Dialogue categories and Frobenius monoids

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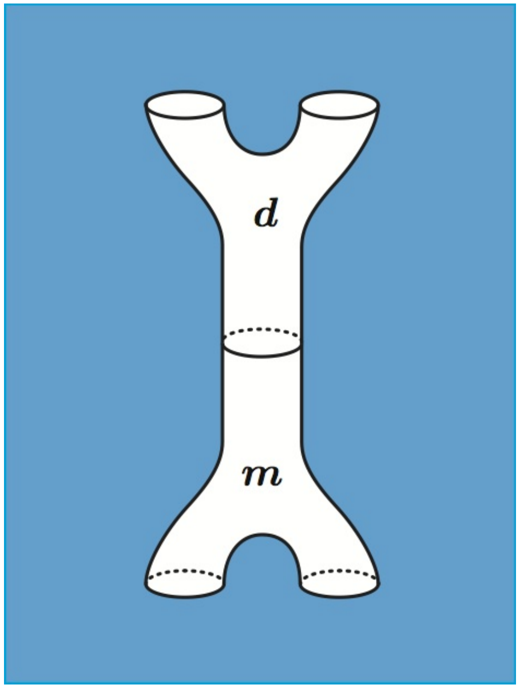
Two [academic] lifes entangled

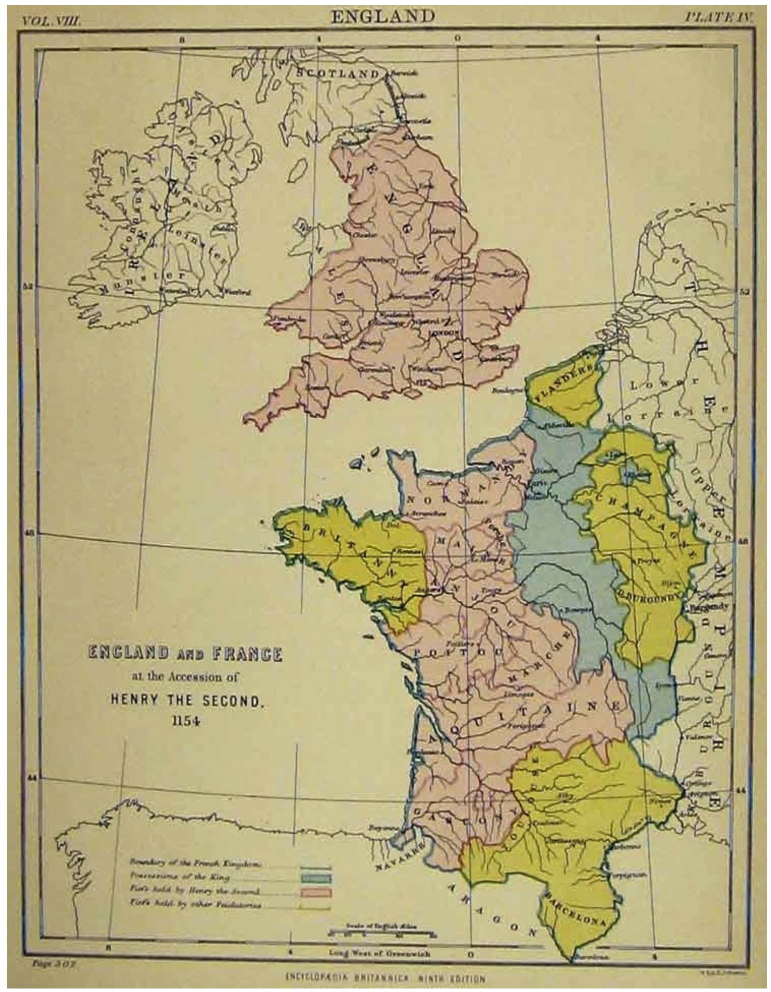
Dialogue games

$\Longleftrightarrow$



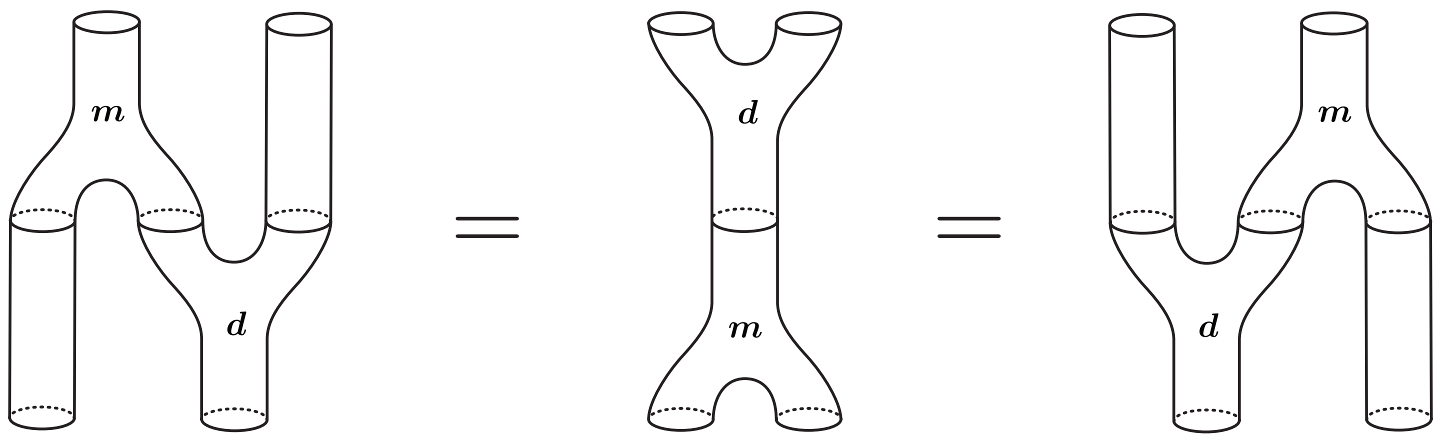
Frobenius algebras

Living on both sides of the Channel



The Australian connection

A Frobenius monoid F is a monoid and a comonoid satisfying



A deep relationship with $*$-autonomous categories
discovered by Brian Day and Ross Street.

Original purpose of tensorial logic

To provide a clear type-theoretic foundation to game semantics

Propositions as types $\Leftrightarrow$ Propositions as games

based on the idea that

game semantics is a diagrammatic syntax of continuations

Continuations

Captures the difference between addition as a **function**

$$\texttt{nat} \times \texttt{nat} \quad \Rightarrow \quad \texttt{nat}$$

and addition as a **sequential algorithm**

$$(\texttt{nat} \Rightarrow \bot) \Rightarrow \bot \quad \times \quad (\texttt{nat} \Rightarrow \bot) \Rightarrow \bot \quad \times \quad (\texttt{nat} \Rightarrow \bot) \quad \Rightarrow \quad \bot$$

This enables to distinguish the **left-to-right** implementation

