# Programming Languages in String Diagrams

( two )

# Asynchronous Games

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## Starting point: game semantics

Every proof of formula $A$ initiates a dialogue where

Proponent tries to convince Opponent

Opponent tries to refute Proponent

An interactive approach to logic and programming languages

# A formal proof of the drinker formula

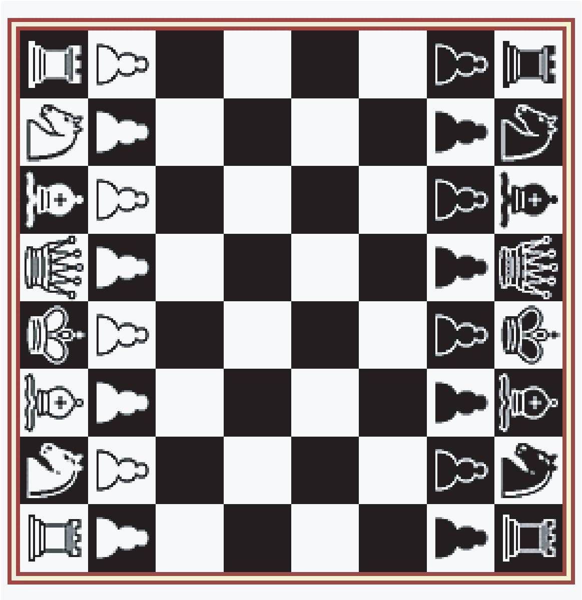
$$
\begin{array}{cl}
\hline
A(x_0) \vdash A(x_0) & \text{Axiom} \\
\hline
A(x_0) \vdash A(x_0), \forall x.A(x) & \text{Right Weakening} \\
\hline
\vdash A(x_0), A(x_0) \Rightarrow \forall x.A(x) & \text{Right} \Rightarrow \\
\hline
\vdash A(x_0), \exists y.\{A(y) \Rightarrow \forall x.A(x)\} & \text{Right } \exists \\
\hline
\vdash \forall x.A(x), \exists y.\{A(y) \Rightarrow \forall x.A(x)\} & \text{Right } \forall \\
\hline
A(y_0) \vdash \forall x.A(x), \exists y.\{A(y) \Rightarrow \forall x.A(x)\} & \text{Left Weakening} \\
\hline
\vdash A(y_0) \Rightarrow \forall x.A(x), \exists y.\{A(y) \Rightarrow \forall x.A(x)\} & \text{Right} \Rightarrow \\
\hline
\vdash \exists y.\{A(y) \Rightarrow \forall x.A(x)\}, \exists y.\{A(y) \Rightarrow \forall x.A(x)\} & \text{Right } \exists \\
\hline
\vdash \exists y.\{A(y) \Rightarrow \forall x.A(x)\} & \text{Contraction}
\end{array}
$$

# Duality

Proponent
Program

plays the game

$A$



Opponent
Environment

plays the game

$\neg A$

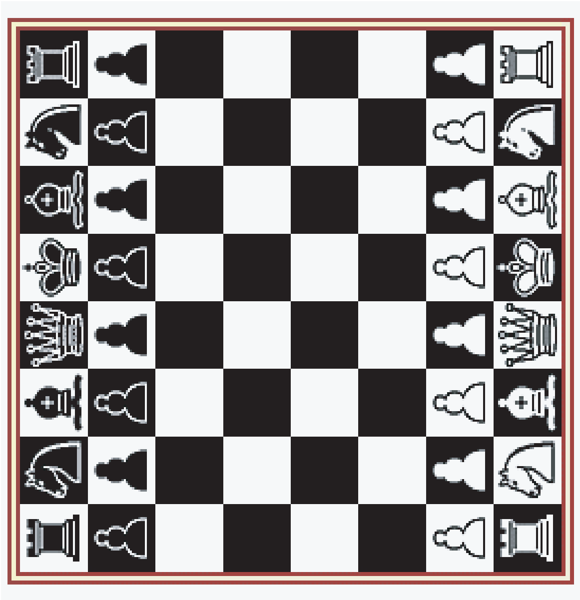
Negation permutes the rôles of Proponent and Opponent

# Duality

Opponent
Environment

plays the game

$\neg A$



Proponent
Program

plays the game

$A$

Negation permutes the rôles of Opponent and Proponent

# Tensor product

$\otimes$

Play the two games in parallel

# Sum

$\oplus$

Proponent selects one component

# Exponentials

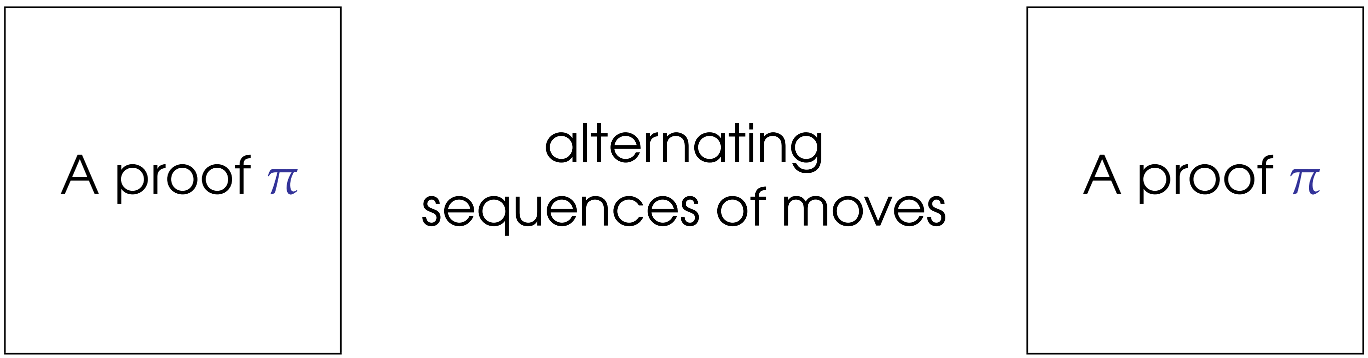
 $\otimes$  $\otimes$  $\otimes$ $\cdots$

Opponent opens as many copies as necessary to beat Proponent

# Sequential games

An idea dating back to André Joyal

# Traditional game semantics




Game semantics: an **interleaving** semantics of proofs.

# Sequential games

A sequential game $(M, P, \lambda)$ consists of

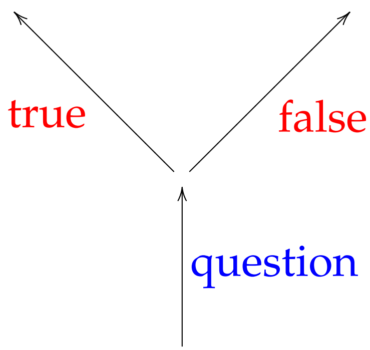
| | |
|---|---|
| $M$ | a set of **moves**, |
| $P \subseteq M^*$ | a set of **plays**, |
| $\lambda : M \to \{-1, +1\}$ | a **polarity** function on moves |

such that every play is **alternating** and **starts by Opponent.**

Alternatively, a sequential game is an alternating decision tree.

