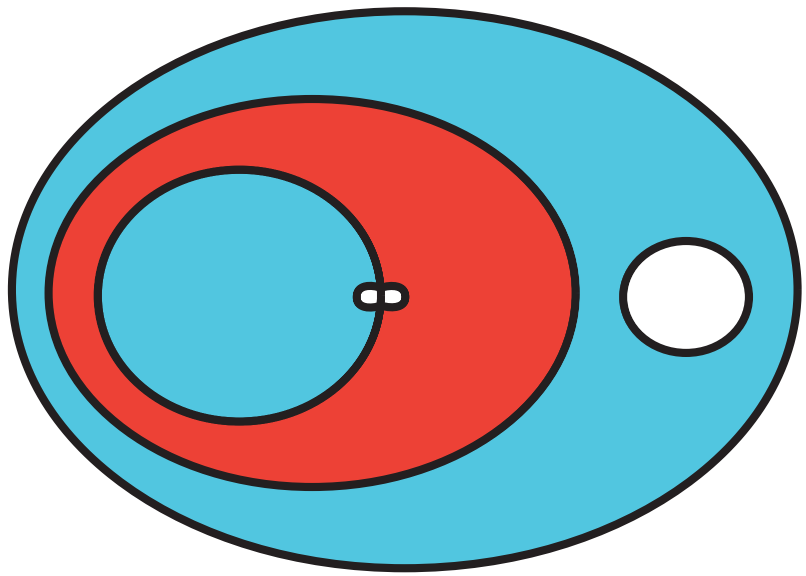

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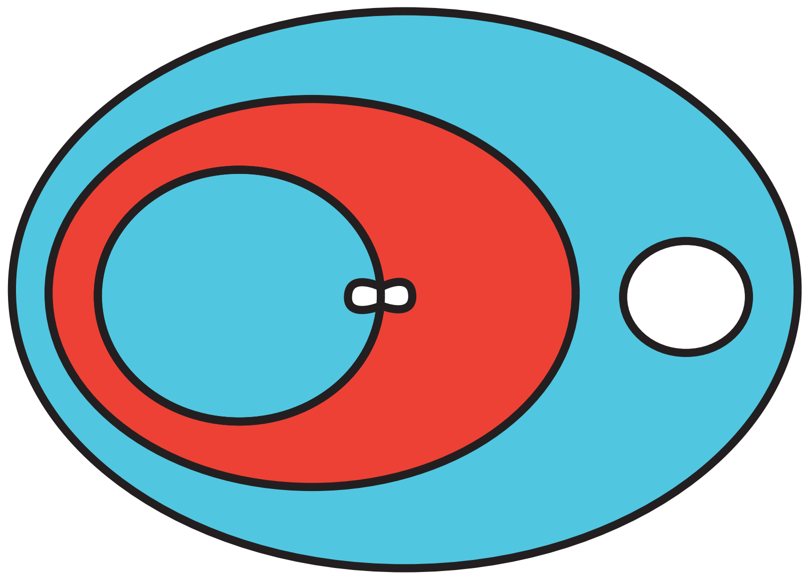

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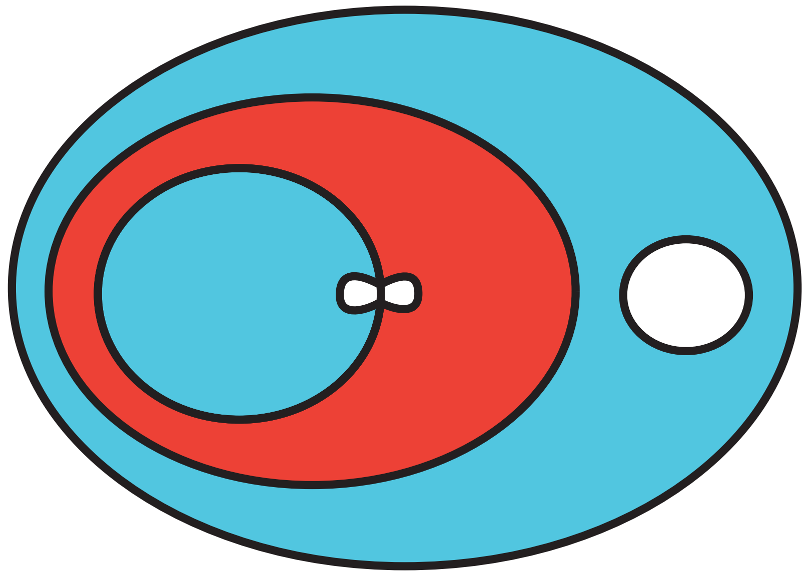