$$\texttt{lradd} \quad = \quad \lambda\varphi.\,\lambda\psi.\,\lambda k.\,\varphi\,(\,\lambda x.\,\psi\,(\,\lambda y.\,k\,(x+y))\,)$$

from the **right-to-left** implementation

$$\texttt{rladd} \quad = \quad \lambda\varphi.\,\lambda\psi.\,\lambda k.\,\psi\,(\,\lambda y.\,\varphi\,(\,\lambda x.\,k\,(x+y))\,)$$

The left-to-right addition

$\neg\neg$ nat	$\times$	$\neg\neg$ nat	$\Rightarrow$	$\neg\neg$ nat
				question
question				
12				
		question		
		5		
				17

$$\texttt{lradd} \quad = \quad \lambda\varphi.\,\lambda\psi.\,\lambda k.\,\varphi\,(\,\lambda x.\,\psi\,(\,\lambda y.\,k\,(x+y))\,)$$

The right-to-left addition

$\neg\neg$ nat $\times$ $\neg\neg$ nat $\Rightarrow$ $\neg\neg$ nat

question

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question
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17

$$\texttt{rladd} \quad = \quad \lambda\varphi.\,\lambda\psi.\,\lambda k.\,\psi\,(\,\lambda y.\,\varphi\,(\,\lambda x.\,k\,(x+y)))$$

Tensorial logic

tensorial logic	=	a logic of tensor and negation
	=	linear logic without $A \cong \neg\neg A$
	=	the syntax of linear continuations
	=	the syntax of dialogue games

A synthesis between linear logic and game semantics

Tensorial logic

- ▷ Every sequent of the logic is of the form:

$$A_1 , \cdots , A_n \quad \vdash \quad B$$

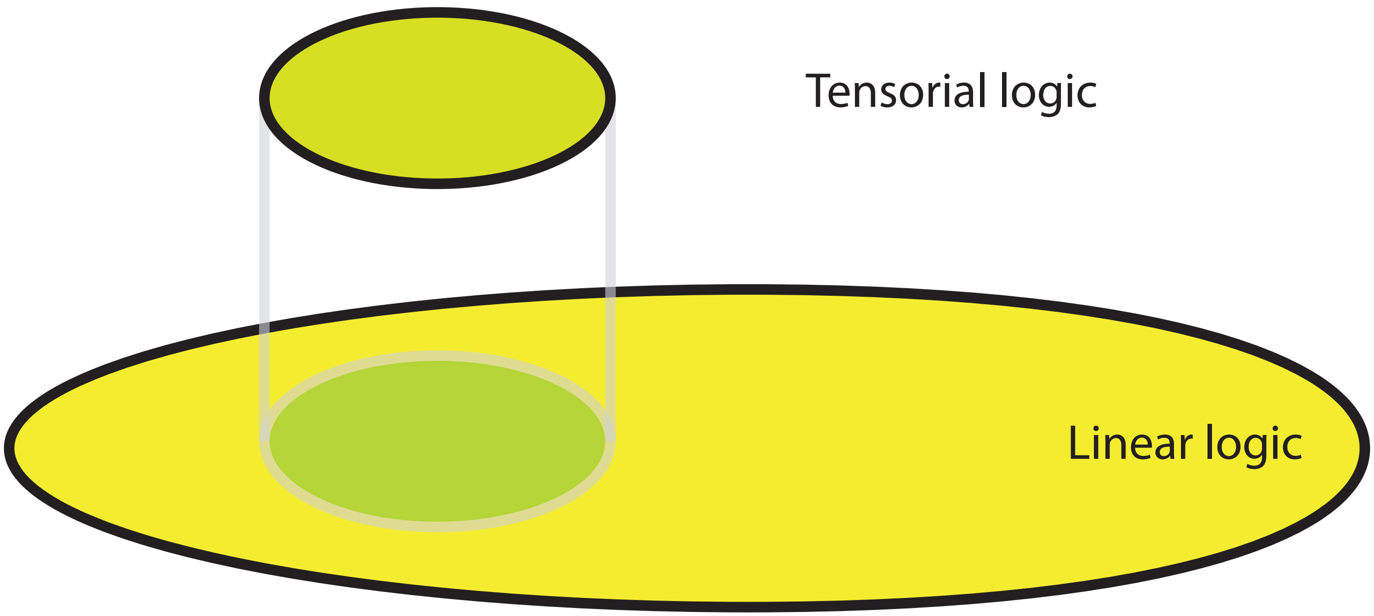
- ▷ Main rules of the logic:

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad \frac{\Gamma , A , B , \Delta \vdash C}{\Gamma , A \otimes B , \Delta \vdash C}$$

$$\frac{\Gamma , A \vdash \bot}{\Gamma \vdash \neg A} \qquad \frac{\Gamma \vdash A}{\Gamma , \neg A \vdash \bot}$$

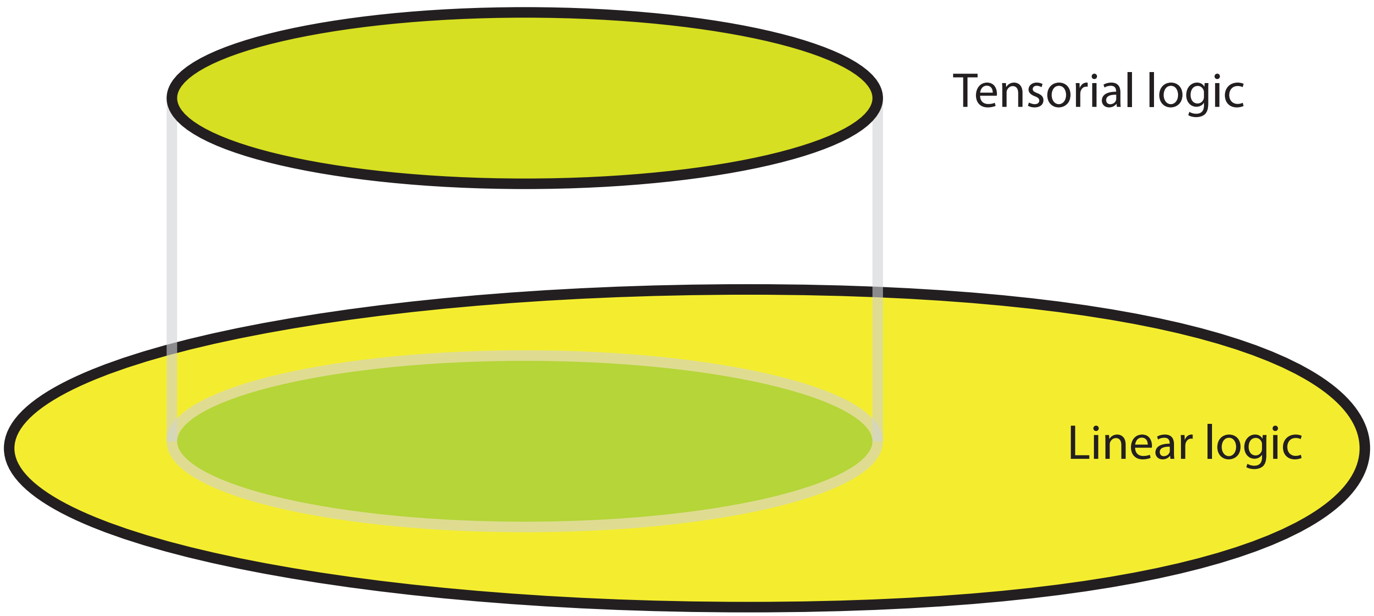
The primitive kernel of logic

A different way to think of polarities



Motto: linear logic is a depolarized tensorial logic

A different way to think of polarities



Motto: linear logic is a depolarized tensorial logic

The left-to-right scheduler

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A} \qquad \dfrac{}{B \vdash B}}{A , B \vdash A \otimes B}\;\text{Right }\otimes}{B , \neg (A \otimes B) , A \vdash}\;\text{Left }\neg}{\neg (A \otimes B) , A \vdash \neg B}\;\text{Right }\neg}{A , \neg\neg B , \neg (A \otimes B) \vdash}\;\text{Left }\neg}{\neg\neg B , \neg (A \otimes B) \vdash \neg A}\;\text{Right }\neg}{\neg (A \otimes B) , \neg\neg A , \neg\neg B \vdash}\;\text{Left }\neg}{\neg\neg A , \neg\neg B \vdash \neg\neg (A \otimes B)}\;\text{Right }\neg}{\neg\neg A \otimes \neg\neg B \vdash \neg\neg (A \otimes B)}\;\text{Left }\otimes
$$