# Sequential games

The boolean game $\mathbb{B}$:




Player in red
Opponent in blue

# Deterministic strategies

A **strategy** $\sigma$ is a set of **alternating plays** of **even-length**

$$s \quad = \quad m_1 \cdots m_{2k}$$

such that:

– $\sigma$ contains **the empty play,**

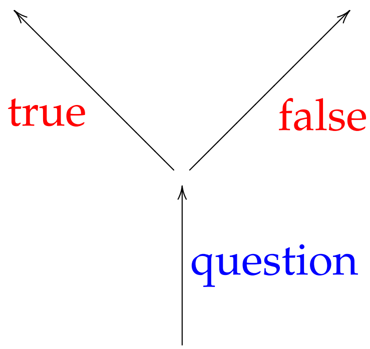
– $\sigma$ is **closed by even-length prefix:**

$$\forall s, \forall m, n \in M, \qquad s \cdot m \cdot n \in \sigma \implies s \in \sigma$$

– $\sigma$ is **deterministic:**

$$\forall s \in \sigma, \forall m, n_1, n_2 \in M, \qquad s \cdot m \cdot n_1 \in \sigma \text{ and } s \cdot m \cdot n_2 \in \sigma \implies n_1 = n_2.$$

# Three strategies on the boolean game $\mathbb{B}$




Player in red
Opponent in blue

## Total strategies

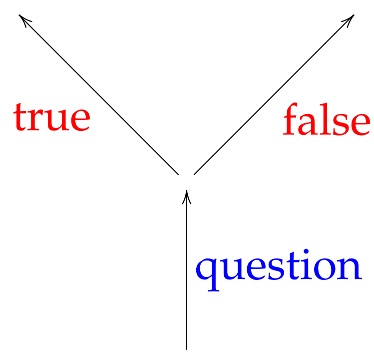
A strategy $\sigma$ is **total** when

– for every play $s$ of the strategy $\sigma$

– for every Opponent move $m$ such that $s \cdot m$ is a play

there exists a Proponent move $n$ such that $s \cdot m \cdot n$ is a play of $\sigma$.

# Two total strategies on the boolean game $\mathbb{B}$




Player in red
Opponent in blue

# Tensor product

Given two sequential games $A$ and $B$, define

$$A \otimes B$$

as the sequential game

$$M_{A \otimes B} = M_A + M_B$$

$$\lambda_{A \otimes B} = [\lambda_A, \lambda_B]$$

$$P_{A \otimes B} = P_A \otimes P_B$$

where $P_A \otimes P_B$ denotes the set of alternating plays in $M^*_{A \otimes B}$ obtained by interleaving a play $s \in P_A$ and a play $t \in P_B$.

# Linear implication

Given two sequential games $A$ and $B$, define

$$A \multimap B$$

as the sequential game

$$M_{A \multimap B} = M_A + M_B$$

$$\lambda_{A \multimap B} = [\, -\lambda_A \,,\, \lambda_B \,]$$

$$P_{A \multimap B} = P_A \multimap P_B$$

where $P_A \multimap P_B$ denotes the set of alternating plays in $M^*_{A \multimap B}$ obtained by interleaving a play $s \in P_A$ and a play $t \in P_B$.

# A category of sequential games

The category **Games** has

– the sequential games as objects

– the strategies of the sequential game

$$\sigma \quad \in \quad A \quad \multimap \quad B$$

as maps

$$\sigma \quad : \quad A \quad \longrightarrow \quad B$$

## The copycat strategy

The identity map

$$id_A \quad : \quad A \quad \longrightarrow \quad A$$

is the **copycat** strategy

$$id_A \quad : \quad A \quad \multimap \quad A$$

defined as

$$id_A \quad = \quad \{ \quad s \ \in \ P_{A \multimap A} \quad | \quad s \ = \ m_1 \cdot m_1 \cdot m_2 \cdot m_2 \cdots \ m_k \cdot m_k \quad \}$$

# The copycat strategy

| $A$ | $\xrightarrow{id}$ | $A$ |
| --- | --- | --- |
| | | $m_1$ |
| $m_1$ | | |
| $m_2$ | | |
| | | $m_2$ |
| | | $m_3$ |
| $m_3$ | | |
| $\vdots$ | | $\vdots$ |
| $m_k$ | | |
| | | $m_k$ |

# Composition

Given two strategies

$$A \xrightarrow{\sigma} B \xrightarrow{\tau} C$$

the composite strategy

$$A \xrightarrow{\sigma\,;\,\tau} C$$

is defined as

$$\sigma\,;\,\tau = \left\{ u \in P_{A \multimap C} \;\middle|\; \exists\, s \in \sigma\,,\, \exists\, t \in \tau \quad \begin{array}{c} u \upharpoonright A = s \upharpoonright A \\ s \upharpoonright B = t \upharpoonright B \\ u \upharpoonright C = t \upharpoonright C \end{array} \right\}$$

The definition of composition is associative.

# Illustration

$$1 \xrightarrow{\text{true}} \mathbb{B} \xrightarrow{id} \mathbb{B}$$

question

question
true

true

# Illustration

$$1 \xrightarrow{\text{false}} \mathbb{B} \xrightarrow{id} \mathbb{B}$$

question
false

question

false

# Illustration

$$1 \xrightarrow{\text{true}} \mathbb{B} \xrightarrow{\text{negation}} \mathbb{B}$$

question true

question false

# An important isomorphism

The sequential game

$$(A \otimes B) \multimap C$$

is isomorphic to the sequential game

$$A \multimap (B \multimap C)$$

for all sequential games $A, B, C$.

Here, isomorphism means tree isomorphism.

# The category of sequential games

**Theorem.**

The category **Games** is symmetric monoidal closed.

As such, it defines a model of linear $\lambda$-calculus.

# Illustration

| $(\mathbb{B}$ | $\multimap$ | $\mathbb{B})$ | $\otimes$ | $\mathbb{B}$ | $\xrightarrow{\text{eval}}$ | $\mathbb{B}$ |
|---|---|---|---|---|---|---|
| | | | | | | question |
| | | | | question | | |
| | | | | bool | | |
| | | | | | | bool |

$$f : \mathbb{B} \multimap \mathbb{B} \;,\; x : \mathbb{B} \;\vdash\; f(x) : \mathbb{B}$$

# Illustration

| $(\mathbb{B}$ | $\multimap$ | $\mathbb{B})$ | $\otimes$ | $\mathbb{B}$ | $\xrightarrow{\text{eval}}$ | $\mathbb{B}$ |
|---|---|---|---|---|---|---|
| | | | | | | question |
| | | question | | | | |
| question | | | | | | |
| | | | | question | | |
| | | | | bool | | |
| bool | | | | | | |
| | | $f(\text{bool})$ | | | | |
| | | | | | | $f(\text{bool})$ |

$$f : \mathbb{B} \multimap \mathbb{B} \;,\; x : \mathbb{B} \vdash f(x) : \mathbb{B}$$

# Illustration

| $(\mathbb{B}$ | $\multimap$ | $\mathbb{B})$ | $\xrightarrow{id}$ | $\mathbb{B}$ | $\multimap$ | $\mathbb{B}$ |
|---|---|---|---|---|---|---|
| | | | | | | question |
| | | question | | | | |
| question | | | | | | |
| | | | | question | | |
| | | | | bool | | |
| bool | | | | | | |
| | | $f$ ( bool ) | | | | |
| | | | | | | $f$ ( bool ) |

$$f : \mathbb{B} \multimap \mathbb{B} \;\vdash\; \lambda x . f(x) : \mathbb{B} \multimap \mathbb{B}$$

# Currification

More generally, the transformation of the term

$$\Gamma \;,\; x : A \vdash f : B$$

into the term

$$\Gamma \vdash \lambda x . f : A \multimap B$$

does not alter the associated strategy, simply reorganizes it.

# Cartesian product

Given two sequential games $A$ and $B$, define

$$A \quad \& \quad B$$

as the sequential game

$$\begin{array}{rcl} M_{A\&B} & = & M_A \quad + \quad M_B \\ \lambda_{A\&B} & = & \lambda_A \quad + \quad \lambda_B \\ P_{A\&B} & = & P_A \quad \oplus \quad P_B \end{array}$$

where $P_A \oplus P_B$ is the **coalesced sum** of the pointed sets $P_A$ and $P_B$.

This means that every **nonempty** play in $A\&B$ is either in $A$ or in $B$.

