An introduction to Yoneda structures

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Bibliography

Ross Street and Bob Walters
Yoneda structures on 2-categories
Journal of Algebra
50:350-379, 1978

Mark Weber
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Covariant and contravariant presheaves

A few opening words on Isbell conjugacy

Ideal completion

Every partial order A generates a free complete $\bigvee$-lattice $\mathscr{P}A$

$$A \longrightarrow \mathscr{P}A$$

whose elements are the downward closed subsets of A, with

$$\varphi \leq_{\mathscr{P}A} \psi \iff \varphi \subseteq \psi.$$

$$\boxed{\mathscr{P}A = A^{op} \Rightarrow \{0,1\}}$$

Free colimit completions of categories

Every small category $\mathscr{A}$ generates a free cocomplete category $\mathscr{P}\,\mathscr{A}$

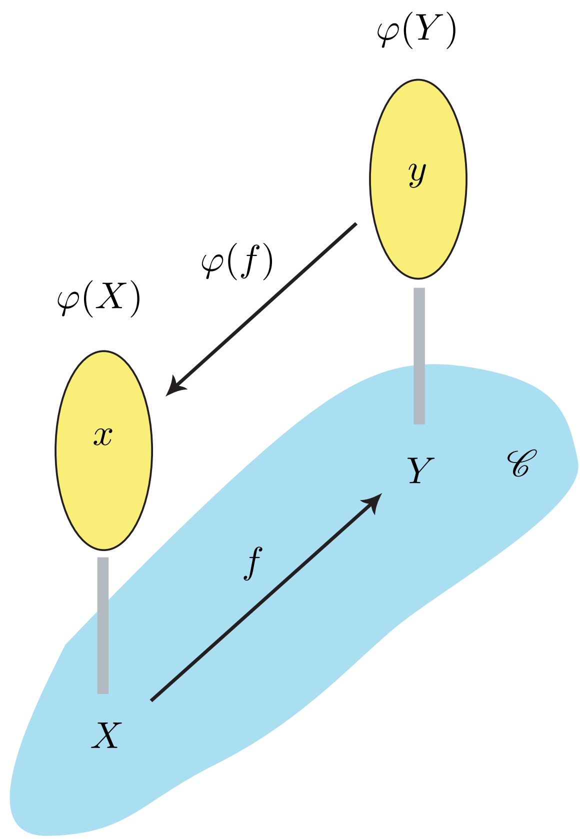
$$\mathscr{A} \quad \longrightarrow \quad \mathscr{P}\,\mathscr{A}$$

whose elements are the presheafs over $\mathscr{A}$, with

$$\varphi \quad \longrightarrow_{\mathscr{P}\,\mathscr{A}} \quad \psi \qquad \Longleftrightarrow \qquad \varphi \quad \stackrel{natural}{\longrightarrow} \quad \psi.$$

$$\mathscr{P}\,\mathscr{A} \quad = \quad \mathscr{A}^{op} \Rightarrow Set$$

Contravariant presheaves



Replaces downward closed sets

Filter completion

Every partial order A generates a free complete $\bigwedge$-lattice $\mathcal{Q}A$

$$A \quad \longrightarrow \quad \mathcal{Q}A$$

whose elements are the upward closed subsets of A, with

$$\varphi \quad \leq_{\mathcal{Q}A} \quad \psi \qquad \Longleftrightarrow \qquad \varphi \quad \supseteq \quad \psi.$$

$$\boxed{\mathcal{Q}A \quad = \quad (\, A \Rightarrow \{0,1\}\,)^{op}}$$

Free limit completions of categories

Every small category $\mathscr{A}$ generates a free complete category $\mathscr{Q}\,\mathscr{A}$

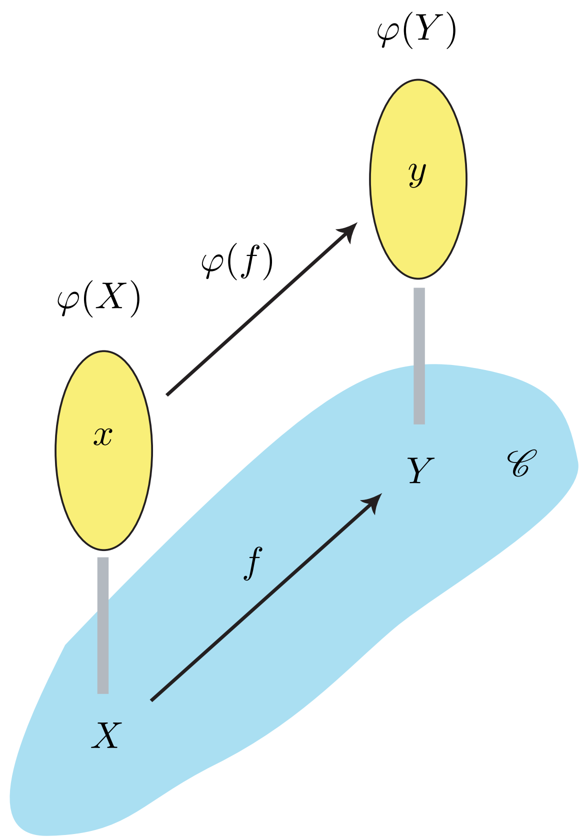
$$\mathscr{A} \quad \longrightarrow \quad \mathscr{Q}\,\mathscr{A}$$

whose elements are the covariant presheafs over $\mathscr{A}$, with

$$\varphi \quad \longrightarrow_{\mathscr{Q}\,\mathscr{A}} \quad \psi \qquad \Longleftrightarrow \qquad \varphi \quad \overset{natural}{\longleftarrow} \quad \psi.$$

$$\boxed{\mathscr{Q}\,\mathscr{A} \quad = \quad (\,\mathscr{P}\,\mathscr{A}^{op}\,)^{op} \quad = \quad (\,\mathscr{A} \;\Rightarrow\; Set\,)^{op}}$$