$$\texttt{lrsched} \quad = \quad \lambda\varphi.\, \lambda\psi.\, \lambda k.\, \varphi\,(\,\lambda x.\, \psi\,(\,\lambda y.\, k\,(x, y))\,)$$

The left-to-right scheduler

$\neg\neg A$	$\times$	$\neg\neg B$	$\Rightarrow$	$\neg\neg A \otimes B$
				question
question				
answer				
		question		
		answer		
				answer

$$\texttt{lrsched} \quad = \quad \lambda\varphi.\, \lambda\psi.\, \lambda k.\, \varphi\,(\,\lambda x.\, \psi\,(\,\lambda y.\, k\,(x, y))\,)$$

The right-to-left scheduler

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A} \quad \dfrac{}{B \vdash B}}{A , B \vdash A \otimes B}\ \text{Right}\ \otimes}{A , B , \neg (A \otimes B) \vdash}\ \text{Left}\ \neg}{B , \neg (A \otimes B) \vdash \neg A}\ \text{Right}\ \neg}{B , \neg (A \otimes B) , \neg\neg A \vdash}\ \text{Left}\ \neg}{\neg (A \otimes B) , \neg\neg A \vdash \neg B}\ \text{Right}\ \neg}{\neg (A \otimes B) , \neg\neg A , \neg\neg B \vdash}\ \text{Left}\ \neg}{\neg\neg A , \neg\neg B \vdash \neg\neg (A \otimes B)}\ \text{Right}\ \neg}{\neg\neg A \otimes \neg\neg B \vdash \neg\neg (A \otimes B)}\ \text{Left}\ \otimes$$

$$\texttt{rlsched} \quad = \quad \lambda \varphi .\, \lambda \psi .\, \lambda k .\, \psi \, (\lambda y .\, \varphi \, (\lambda x .\, k \, (x, y)) \,)$$

The right-to-left scheduler

$\neg\neg\ A$	$\times$	$\neg\neg\ B$	$\Rightarrow$	$\neg\neg\ A \otimes B$
				`question`
		`question`		
		`answer`		
`question`				
`answer`				
				`answer`

$$\texttt{rlsched} \quad = \quad \lambda\varphi.\, \lambda\psi.\, \lambda k.\, \psi\, (\, \lambda y.\, \varphi\, (\, \lambda x.\, k\, (x, y))\,)$$

Dialogue categories

A functorial bridge between proofs and knots

Dialogue categories

A **monoidal category** with a **left duality**

A natural bijection between the set of maps

$$A \otimes B \quad \longrightarrow \quad \bot$$

and the set of maps

$$B \quad \longrightarrow \quad A \multimap \bot$$

A familiar situation in tensorial algebra

Dialogue categories

A **monoidal category** with a **right duality**

A natural bijection between the set of maps

$$A \otimes B \quad \longrightarrow \quad \bot$$

and the set of maps

$$A \quad \longrightarrow \quad \bot \multimap B$$

A familiar situation in tensorial algebra

Dialogue categories

Definition. A dialogue category is a monoidal category $\mathscr{C}$ equipped with

- ▷ an object $\bot$
- ▷ two natural bijections

$$\varphi_{A,B} \quad : \quad \mathscr{C}(A \otimes B, \bot) \quad \longrightarrow \quad \mathscr{C}(B, A \multimap \bot)$$

$$\psi_{A,B} \quad : \quad \mathscr{C}(A \otimes B, \bot) \quad \longrightarrow \quad \mathscr{C}(A, \bot \multimapinv B)$$

Helical dialogue categories

A dialogue category equipped with a family of bijections

$$wheel_{A,B} \quad : \quad \mathscr{C}(A \otimes B, \bot) \quad \longrightarrow \quad \mathscr{C}(B \otimes A, \bot)$$