## The cartesian product

For every sequential game $X$, there exists an isomorphism

$$X \multimap A \& B \quad \cong \quad (X \multimap A) \;\&\; (X \multimap B)$$

This means that the game $A\&B$ is the **cartesian product** of $A$ and $B$.

## The cartesian product

For every game $X$, there exists a bijection between the strategies

$$X \longrightarrow A\&B$$

and the pair of strategies

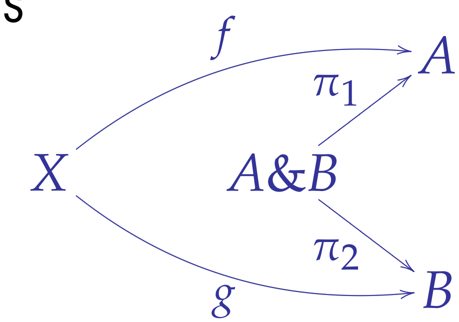
$$X \longrightarrow A \qquad\qquad X \longrightarrow B.$$

# The cartesian product

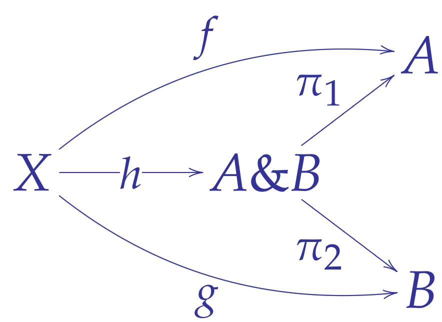
The cartesian product $A\&B$ of two objects $A$ and $B$ is an object equipped with two maps

$$\pi_1 \quad : \quad A\&B \quad \longrightarrow \quad A \qquad\qquad \pi_2 \quad : \quad A\&B \quad \longrightarrow \quad B$$

such that for all diagrams




there exists a unique map $h : X \longrightarrow A \times B$ making the diagram




commute.

# The exponential game

Given a sequential game $A$, define

$$!A$$

as the game

$$\begin{array}{rcl} M_{!A} & = & \mathbb{N} \times M_A \\ \lambda_{!A} & : & (i,m) \mapsto \lambda_A(m) \\ P_{!A} & = & !P_A \end{array}$$

where $!P_A$ is the set of alternating plays in $M_{!A}$ such that for all $i \in \mathbb{N}$

1. the restriction $s \upharpoonright i$ of $s$ to the $i$-th component is a play of $A$
2. for every prefix $u$ of $s$ : $u \upharpoonright i = \varepsilon \Rightarrow u \upharpoonright i+1 = \varepsilon$.

# Two important isomorphisms

Define

$$A \Rightarrow B \quad = \quad !A \multimap B$$

Observe the isomorphism

$$!(A \& B) \quad \cong \quad !A \otimes !B$$

and the isomorphism

$$\begin{array}{rcl} (A \& B) \Rightarrow C & = & !(A \& B) \multimap C \\ & \cong & (!A \otimes !B) \multimap C \\ & \cong & !A \multimap (!B \multimap C) \\ & = & A \Rightarrow (B \Rightarrow C) \end{array}$$

## A model of the simply-typed $\lambda$-calculus

**Fact.**

Every simply-typed $\lambda$-term

$$x_1 : A_1 \quad , \quad \cdots \quad , \quad x_k : A_k \quad \vdash \quad M \quad : \quad B$$

may be interpreted as a strategy in the game

$$!A_1 \quad \otimes \quad \cdots \quad \otimes \quad !A_k \quad \longrightarrow \quad B$$

Here, the modality $!$ enables to backtrack in the sequential games

# Illustration

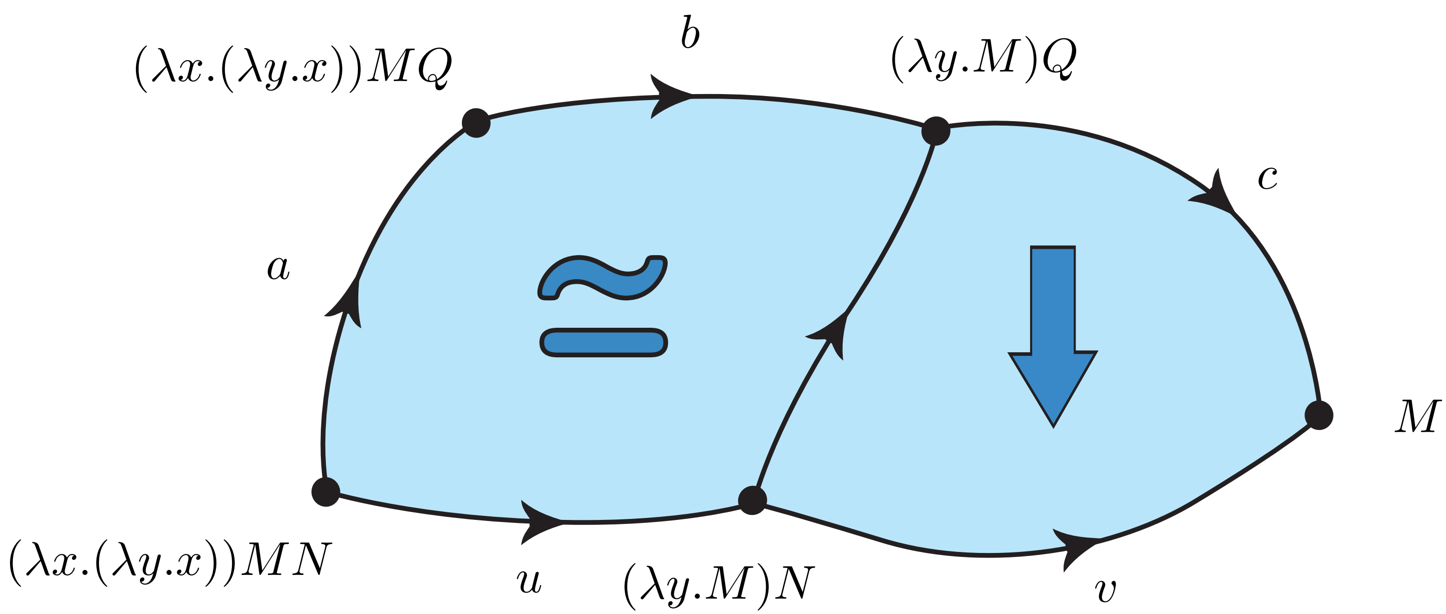
| $!(\mathbb{N}$ | $\multimap$ | $\mathbb{N})$ | $\otimes$ | $\mathbb{N}$ | $\xrightarrow{M}$ | $\mathbb{N}$ |
|---|---|---|---|---|---|---|
| | | | | | | question |
| | | $\text{question}_1$ | | | | |
| $\text{question}_1$ | | | | | | |
| | | $\text{question}_2$ | | | | |
| $\text{question}_2$ | | | | | | |
| | | | | question | | |
| | | | | $n$ | | |
| $n$ | | | | | | |
| | | $f(n)$ | | | | |
| $f(n)$ | | | | | | |
| | | $f\,f\,(n)$ | | | | |
| | | | | | | $f\,f\,(n)$ |

$$f : !(\mathbb{N} \multimap \mathbb{N}),\ x : \mathbb{N} \vdash f\ f\ (x) : \mathbb{N}$$

# Asynchronous games

An asynchronous reconstruction of the $\lambda$-calculus

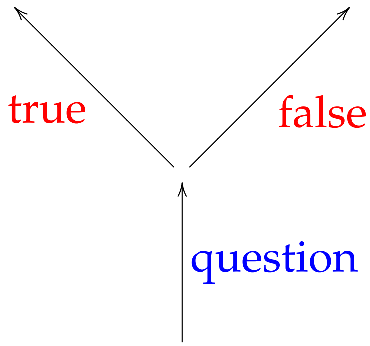
# Geometry of rewriting




Can we do the same thing for open systems ?