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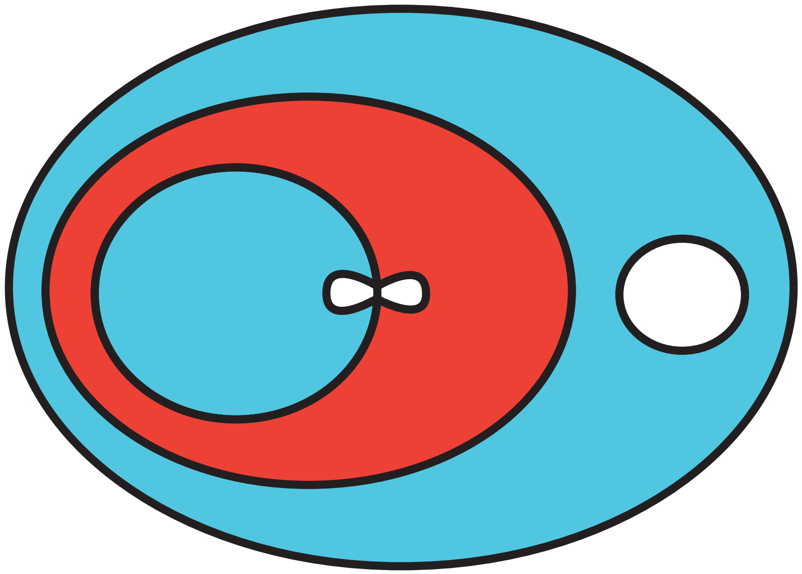

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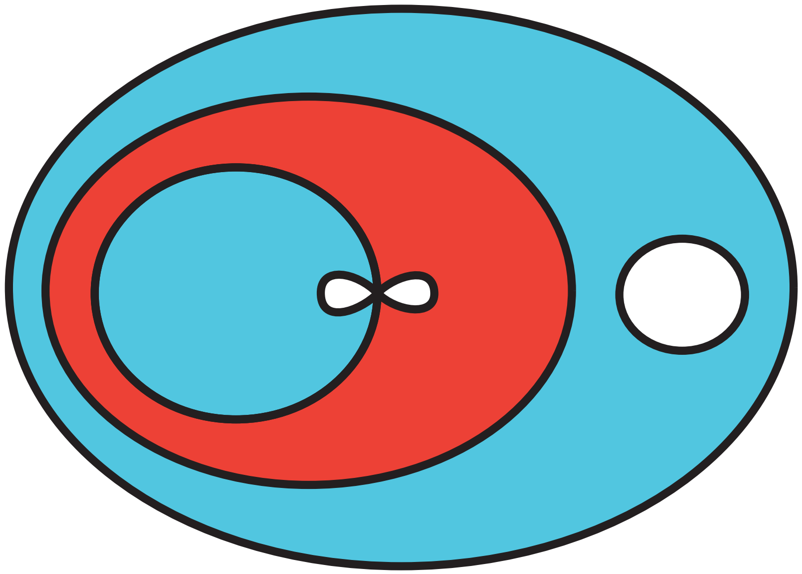

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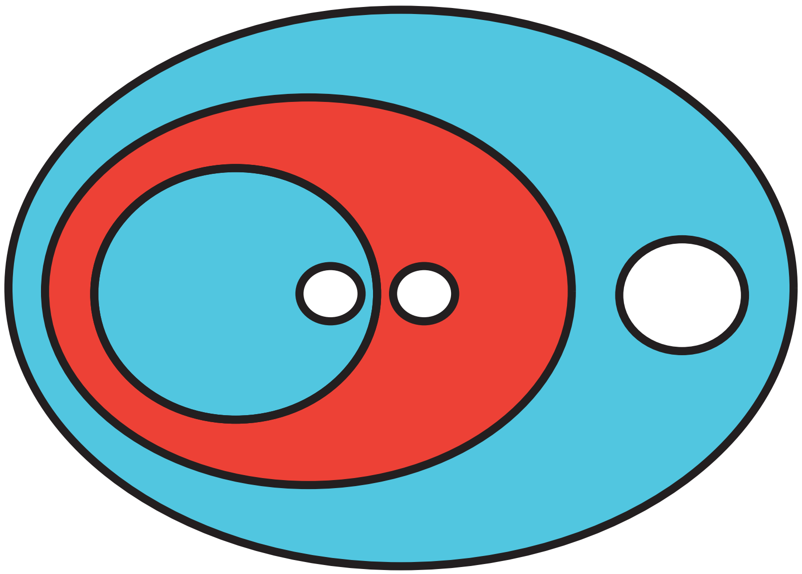