Covariant presheaves



Replaces upward closed sets

Related to the Dedekind-MacNeille completion

A Galois connection

$$\mathscr{P}A \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{Q}A \qquad (\bot)$$

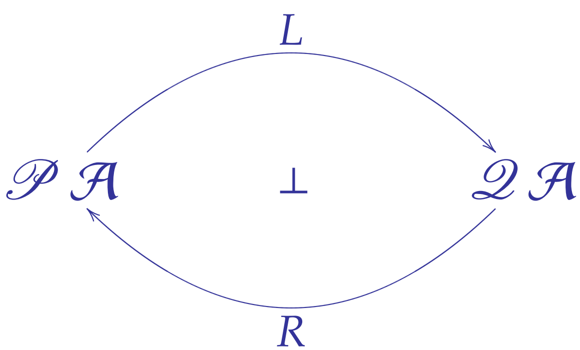
$$\begin{aligned} L(\varphi) &= \{\ y \mid \forall x \in \varphi,\ \ x \leq_A y\ \} \\ R(\psi) &= \{\ x \mid \forall y \in \psi,\ \ x \leq_A y\ \} \end{aligned}$$

$$\varphi \subseteq R(\psi) \quad \Longleftrightarrow \quad \forall x \in \varphi, y \in \psi,\ \ x \leq_A y \quad \Longleftrightarrow \quad L(\varphi) \supseteq \psi$$

The completion keeps the pairs (φ, ψ) such that $\psi = L(\varphi)$ and $\varphi = R(\psi)$

The Isbell conjugacy

An adjunction



$$L(\varphi) \; : \; Y \; \mapsto \; \mathscr{P}\mathcal{A}(\varphi, Y) \; = \; \int_{x \in \mathcal{A}} \varphi(X) \Rightarrow hom(X, Y)$$

$$R(\psi) \; : \; X \; \mapsto \; \mathscr{Q}\mathcal{A}(X, \psi) \; = \; \int_{Y \in \mathcal{A}} \psi(Y) \Rightarrow hom(X, Y)$$

$$\mathscr{P}\mathcal{A}(\varphi, R(\psi)) \;\cong\; \int_{X, Y \in \mathcal{A}} \varphi(X) \times \psi(Y) \Rightarrow hom(X, Y) \;\cong\; \mathscr{Q}\mathcal{A}(L(\varphi), \psi)$$

Yoneda structures

An axiomatic approach by Street and Walters

General idea

Suppose a universe $\mathscr{U}_1$ inside a larger universe $\mathscr{U}_2$.

– **Set** is the category of sets in the universe $\mathscr{U}_1$,

– $\mathscr{CAT}$ is the 2-category of categories in the universe $\mathscr{U}_2$.

In particular, the category **Set** is an object of $\mathscr{CAT}$.

Admissible functors

An object $\mathcal{A}$ of $\mathcal{CAT}$ is called **admissible** when its homsets

$$\mathcal{A}\,(A, A')$$

are all in the category **Set**. More generally, an arrow in $\mathcal{CAT}$

$$F \quad : \quad \mathcal{A} \longrightarrow \mathcal{B}$$

is called **admissible** when the homsets

$$\mathcal{B}\,(FA, B)$$

are all in the category **Set**.

Yoneda structures

A Yoneda structure in a 2-category $\mathscr{K}$ is defined as

– a class of **admissible** arrows such that the composite arrow

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$$

is admissible whenever the arrow

$$\mathcal{B} \xrightarrow{G} \mathcal{C}$$

is admissible.

Admissible objects

An object $\mathcal{A}$ is called **admissible** when the identity arrow

$$id_{\mathcal{A}} \quad : \quad \mathcal{A} \quad \longrightarrow \quad \mathcal{A}$$

is admissible.

In the case of the 2-category $\mathcal{CAT}$ equipped with its Yoneda structure:

admissible objects = locally small categories

Yoneda structures

– every admissible object

$$\mathscr{A}$$

induces an admissible arrow

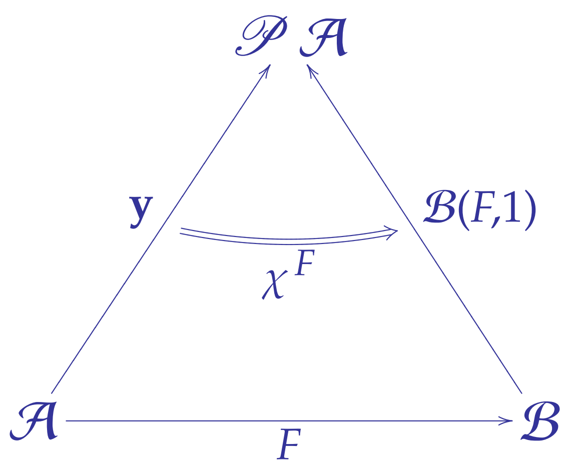
$$\mathbf{y}_{\mathscr{A}} \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{P}\mathscr{A}$$