# The traditional interleaving semantics

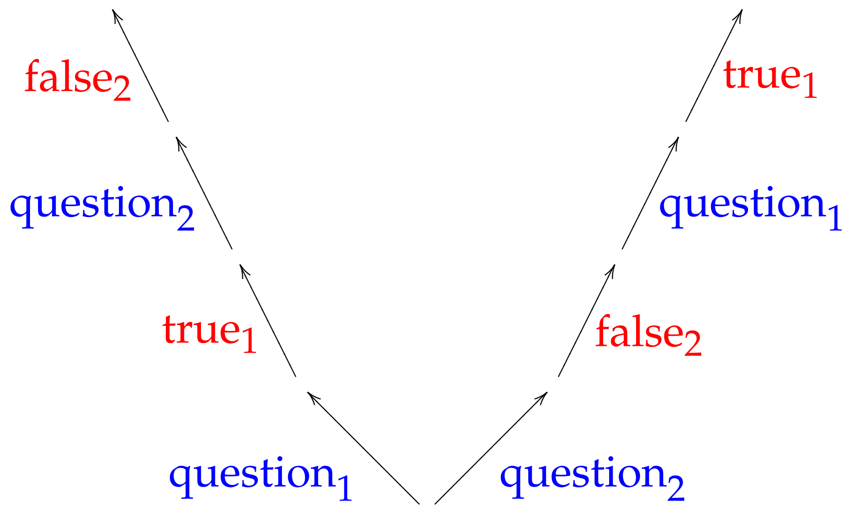
The boolean game $\mathbb{B}$:



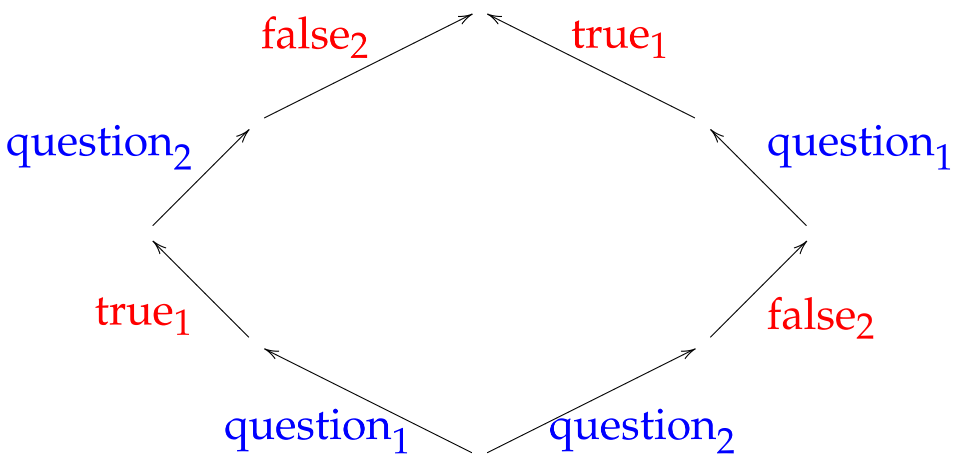

Player in red
Opponent in blue

# The traditional interleaving semantics

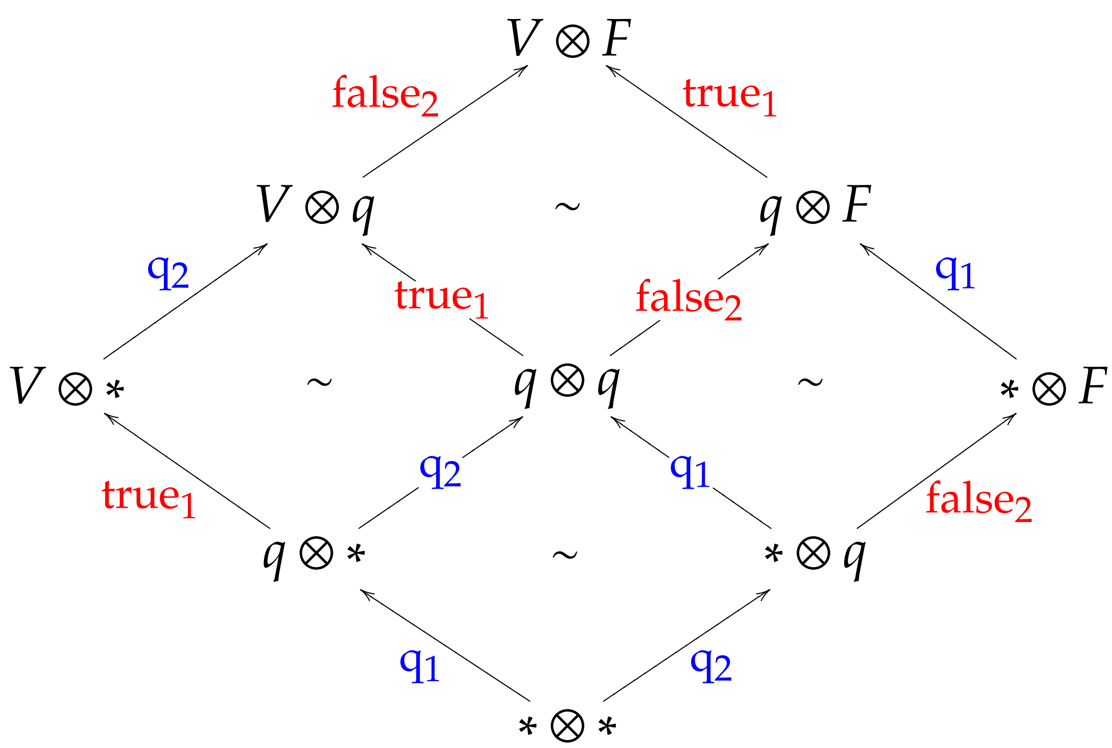
The tensor product of two boolean games $\mathbb{B}_1$ et $\mathbb{B}_2$:



# A step towards true concurrency: bend the branches!



# True concurrency: tile the diagram!



## Asynchronous game semantics

A proof $\pi$

trajectories in
an asynchronous graph

A proof $\pi$

The phenomenon refined: a **truly concurrent** semantics of proofs.

# Event structures

An **event structure** $(M, \leq, \#)$ consists of:

— a partially ordered set $(M, \leq)$ where

$$m\downarrow \quad = \quad \{ n \in M \mid n \leq m \}$$

is **finite** for every event $m \in M$,

— a binary symmetric irreflexive relation $\#$ satisfying

$$\forall m, n, p \in M, \qquad m \# n \leq p \quad \Rightarrow \quad m \# p.$$

## The asynchronous graph

A **position** is a **compatible downward-closed** subset of $M$.

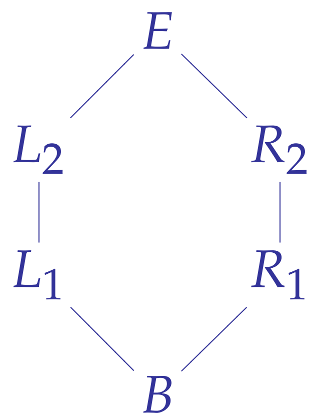
The **asynchronous graph** has:

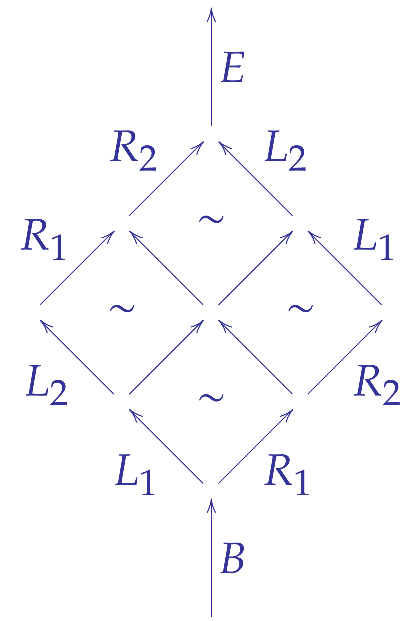
— the finite positions as vertices

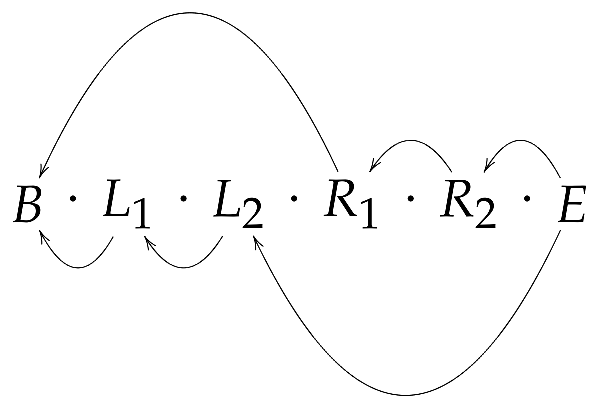
— the triples $(x, m, y)$ such that $y = x \uplus \{m\}$ as edges $x \longrightarrow y$.