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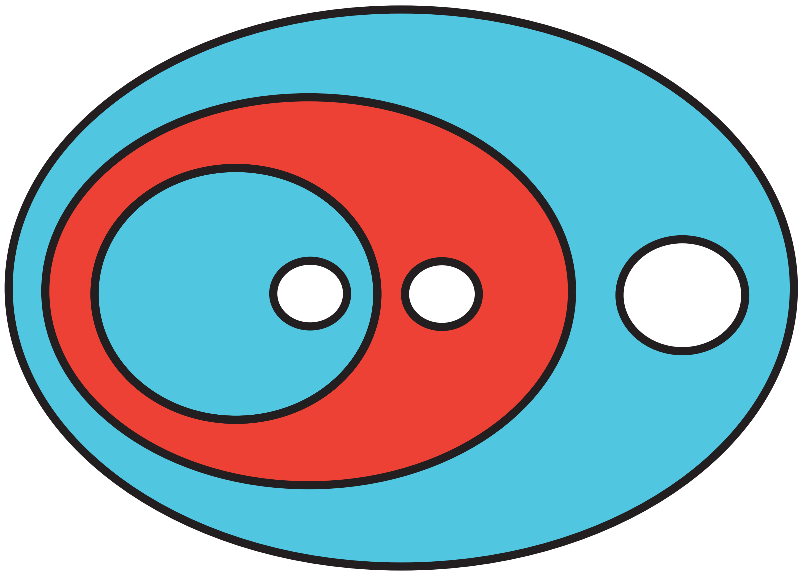

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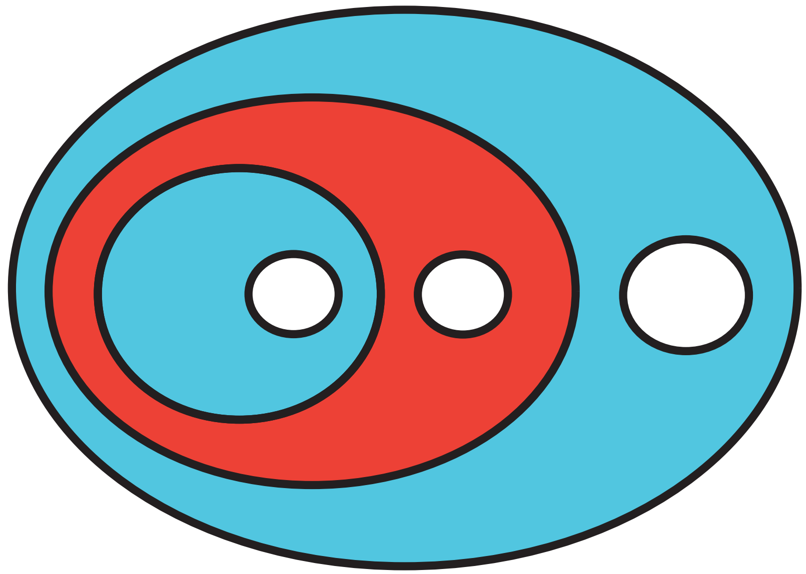

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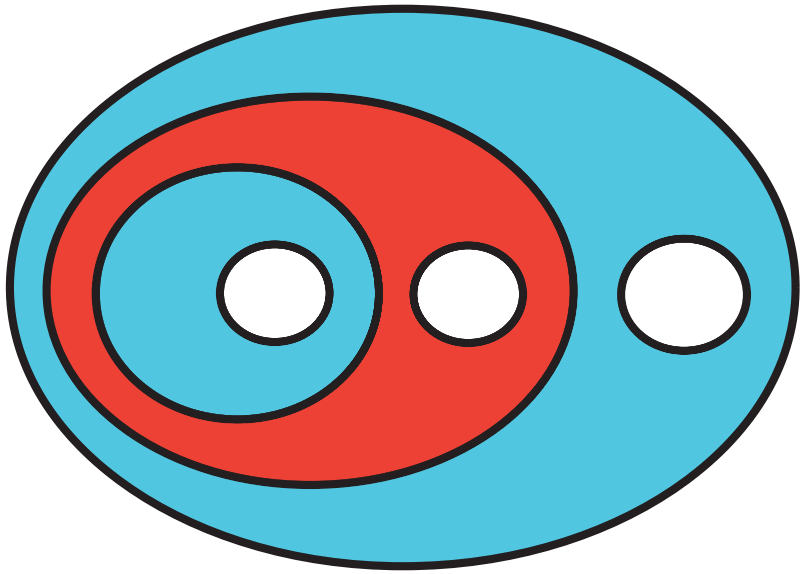