Yoneda structures

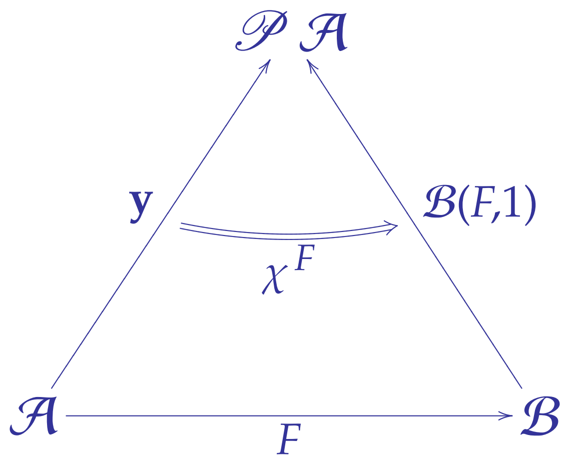
– every admissible arrow

$$F \quad : \quad \mathcal{A} \quad \longrightarrow \quad \mathcal{B}$$

from an admissible object $\mathcal{A}$ induces a diagram



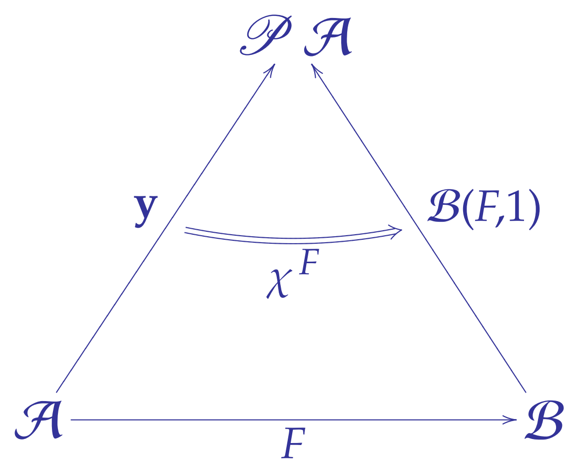
In the traditional case



$$\mathcal{B}(F,1) \quad : \quad b \quad \mapsto \quad \lambda a \,.\, \mathcal{B}(Fa, b)$$

$$\chi^F_{a_2} \quad : \quad \lambda a_1 \,.\, \mathcal{A}(a_1, a_2) \quad \longrightarrow \quad \lambda a_1 \,.\, \mathcal{B}(Fa_1, Fa_2)$$

In the traditional case

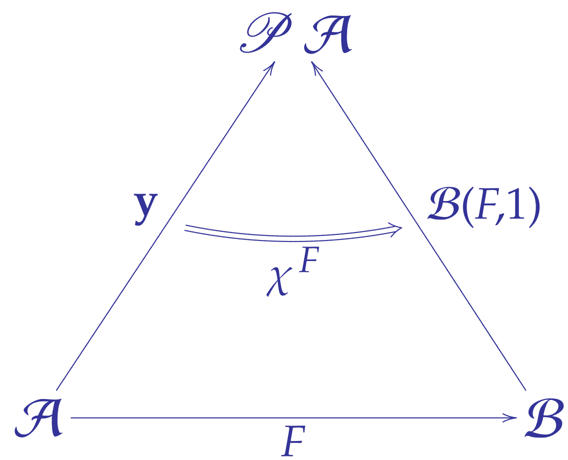


$$\mathcal{B}(F,1) \quad : \quad b \quad \mapsto \quad \lambda a\,.\,\mathcal{B}(Fa,b)$$

$$\chi^F_{a_1 a_2} \quad : \quad \mathcal{A}(a_1,a_2) \quad \longrightarrow \quad \mathcal{B}(Fa_1,Fa_2)$$

Axiom 1

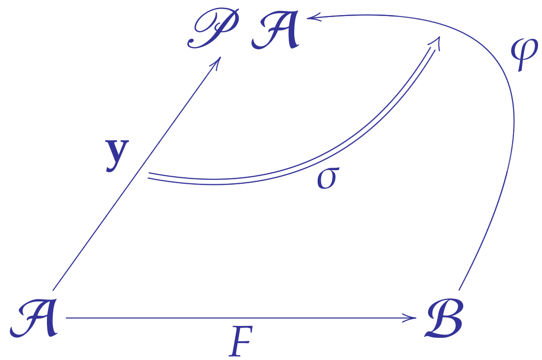
For $\mathcal{A}$ and F accessible, the 2-cell



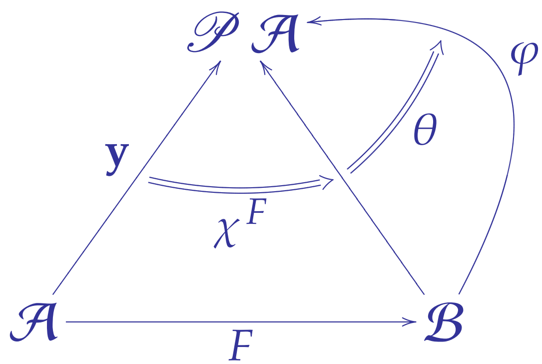
exhibits the arrow $\mathcal{B}(F, 1)$ as a left extension of $\mathbf{y}_{\mathcal{A}}$ along the arrow F.