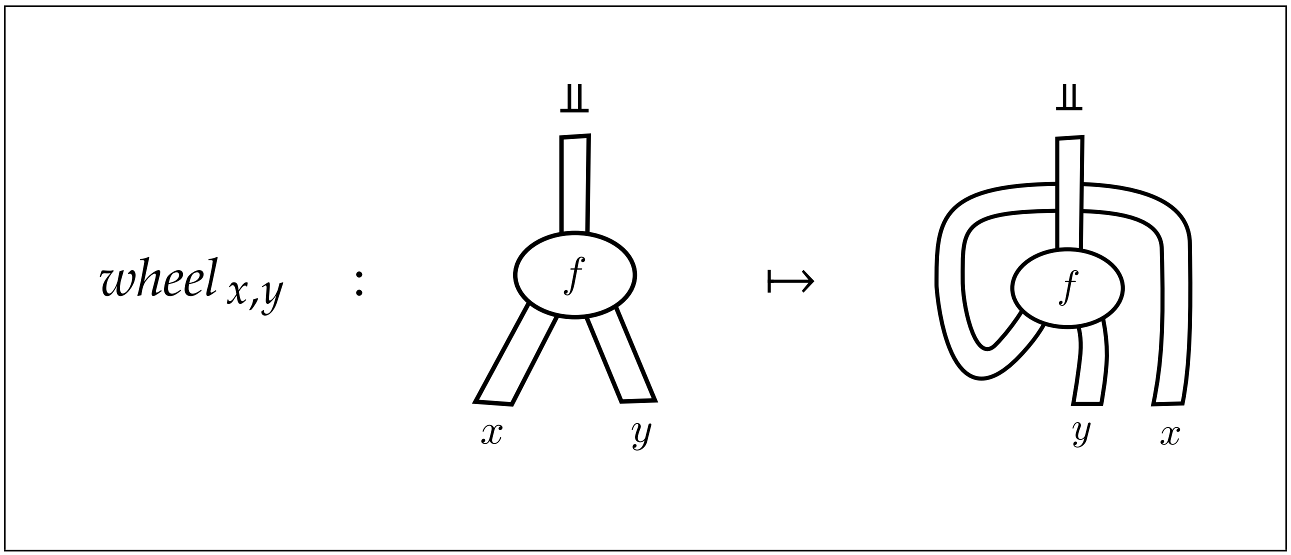
natural in A and B making the diagram

$$\begin{array}{ccc}
\mathscr{C}((B \otimes C) \otimes A, \bot) & \xrightarrow{\;associativity\;} & \mathscr{C}(A \otimes (C \otimes B), \bot) \\
\uparrow {\scriptstyle wheel_{A,B\otimes C}} & & \downarrow {\scriptstyle wheel_{B,C\otimes A}} \\
\mathscr{C}(A \otimes (B \otimes C)) & & \mathscr{C}((C \otimes A) \otimes B, \bot) \\
\downarrow {\scriptstyle associativity} & & \uparrow {\scriptstyle associativity} \\
\mathscr{C}((A \otimes B) \otimes C, \bot) & \xrightarrow{\;wheel_{A\otimes B,C}\;} & \mathscr{C}(C \otimes (A \otimes B), \bot)
\end{array}$$

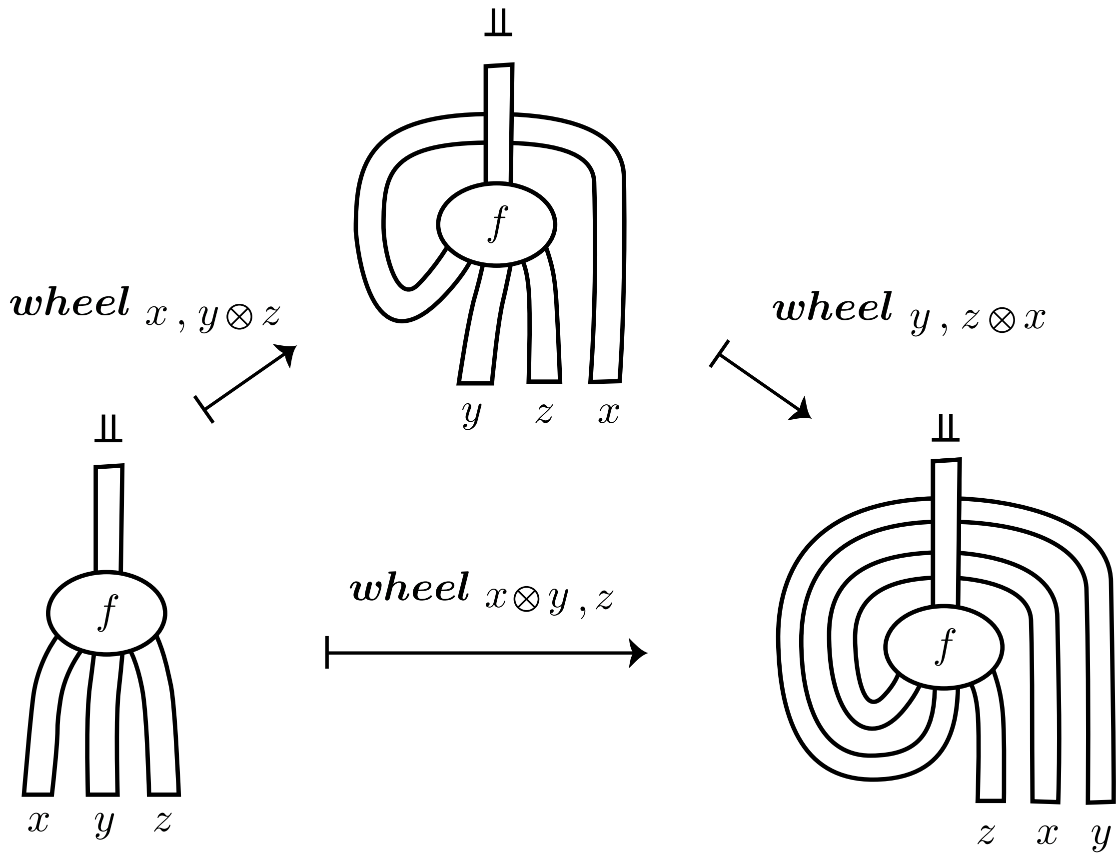
commutes.