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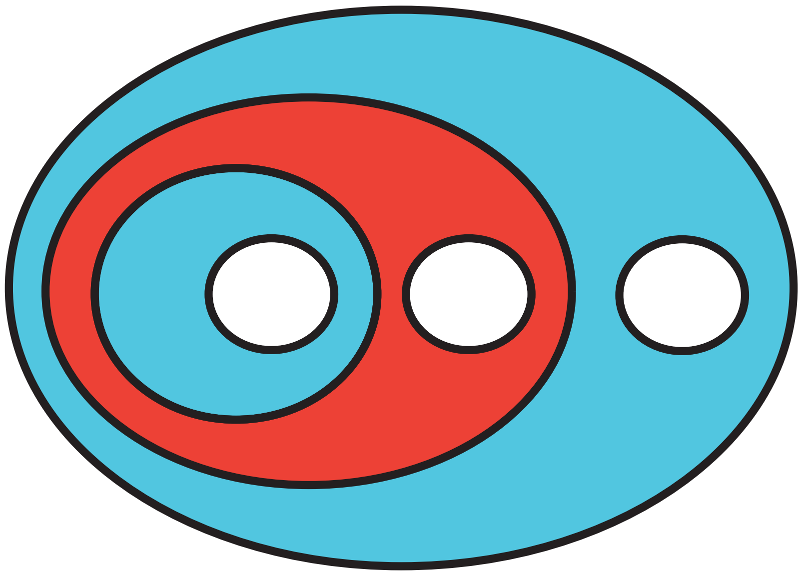

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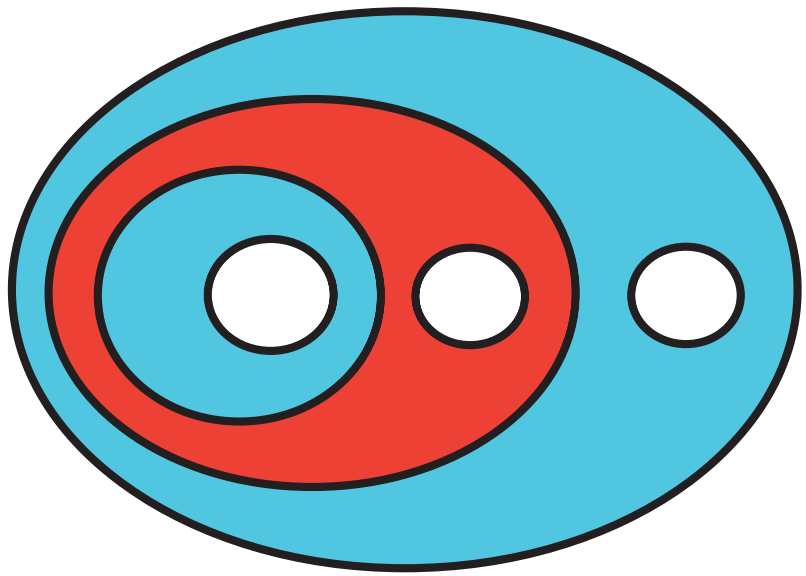