Axiom 1

In other words, every 2-cell

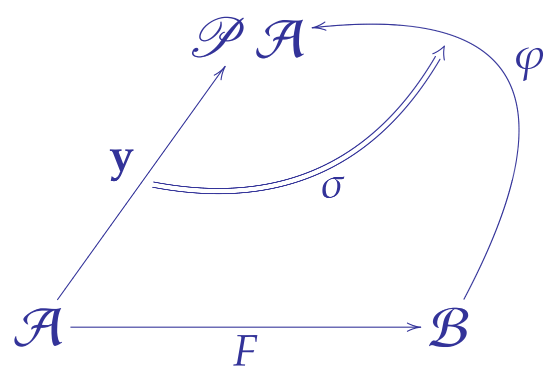


factors uniquely as



In the traditional case

A natural transformation



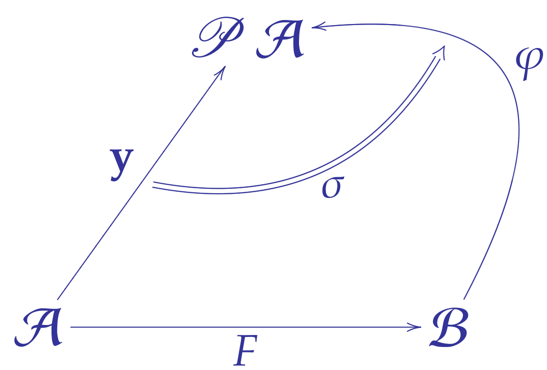
is the same thing as a family of functions

$$\sigma_{a_1 a_2} \quad : \quad \mathcal{A}(a_1, a_2) \quad \longrightarrow \quad \varphi(Fa_2)(a_1)$$

natural in a_1 contravariantly and in a_2 covariantly.

In the traditional case

A natural transformation



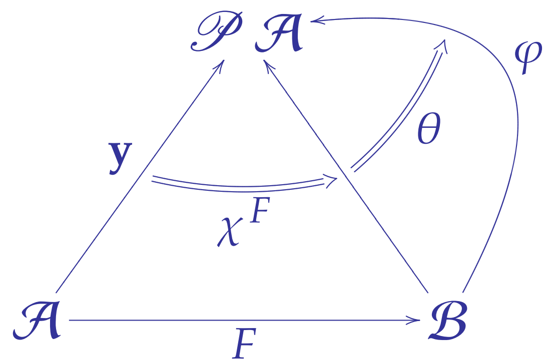
is the same thing as a family of functions

$$\sigma_{a_1 a_2} \quad : \quad \mathcal{A}(a_1, a_2) \quad \longrightarrow \quad \varphi(Fa_2)(a_1)$$

natural in a_1 contravariantly and in a_2 covariantly.

In the traditional case

A natural transformation θ in



is the same thing as a family of functions

$$\theta_{a\,b} \quad : \quad \mathcal{B}(Fa, b) \quad \longrightarrow \quad \varphi(b)(a)$$

natural in a contravariantly and in b covariantly.

In the traditional case

This means that every natural transformation

$$\sigma_{a_1 a_2} \quad : \quad \mathcal{A}(a_1, a_2) \quad \longrightarrow \quad \varphi(Fa_2)(a_1)$$

factors as

$$\chi^F_{a_1 a_2} \quad : \quad \mathcal{A}(a_1, a_2) \quad \longrightarrow \quad \mathcal{B}(Fa_1, Fa_2)$$

followed by

$$\theta_{a_1 Fa_2} \quad : \quad \mathcal{B}(Fa_1, Fa_2) \quad \longrightarrow \quad \varphi(Fa_2)(a_1)$$

for a unique natural transformation θ which remains to be defined.

Existence

Given the natural transformation σ, the natural transformation

$$\theta_{a\,b} \quad : \quad \mathcal{B}(Fa, b) \quad \longrightarrow \quad \varphi\,(Fa)\,(b)$$

transports every morphism

$$f \quad : \quad F\,a \quad \longrightarrow \quad b$$

to the result of the action of f on the element

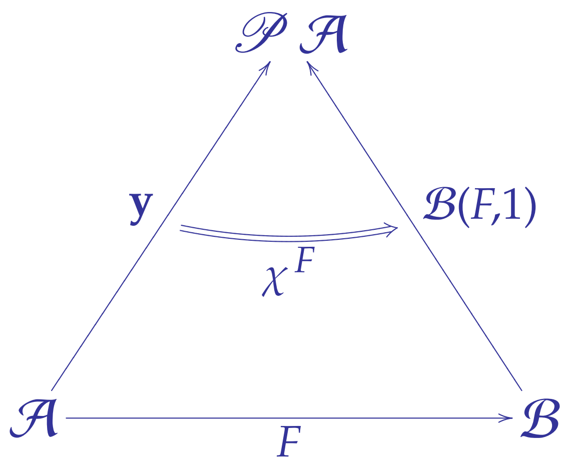
$$\sigma_{a\,a}\,(\,a \xrightarrow{id} a\,) \quad \in \quad \varphi\,(Fa)\,(a)$$

Uniqueness

...