Helical dialogue categories

The wheel should be understood diagrammatically as:



The coherence diagram

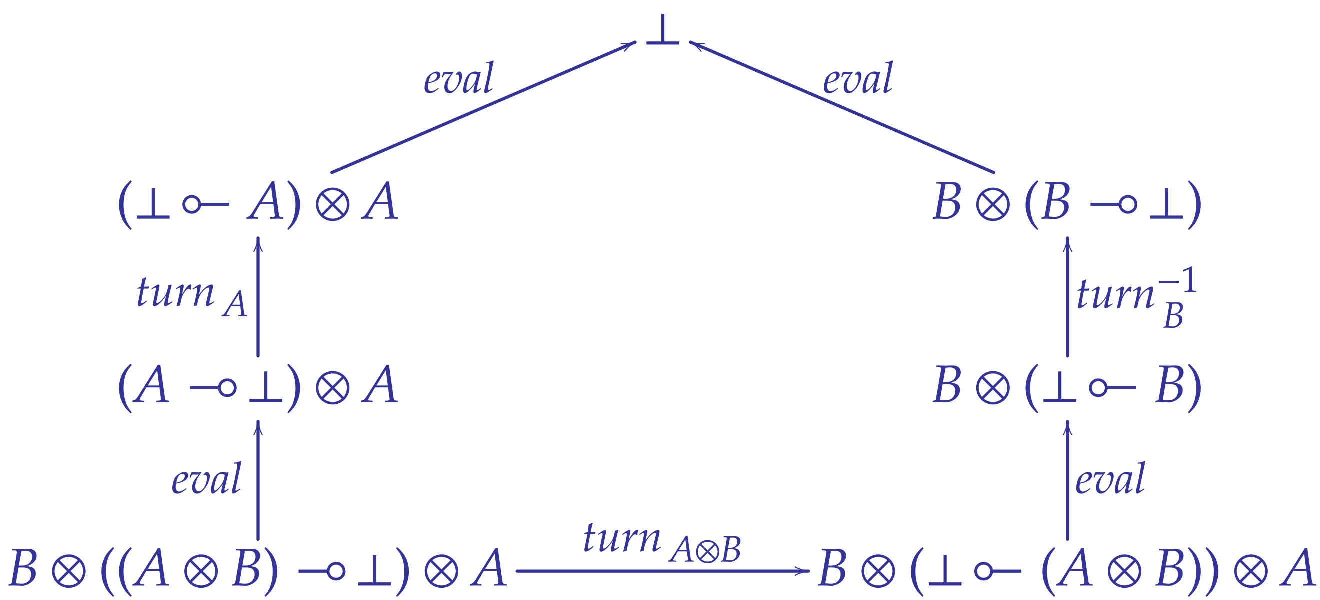


An equivalent formulation

A dialogue category equipped with a natural isomorphism

$$turn_A \quad : \quad A \multimap \bot \quad \longrightarrow \quad \bot \multimapinv A$$

making the diagram below commute:



The free dialogue category

The objects of the category **free-dialogue**($\mathscr{C}$) are the **formulas** of tensorial logic:

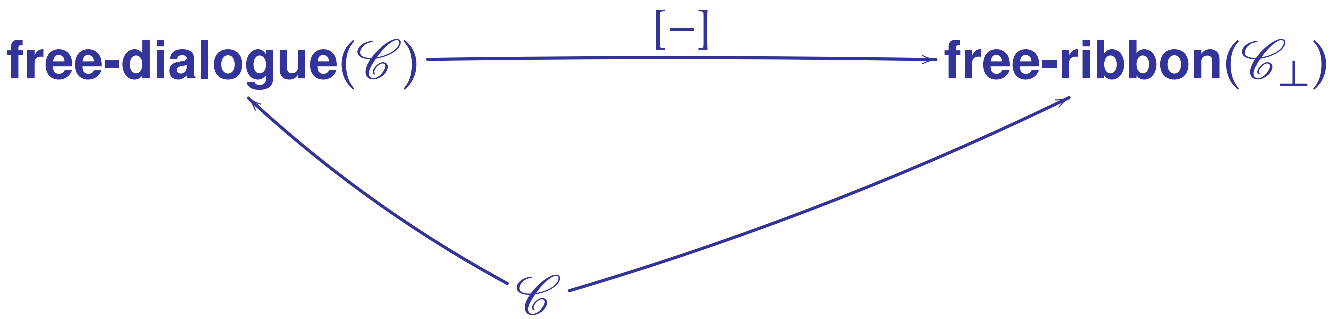
$$A, B \quad ::= \quad X \quad | \quad A \otimes B \quad | \quad A \multimap \bot \quad | \quad \bot \multimapinv A \quad | \quad 1$$

where X is an object of the category $\mathscr{C}$.