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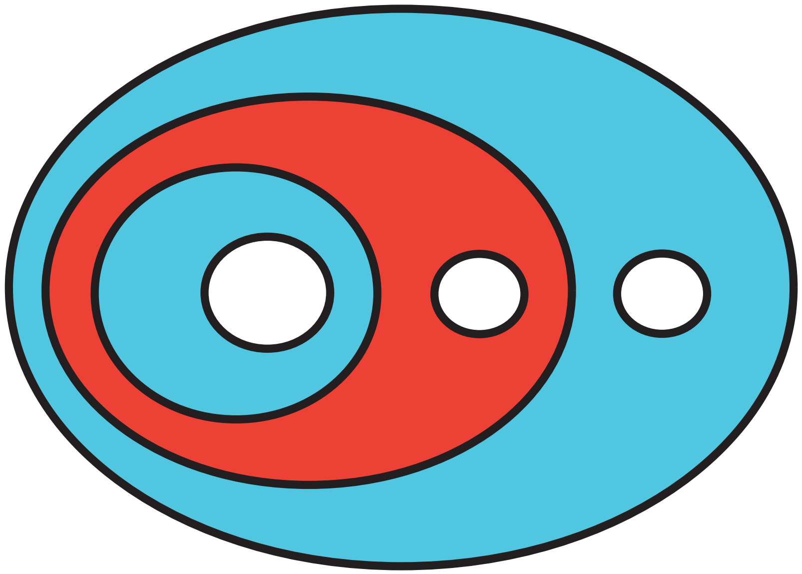

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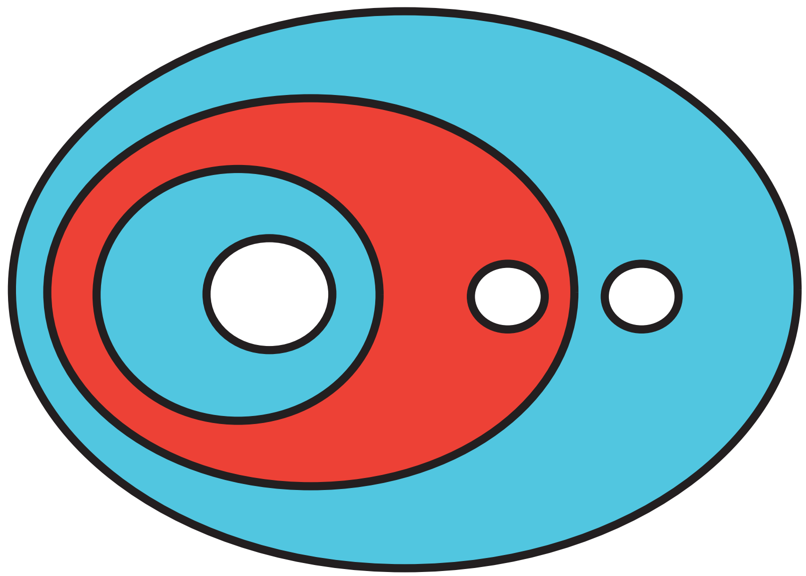

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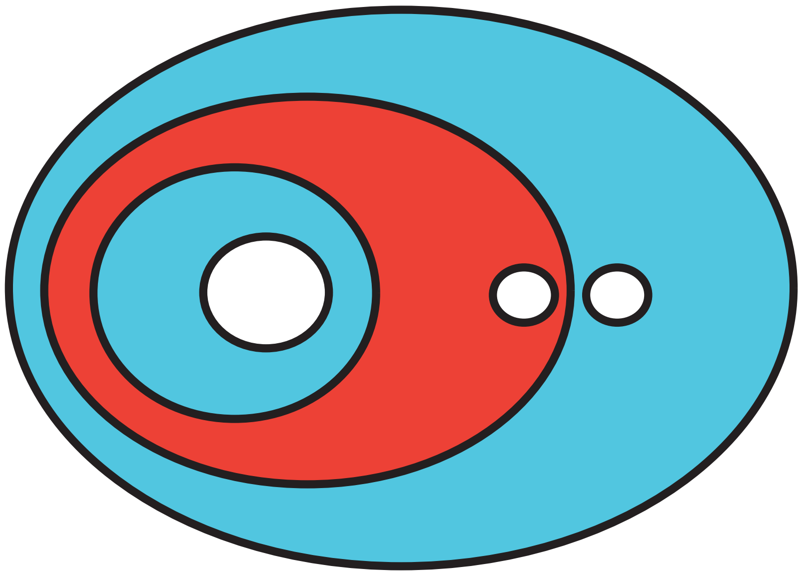