Axiom 2

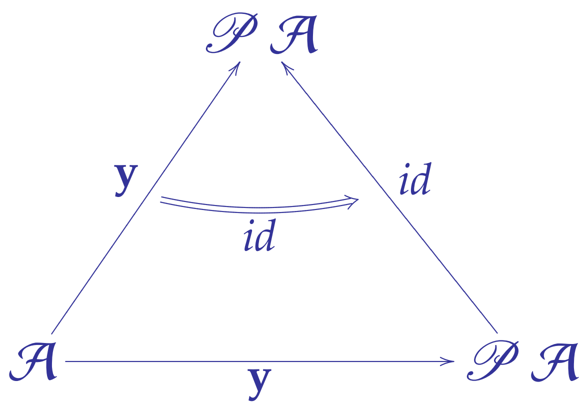
For $\mathcal{A}$ and F accessible, the 2-cell



exhibits the arrow F as an absolute left lifting of $\mathbf{y}_{\mathcal{A}}$ through $\mathcal{B}(F, 1)$.

Axiom 3(i)

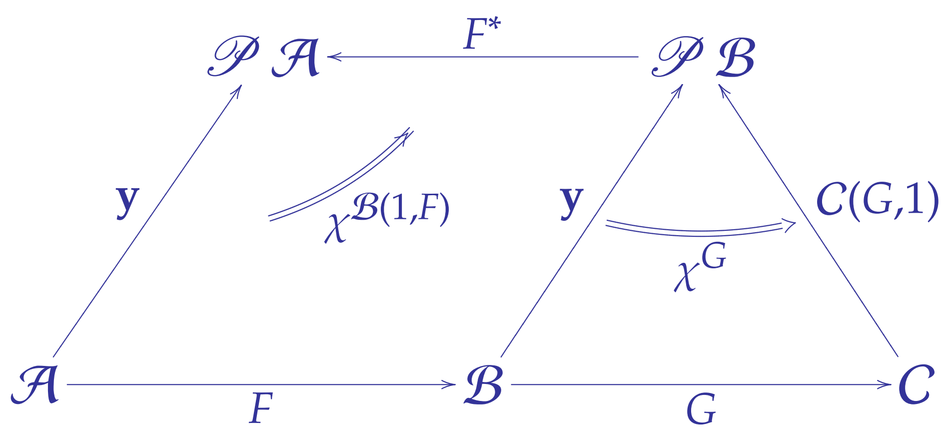
For $\mathcal{A}$ accessible, the identity 2-cell



exhibits the identity arrow as a left extension of $\mathbf{y}_{\mathcal{A}}$ along $\mathbf{y}_{\mathcal{A}}$.

Axiom 3(ii)

For $\mathcal{A}, \mathcal{B}, F, G$ accessible, the 2-cell



exhibits the arrow $F^* \circ \mathcal{C}(G, 1)$ as a left extension of $\mathbf{y}_{\mathcal{A}}$ along $G \circ F$.

The inverse arrow

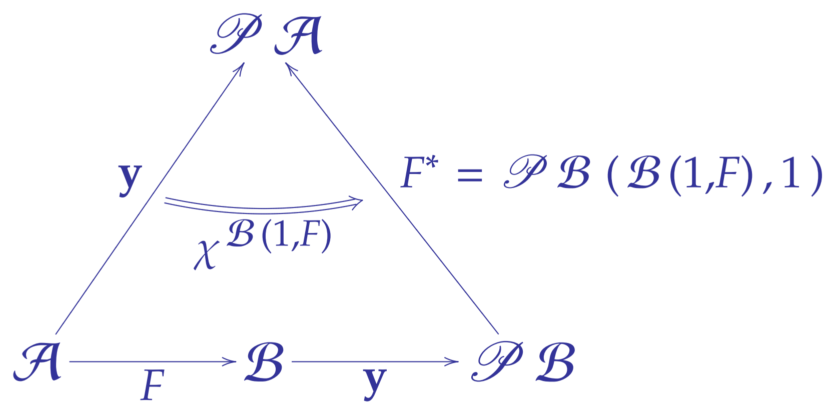
Given an arrow between admissible objects $\mathcal{A}$ and $\mathcal{B}$

$$F \quad : \quad \mathcal{A} \quad \longrightarrow \quad \mathcal{B}$$

the arrow

$$F^* \quad : \quad \mathscr{P}\mathcal{B} \quad \longrightarrow \quad \mathscr{P}\mathcal{A}$$

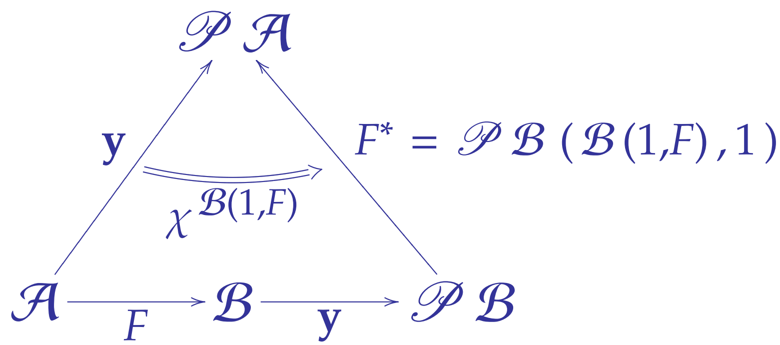
is defined as follows:



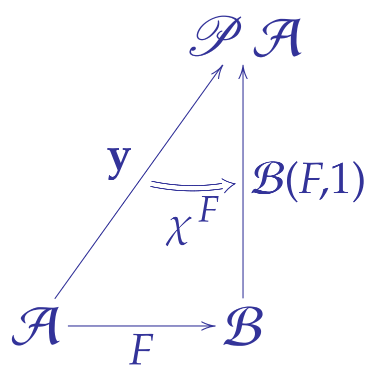
where $\mathcal{B}(1, F)$ denotes the composite arrow $\mathbf{y} \circ F$.