The asynchronous graph is **pointed** with the empty position $*$ as root.

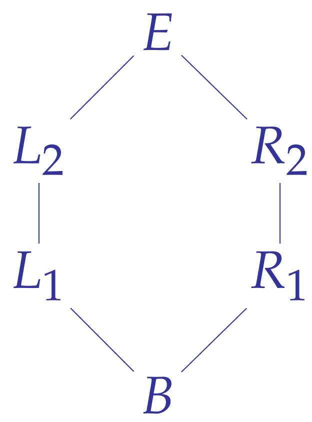
# Event structures

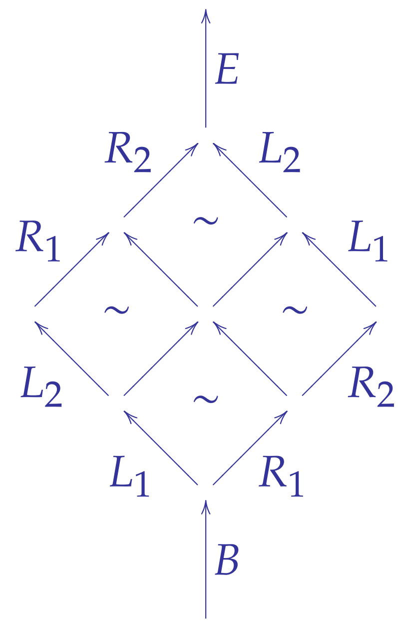


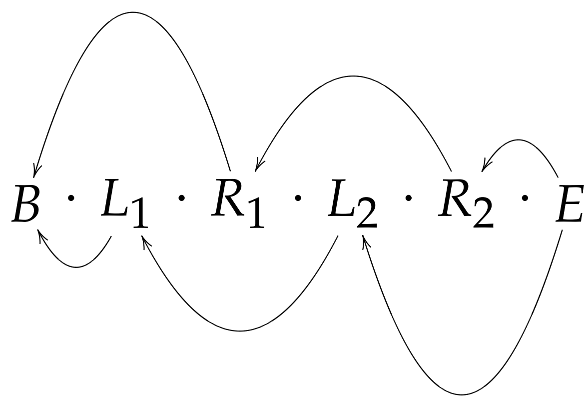



# Event structures



## Asynchronous games

An **asynchronous game** is an event structure

$$(M, \leq, \#)$$

equipped with a **polarity** function

$$\lambda \quad : \quad M \longrightarrow \{-1, +1\}$$

indicating whether a move is Player (+1) or Opponent (−1).

## Legal plays

A **legal play** is a path

$$* \xrightarrow{m_1} x_1 \xrightarrow{m_2} x_2 \xrightarrow{m_3} \cdots x_{k-1} \xrightarrow{m_k} x_k$$

starting from the empty position $*$ and satisfying:

$$\forall i \in [1, \ldots, k] \qquad \lambda(m_i) = (-1)^i$$

So, a legal play is **alternating** and starts by an **Opponent move.**

# Strategies

A **strategy** is a set of **legal plays of even length**, such that:

— $\sigma$ contains **the empty play,**

— $\sigma$ is **closed under even-length prefix**

$$s \cdot m \cdot n \in \sigma \;\Rightarrow\; s \in \sigma,$$

— $\sigma$ is **deterministic**

$$s \cdot m \cdot n_1 \in \sigma \quad \text{and} \quad s \cdot m \cdot n_2 \in \sigma \quad \Rightarrow \quad n_1 = n_2.$$

A strategy plays according to the current play.

# Innocence: strategies with partial information

**Full definability result** (Hyland, Ong, Nickau 1994)

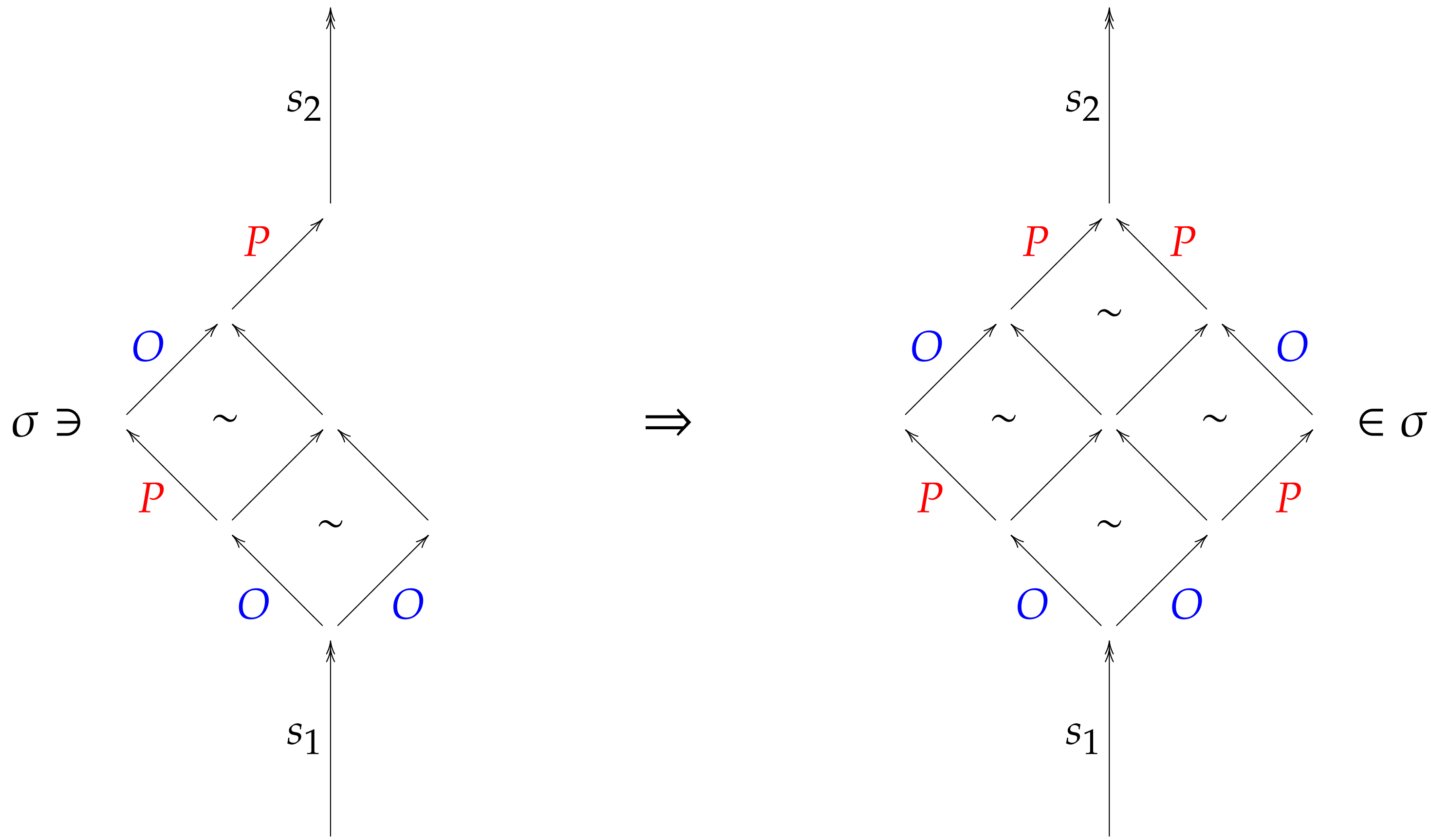
Innocence characterizes the interactive behavior of $\lambda$-terms.