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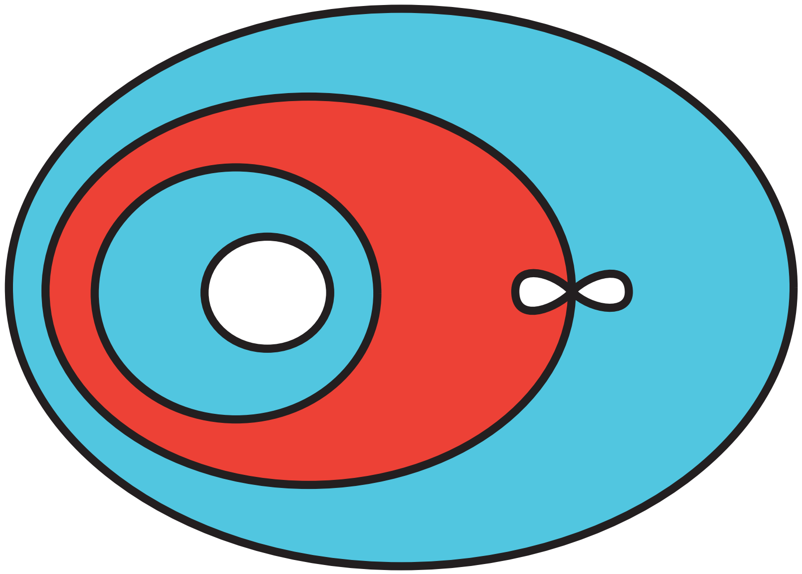

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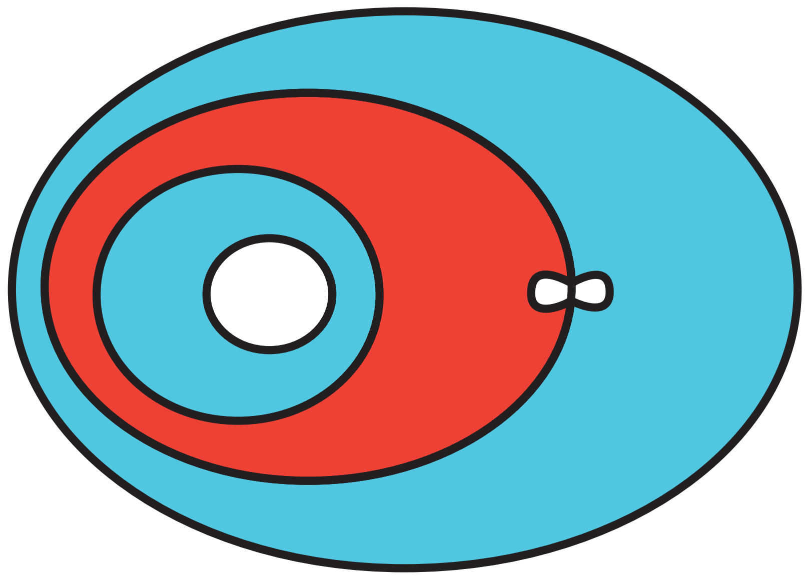

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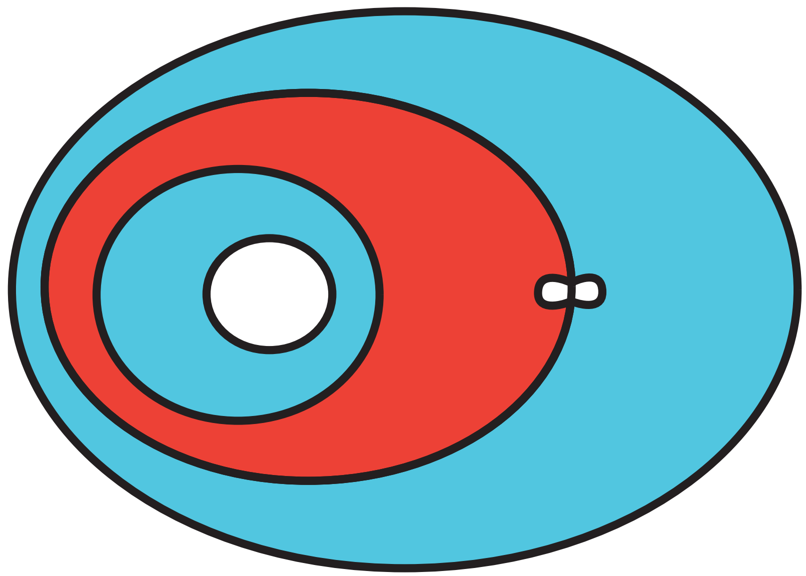