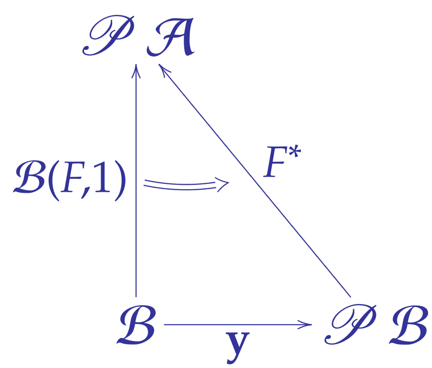
The inverse arrow

The 2-dimensional cell



factors as a pair of Kan extensions:





Existential image

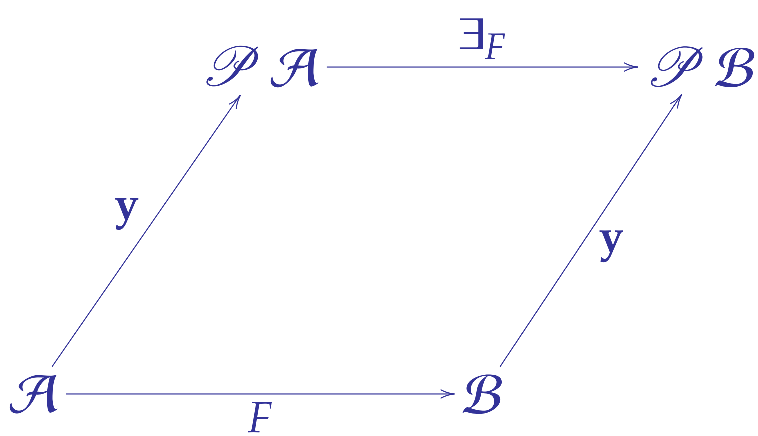
Given an arrow between admissible objects $\mathcal{A}$ and $\mathcal{B}$

$$F \quad : \quad \mathcal{A} \quad \longrightarrow \quad \mathcal{B}$$

the arrow

$$\exists_F \quad : \quad \mathscr{P}\,\mathcal{A} \quad \longrightarrow \quad \mathscr{P}\,\mathcal{B}$$

is defined as follows:



Universal image

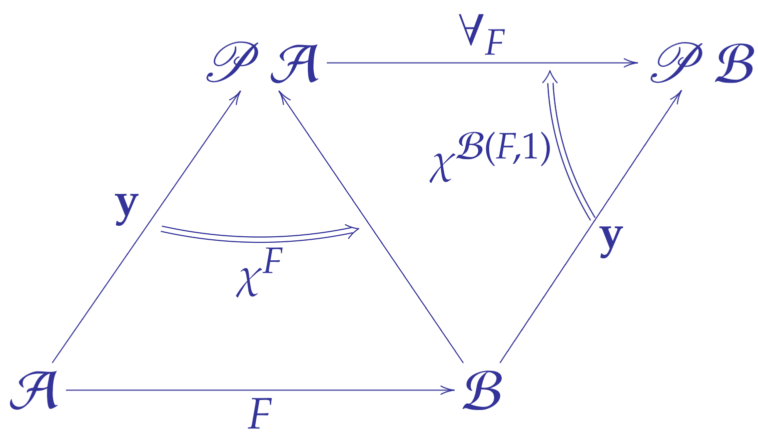
Given an arrow between admissible objects $\mathscr{A}$ and $\mathscr{B}$

$$F \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}$$

the arrow

$$\forall_F \quad : \quad \mathscr{P}\,\mathscr{A} \quad \longrightarrow \quad \mathscr{P}\,\mathscr{B}$$

is defined as follows:



Monads with arity

An idea by Mark Weber

Category with arity

A fully faithful and dense functor

$$i_0 \quad : \quad \Theta_0 \quad \longrightarrow \quad \mathcal{A}$$

where Θ_0 is a small category.

Category with arity

This induces a fully faithful functor

$$\mathcal{A}(i_0, 1) \quad : \quad \mathcal{A} \quad \longrightarrow \quad \mathscr{P}\,\Theta_0$$

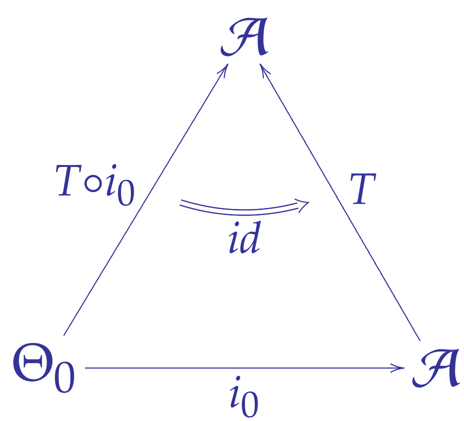
which transports every object A of the category $\mathcal{A}$ into the presheaf

$$\begin{array}{ccccc} \mathcal{A}(i_0, A) & : & \Theta_0 & \longrightarrow & \textbf{Set} \\ & & p & \mapsto & \mathcal{A}(i_0 p, A) \end{array}$$

Monads with arity

A monad T on a category with arity $(\mathcal{A}, i_0)$ such that:

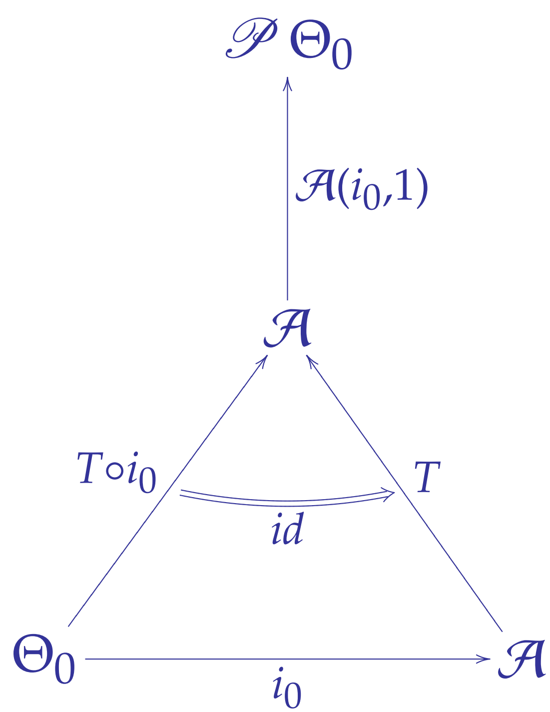
(1) the natural transformation exhibits the functor T



as a left kan extension of the functor $T \circ i_0$ along the functor i_0,

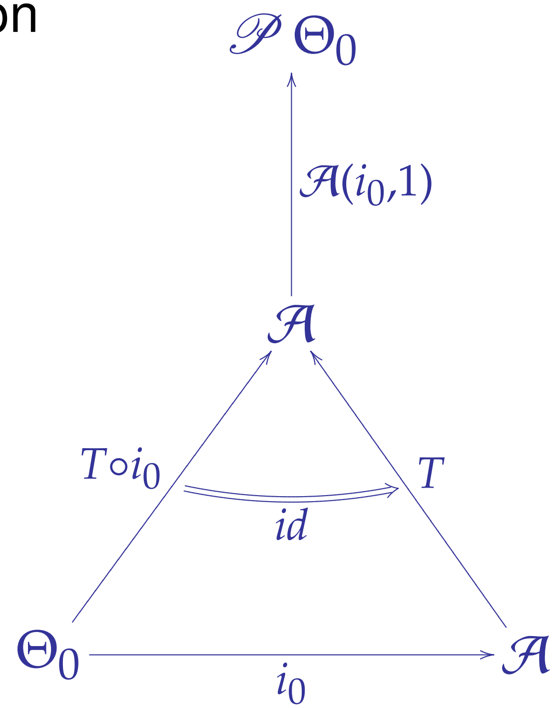
Monads with arity

(2) this left kan extension is preserved by the inclusion functor to $\mathscr{P}\Theta_0$.



Equivalently...

The natural transformation



exhibits $\mathcal{A}(i_0, 1) \circ T$ as a left kan extension of $\mathcal{A}(i_0, 1) \circ T \circ i_0$ along i_0.

Equivalently...

For every object A, the canonical morphism

$$\int^{p \in \Theta_0} \mathcal{A}(i_0 n, T i_0 p) \times \mathcal{A}(i_0 p, A) \quad \longrightarrow \quad \mathcal{A}(i_0 n, TA)$$

is an isomorphism.

Unique factorization up to zig-zag

Every morphism

$$i_0 n \quad \longrightarrow \quad TA$$

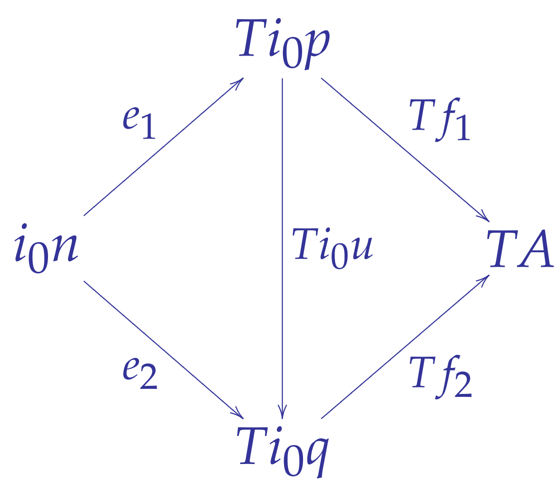
in the category $\mathscr{A}$ decomposes as

$$i_0 n \quad \xrightarrow{e} \quad Ti_0 p \quad \xrightarrow{Tf} \quad TA$$

for a pair of morphisms $e : i_0 n \to Ti_0 p$ and $f : i_0 p \to A$.

Unique factorization up to zig-zag

The factorization should be unique up to zig-zag of



An abstract Segal condition

A general theorem by Mark Weber
axiomatizing a theorem by Clemens Berger
on higher dimensional categories

Motivating example: the free category monad

$$\begin{array}{ccc} \Delta & \longrightarrow & \mathit{Cat} \\ \uparrow & & \uparrow \\ \Delta_0 & \longrightarrow & \mathit{Graph} \end{array}$$

Morphism between categories with arity

A morphism between categories with arity

$$(F, \ell) \quad : \quad (\mathcal{A}, i_0) \quad \longrightarrow \quad (\mathcal{B}, i_1)$$

is defined as a pair of functors (F, ℓ) making the diagram

$$\begin{array}{ccc} \Theta_1 & \xrightarrow{i_1} & \mathcal{B} \\ {\scriptstyle \ell}\uparrow & & \uparrow{\scriptstyle F} \\ \Theta_0 & \xrightarrow{i_0} & \mathcal{A} \end{array}$$

commute.