An innocent strategy plays according to the current view.

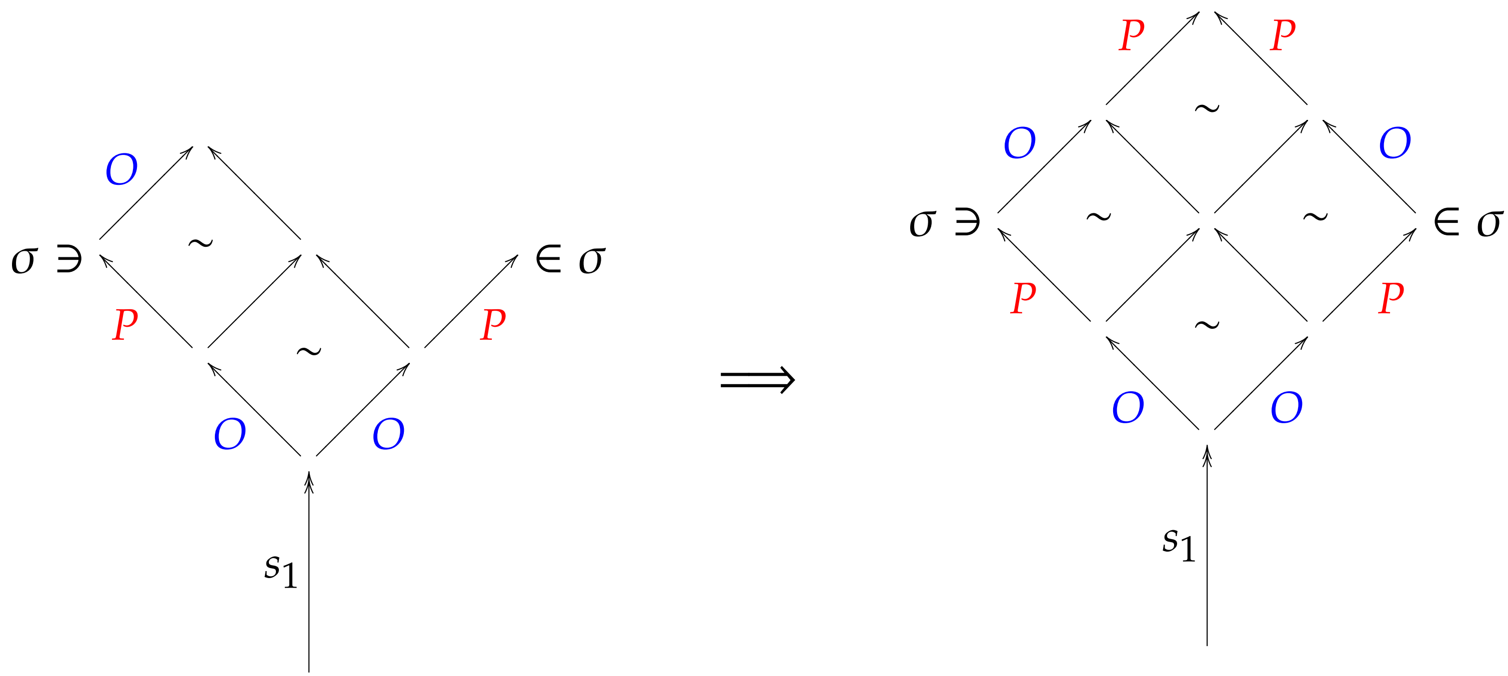
# A positionality theorem for innocent strategies

Starting from a local criterion of innocence

# Backward innocence



# Forward innocence



# Innocent strategies are positional

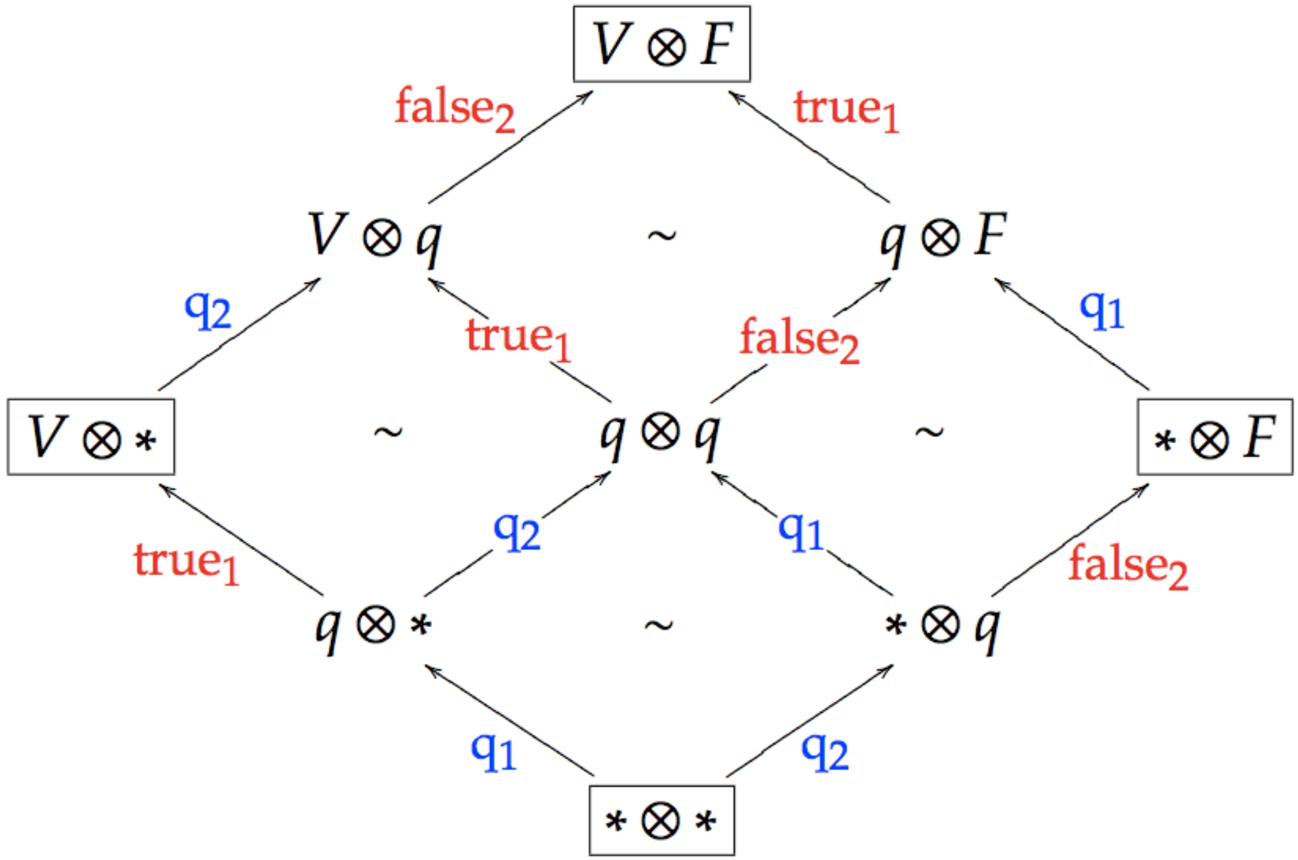
Definition. A strategy $\sigma$ is **positional** when for every two plays $s_1$ and $s_2$ with same target $x$:

$$s_1 \in \sigma \quad \text{and} \quad s_2 \in \sigma \quad \text{and} \quad s_1 \cdot t \in \sigma \quad \Rightarrow \quad s_2 \cdot t \in \sigma$$