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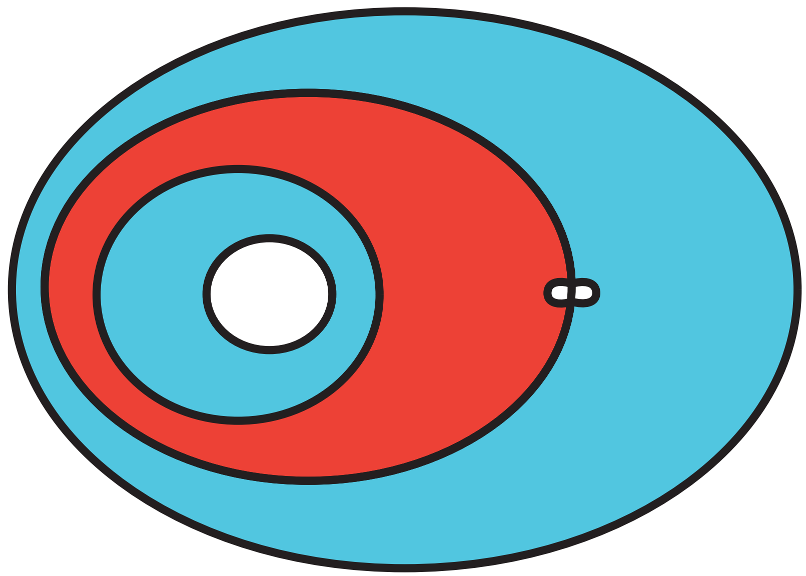

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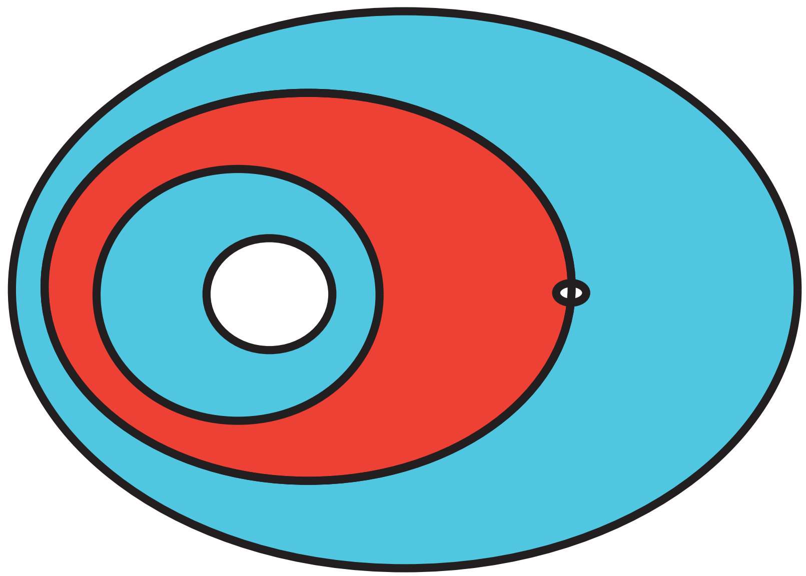

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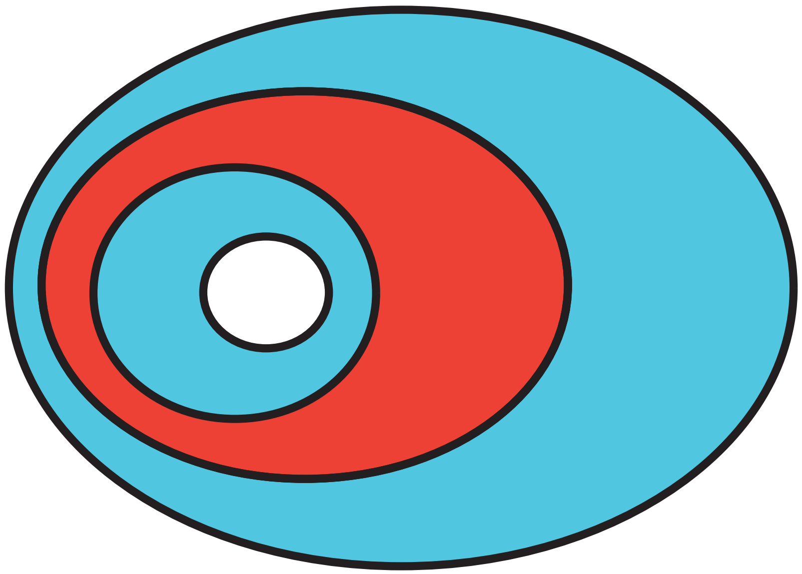

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Thank you