Morphism between categories with arity

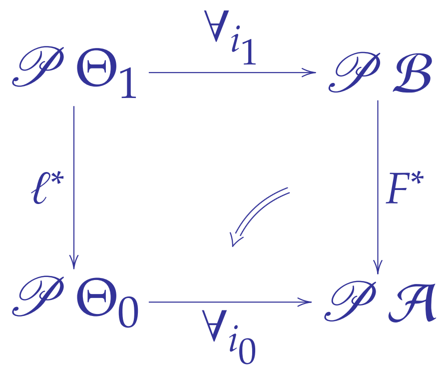
This induces a commutative diagram

$$\begin{array}{ccc} \mathscr{P}\,\Theta_1 & \xleftarrow{\;i_1^*\;} & \mathscr{P}\,\mathcal{B} \\ {\scriptstyle \ell^*}\big\downarrow & \overset{id}{\Leftarrow} & \big\downarrow{\scriptstyle F^*} \\ \mathscr{P}\,\Theta_0 & \xleftarrow[\;i_0^*\;]{} & \mathscr{P}\,\mathcal{A} \end{array}$$

which is required to be an exact square in the sense of Guitart.

Morphism between categories with arity

This means that the Beck-Chevalley condition holds, which states that the canonical natural transformation



is reversible.

Proposition A.

For every morphism (F, ℓ) between categories with arity

$$(F, \ell) \quad : \quad (\mathcal{A}, i_0) \quad \longrightarrow \quad (\mathcal{B}, i_1)$$

the adjunction $i_1^* \dashv \forall_{i_1}$ induces an adjunction between

- the full subcategory of presheaves of $\mathcal{B}$ whose restriction along F is representable in $\mathcal{A}$,
- the full subcategory of presheaves of Θ_1 whose restriction along ℓ is representable along i_0.

Moreover, this adjunction defines an equivalence when the functor F is essentially surjective.

Proposition B.

Every monad T with arity i_0 induces a commutative diagram

$$\begin{array}{ccc} \Theta_T & \xrightarrow{i_T} & \mathcal{A}_T \\ {\scriptstyle \ell}\uparrow & & \uparrow{\scriptstyle F} \\ \Theta_0 & \xrightarrow{i_0} & \mathcal{A} \end{array}$$

where the pair (F, ℓ) defines a morphism

$$(F, \ell) \quad : \quad (\mathcal{A}, i_0) \quad \longrightarrow \quad (\mathcal{A}_T, i_T)$$

of categories with arity.

Algebraic theories with arity

A 2-dimensional approach to Lawvere theories

Algebraic theory with arity

An algebraic theory $\mathbb{L}$ with arity

$$i_0 : \Theta_0 \longrightarrow \mathscr{A}$$

is an identity-on-object functor

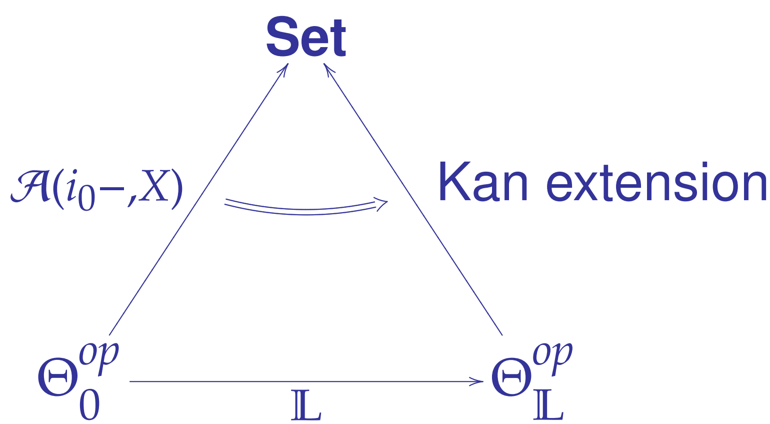
$$\mathbb{L} \quad : \quad \Theta_0 \quad \longrightarrow \quad \Theta_{\mathbb{L}}$$

such that the endofunctor

$$\mathscr{P}\,\Theta_0 \xrightarrow{\exists_{\mathbb{L}}} \mathscr{P}\,\Theta_{\mathbb{L}} \xrightarrow{\mathbb{L}^*} \mathscr{P}\,\Theta_0$$

maps a presheaf representable along i_0 to a presheaf representable along i_0.

Algebraic theories with arity



Model of an algebraic theory

A model A of a Lawvere theory $\mathbb{L}$ is a presheaf

$$A \quad : \quad \Theta_{\mathbb{L}}^{op} \quad \longrightarrow \quad \mathbf{Set}$$

such that the induced presheaf

$$\Theta_0^{op} \xrightarrow{\quad i_0 \quad} \Theta_{\mathbb{L}}^{op} \xrightarrow{\quad A \quad} \mathbf{Set}$$

is representable along i_0.

Main theorem

the category of algebraic theories with arity $(\mathcal{A}, i_0)$

is equivalent to

the category of monads with arity $(\mathcal{A}, i_0)$