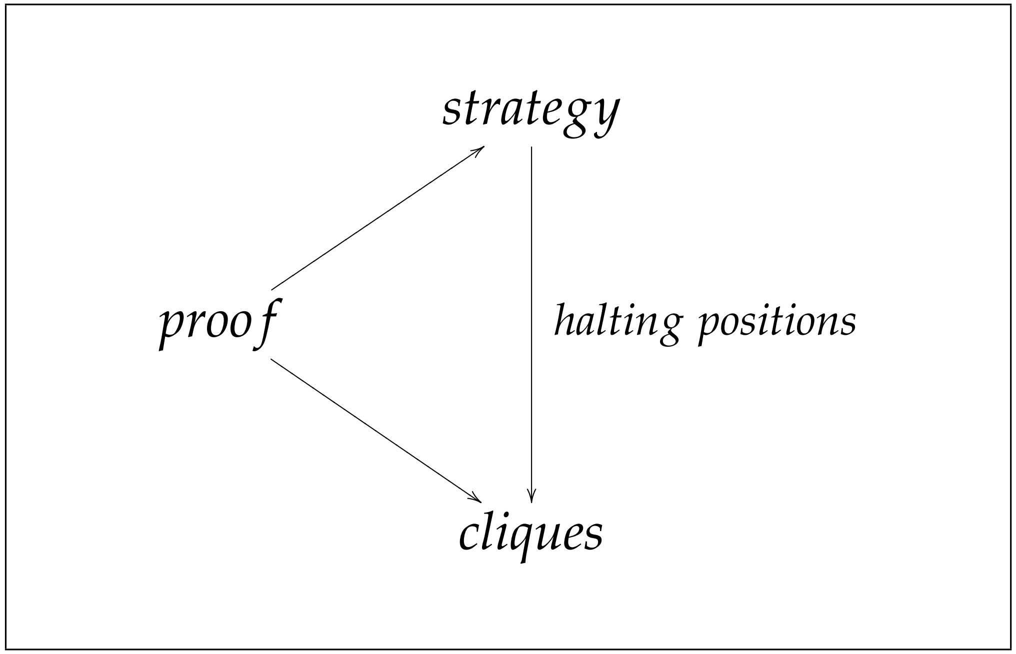
**Theorem** (by an easy diagrammatic proof)
Every innocent strategy $\sigma$ is positional

**More:** An innocent strategy is characterized by the positions it reaches.

# An illustration: the strategy (true ⊗ false)



# Hence, the schism between games and linear logic




Imagine that a discussion is defined by the set of its final states

# Tensorial logic

A primitive logic of tensor and negation

## Five primitive components of logic

| | | |
|---|---|---|
| [1] | the negation | $\neg$ |
| [2] | the linear conjunction | $\otimes$ |
| [3] | the repetition modality | $!$ |
| [4] | the existential quantification | $\exists$ |
| [5] | the least fixpoint | $\mu$ |

Logic = Data Structure + Duality

# Tensorial logic

$$\frac{}{A \vdash A} \quad \text{axiom}$$

$$\frac{\Gamma \vdash A \qquad A, \Delta \vdash B}{\Gamma, \Delta \vdash B} \quad \text{cut}$$

$$\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash} \quad \text{left } \neg$$

$$\frac{\Gamma, A \vdash}{\Gamma \vdash \neg A} \quad \text{right } \neg$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} \quad \text{left } \otimes$$

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \quad \text{right } \otimes$$

$$\frac{\Gamma \vdash A}{\Gamma, \mathbf{true} \vdash A} \quad \text{left } \mathbf{true}$$

$$\frac{}{\vdash \mathbf{true}} \quad \text{right } \mathbf{true}$$

# Tensorial logic

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}$$ weakening

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}$$ contraction

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}$$ dereliction

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}$$ promotion

$$\frac{\Gamma, A(x) \vdash B}{\Gamma, \exists x.A \vdash B}$$ left $\exists$

$$\frac{\Gamma \vdash A(t)}{\Gamma \vdash \exists x.A}$$ right $\exists$

# Tensorial logic

The **boolean** in linear logic:

$$\mathbb{B} \quad := \quad (1 \oplus 1)$$

The **boolean** in tensorial logic:

$$\mathbb{B} \quad := \quad \neg\neg \quad (1 \oplus 1)$$

The **classical boolean** in tensorial logic:

$$\mathbb{C} \quad := \quad \neg \quad ! \quad \neg \quad (1 \oplus 1)$$

## The left-to-right implementation of conjunction

$$\mathbb{B} \quad \otimes \quad \mathbb{B} \quad \longrightarrow \quad \mathbb{B}$$

question (in the result $\mathbb{B}$)

question
$\text{answer}_1$ (in the first $\mathbb{B}$)

question
$\text{answer}_2$ (in the second $\mathbb{B}$)

$\text{answer}_1 \wedge \text{answer}_2$ (in the result $\mathbb{B}$)

## The right-to-left implementation of conjuction

$$\mathbb{B} \quad \otimes \quad \mathbb{B} \quad \longrightarrow \quad \mathbb{B}$$

question

question

$\text{answer}_2$

question

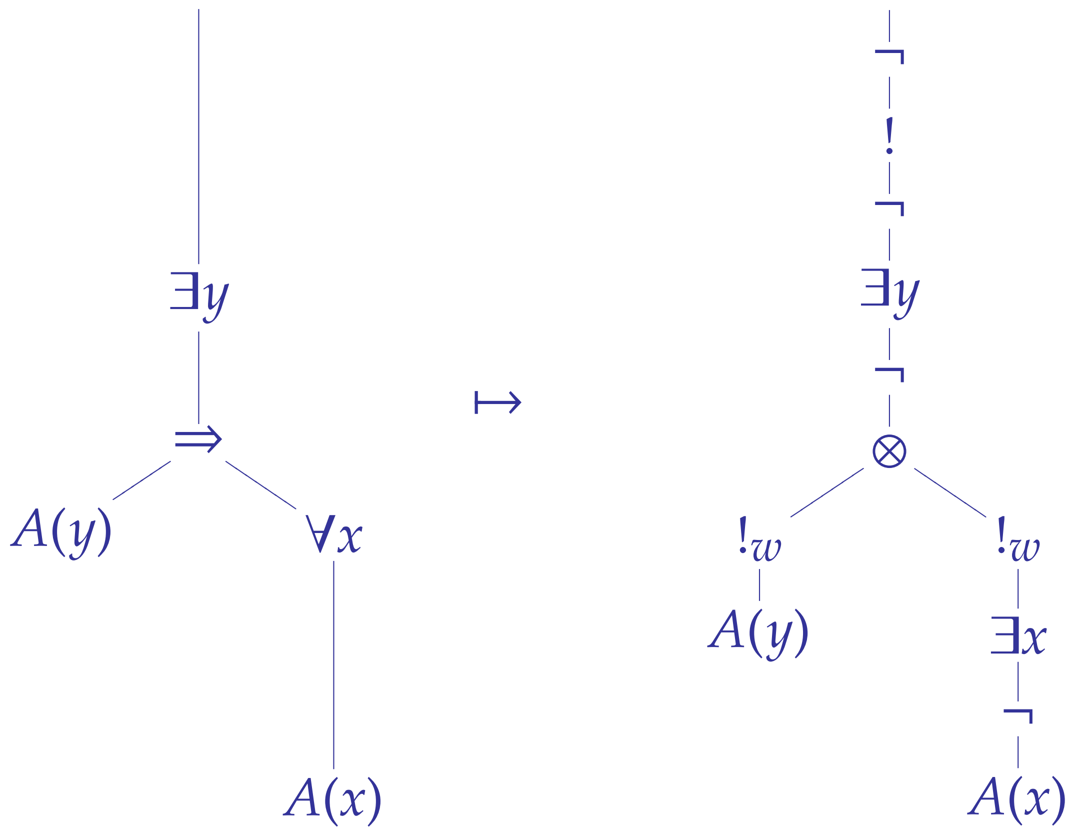
$\text{answer}_1$

$\text{answer}_1 \wedge \text{answer}_2$

# A formal proof of the drinker formula

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A(x_0) \vdash A(x_0)}\ \text{Axiom}}{A(x_0) \vdash A(x_0), \forall x.A(x)}\ \text{Right Weakening}}{\vdash A(x_0), A(x_0) \Rightarrow \forall x.A(x)}\ \text{Right} \Rightarrow}{\vdash A(x_0), \exists y.\{A(y) \Rightarrow \forall x.A(x)\}}\ \text{Right} \exists}{\vdash \forall x.A(x), \exists y.\{A(y) \Rightarrow \forall x.A(x)\}}\ \text{Right} \forall}{A(y_0) \vdash \forall x.A(x), \exists y.\{A(y) \Rightarrow \forall x.A(x)\}}\ \text{Left Weakening}}{\vdash A(y_0) \Rightarrow \forall x.A(x), \exists y.\{A(y) \Rightarrow \forall x.A(x)\}}\ \text{Right} \Rightarrow}{\vdash \exists y.\{A(y) \Rightarrow \forall x.A(x)\}, \exists y.\{A(y) \Rightarrow \forall x.A(x)\}}\ \text{Right} \exists}{\vdash \exists y.\{A(y) \Rightarrow \forall x.A(x)\}}\ \text{Contraction}
$$

# The drinker formula in tensorial logic



# A proof of the drinker formula in tensorial logic

Axiom

$$A(x_0) \quad \vdash \quad A(x_0)$$

Negation

$$\neg A(x_0) \;,\; A(x_0) \quad \vdash$$

Dereliction

$$\neg A(x_0) \;,\; !_w A(x_0) \quad \vdash$$

Weakening

$$\neg A(x_0) \;,\; !_w A(x_0) \;,\; !_w \exists x \neg A(x) \quad \vdash$$

Negation

$$\neg A(x_0) \quad \vdash \quad \neg \{ !_w A(x_0) \otimes !_w \exists x \neg A(x) \}$$

Right $\exists$

$$\neg A(x_0) \quad \vdash \quad \exists y \neg \{ !_w A(y) \otimes !_w \exists x \neg A(x) \}$$

Negation

$$\neg \exists y \neg \{ !_w A(y) \otimes !_w \exists x \neg A(x) \} \;,\; \neg A(x_0) \quad \vdash$$

Left $\exists$

$$\neg \exists y \neg \{ !_w A(y) \otimes !_w \exists x \neg A(x) \} \;,\; \exists x \neg A(x) \quad \vdash$$

Dereliction

$$\neg \exists y \neg \{ !_w A(y) \otimes !_w \exists x \neg A(x) \} \;,\; !_w \exists x \neg A(x) \quad \vdash$$

Weakening

$$\neg \exists y \neg \{ !_w A(y) \otimes !_w \exists x \neg A(x) \} \;,\; !_w A(y_0) \;,\; !_w \exists x.\neg A(x) \quad \vdash$$

Negation

$$\neg \exists y.\neg \{ !_w A(y) \otimes !_w \exists x.\neg A(x) \} \quad \vdash \quad \neg \{ !_w A(y_0) \otimes !_w \exists x \neg A(x) \}$$

Right $\exists$

$$\neg \exists y \neg \{ !_w A(y) \otimes !_w \exists x \neg A(x) \} \quad \vdash \quad \exists y.\neg \{ !_w A(y) \otimes !_w \exists x.\neg A(x) \}$$

Negation

$$\neg \exists y \neg \{ !_w A(y) \otimes !_w \exists x \neg A(x) \} \;,\; \neg \exists y \neg \{ !_w A(y) \otimes !_w \exists x \neg A(x) \} \quad \vdash$$

Dereliction

$$! \neg \exists y \neg \{ !_w A(y) \otimes !_w \exists x \neg A(x) \} \;,\; ! \neg \exists y \neg \{ !_w A(y) \otimes !_w \exists x \neg A(x) \} \quad \vdash$$

Contraction

$$! \neg \exists y \neg \{ !_w A(y) \otimes !_w \exists x \neg A(x) \} \quad \vdash$$

Negation

$$\vdash \quad \neg ! \neg \exists y \neg \{ !_w A(y) \otimes !_w \exists x \neg A(x) \}$$

# A proof of the drinker formula in tensorial logic

$$A(x_0) \quad \vdash \quad A(x_0)$$

Axiom

Negation

$$\neg A(x_0) \;,\; A(x_0) \quad \vdash$$

$$\neg A(x_0) \;,\; !_w A(x_0) \quad \vdash$$

$$\neg A(x_0) \;,\; !_w A(x_0) \;,\; !_w \exists x \neg A(x) \quad \vdash$$

Negation$_2$

$$\neg A(x_0) \quad \vdash \quad \neg \{ !_w A(x_0) \otimes !_w \exists x \neg A(x) \}$$

Right $\exists$

$$\neg A(x_0) \quad \vdash \quad \exists y \neg \{ !_w A(y) \otimes !_w \exists x \neg A(x) \}$$

Negation$_1$

$$\neg \exists y \neg \{ !_w A(y) \otimes !_w \exists x \neg A(x) \} \;,\; \neg A(x_0) \quad \vdash$$

Left $\exists$

$$\neg \exists y \neg \{ !_w A(y) \otimes !_w \exists x \neg A(x) \} \;,\; \exists x \neg A(x) \quad \vdash$$

$$\neg \exists y \neg \{ !_w A(y) \otimes !_w \exists x \neg A(x) \} \;,\; !_w \exists x \neg A(x) \quad \vdash$$

$$\neg \exists y \neg \{ !_w A(y) \otimes !_w \exists x \neg A(x) \} \;,\; !_w A(y_0) \;,\; !_w \exists x . \neg A(x) \quad \vdash$$

Negation$_2$

$$\neg \exists y . \neg \{ !_w A(y) \otimes !_w \exists x . \neg A(x) \} \quad \vdash \quad \neg \{ !_w A(y_0) \otimes !_w \exists x \neg A(x) \}$$

Right $\exists$

$$\neg \exists y \neg \{ !_w A(y) \otimes !_w \exists x \neg A(x) \} \quad \vdash \quad \exists y . \neg \{ !_w A(y) \otimes !_w \exists x . \neg A(x) \}$$

Negation$_1$

$$\neg \exists y \neg \{ !_w A(y) \otimes !_w \exists x \neg A(x) \} \;,\; \neg \exists y \neg \{ !_w A(y) \otimes !_w \exists x \neg A(x) \} \quad \vdash$$

$$! \neg \exists y \neg \{ !_w A(y) \otimes !_w \exists x \neg A(x) \} \;,\; ! \neg \exists y \neg \{ !_w A(y) \otimes !_w \exists x \neg A(x) \} \quad \vdash$$

Contraction

$$! \neg \exists y \neg \{ !_w A(y) \otimes !_w \exists x \neg A(x) \} \quad \vdash$$

Negation$_0$

$$\vdash \quad \neg ! \neg \exists y \neg \{ !_w A(y) \otimes !_w \exists x \neg A(x) \}$$