The morphisms are the **proofs** of the logic modulo equality.

A proof-as-tangle theorem

Every category $\mathscr{C}$ of atomic formulas induces a functor $[-]$ such that



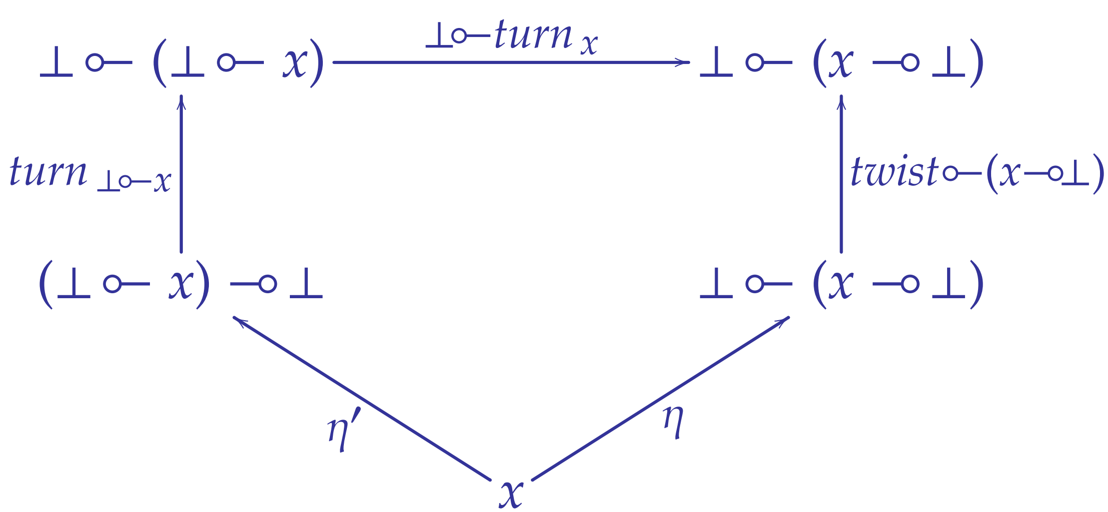
where $\mathscr{C}_\perp$ is the category $\mathscr{C}$ extended with an object $\perp$.

Theorem. The functor $[-]$ is faithful.

$\longrightarrow$ a topological foundation for game semantics

An illustration

Imagine that we want to check that the diagram



commutes in every balanced dialogue category.

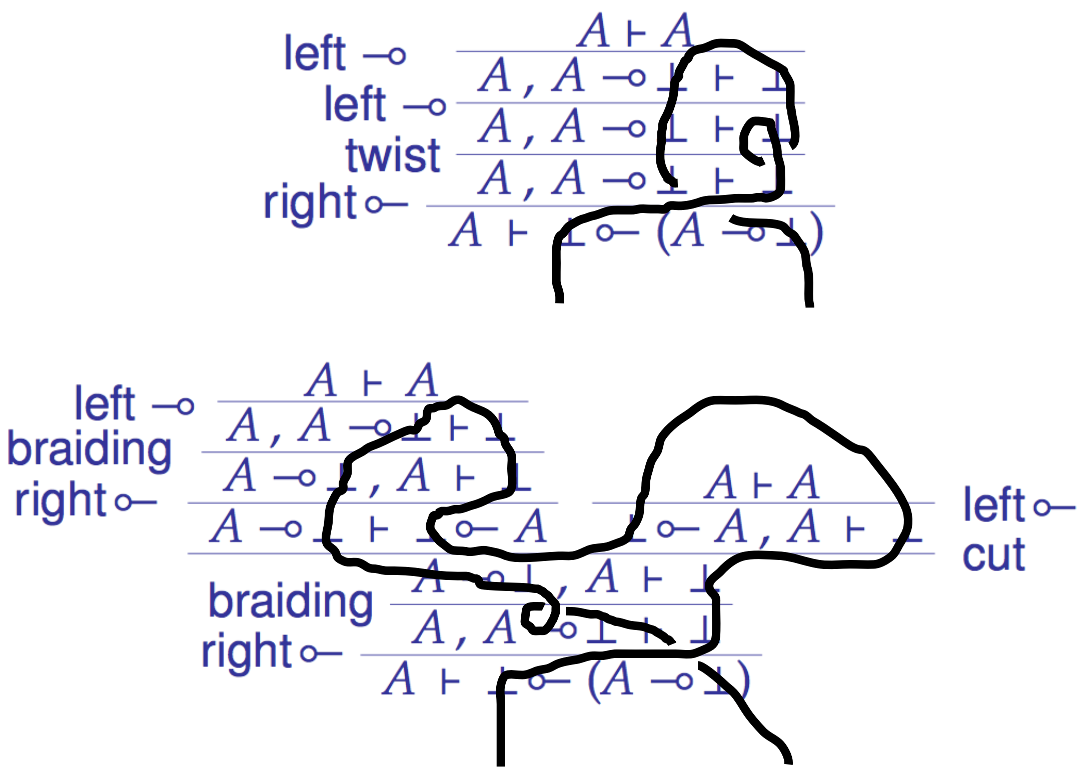
An illustration

Equivalently, we want to check that the two derivation trees are equal:

$$
\text{right}\,\multimapinv\;
\dfrac{
  \text{twist}\;
  \dfrac{
    \text{left}\,\multimap\;
    \dfrac{
      \text{left}\,\multimap\;
      \dfrac{A \vdash A}{A\,,\,A \multimap \bot \;\vdash\; \bot}
    }{A\,,\,A \multimap \bot \;\vdash\; \bot}
  }{A\,,\,A \multimap \bot \;\vdash\; \bot}
}{A \;\vdash\; \bot \multimapinv (A \multimap \bot)}
$$

$$
\text{right}\,\multimapinv\;
\dfrac{
  \text{braiding}\;
  \dfrac{
    \dfrac{
      \text{right}\,\multimapinv\;
      \dfrac{
        \text{braiding}\;
        \dfrac{
          \text{left}\,\multimap\;
          \dfrac{A \;\vdash\; A}{A\,,\,A \multimap \bot \vdash \bot}
        }{A \multimap \bot\,,\,A \;\vdash\; \bot}
      }{A \multimap \bot \;\vdash\; \bot \multimapinv A}
      \qquad
      \dfrac{A \vdash A}{\bot \multimapinv A\,,\,A \;\vdash\; \bot}\;\text{left}\,\multimapinv
    }{A \multimap \bot\,,\,A \;\vdash\; \bot}\;\text{cut}
  }{A\,,\,A \multimap \bot \;\vdash\; \bot}
}{A \;\vdash\; \bot \multimapinv (A \multimap \bot)}
$$

An illustration



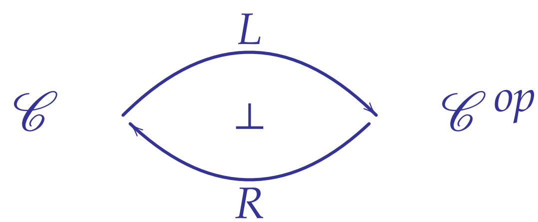
equality of proofs $\Longleftrightarrow$ equality of tangles

Dialogue chiralities

A symmetric account of dialogue categories

The self-adjunction of negations

Negation defines a pair of **adjoint functors**



witnessed by the series of bijection:

$$\mathscr{C}(A, \neg B) \quad \cong \quad \mathscr{C}(B, \neg A) \quad \cong \quad \mathscr{C}^{op}(\neg A, B)$$

The symmetry of logic

Eloise speaks to Abelard who speaks to Eloise who speaks to...

From categories to chiralities

This leads to a slightly bizarre idea:

decorrelate the category $\mathscr{C}$ from its opposite category $\mathscr{C}^{op}$

So, let us define a **chirality** as a pair of categories $(\mathscr{A}, \mathscr{B})$ such that

$$\mathscr{A} \cong \mathscr{C} \qquad \mathscr{B} \cong \mathscr{C}^{op}$$

for some category $\mathscr{C}$.

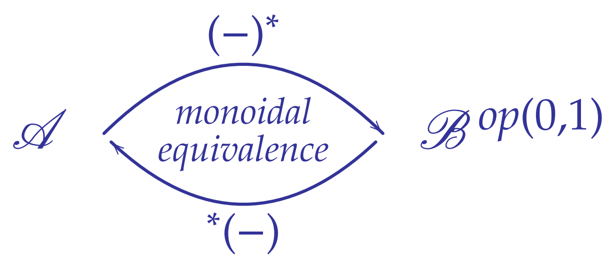
Here $\cong$ means **equivalence** of category

Dialogue chiralities

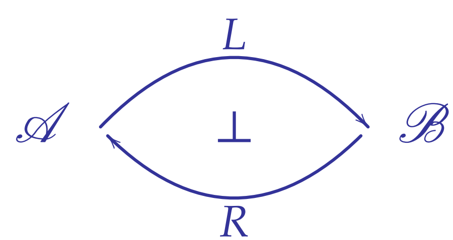
A **dialogue chirality** is a pair of monoidal categories

$$(\mathscr{A}, \owedge, \mathrm{true}) \qquad\qquad (\mathscr{B}, \ovee, \mathrm{false})$$

with a monoidal equivalence



together with an adjunction



Dialogue chiralities

and two natural bijections

$$\chi^{L}_{m,a,b} \quad : \quad \langle\, m \owedge a \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, a \,|\, m^{*} \ovee b \,\rangle$$

$$\chi^{R}_{m,a,b} \quad : \quad \langle\, a \owedge m \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, a \,|\, b \ovee m^{*} \,\rangle$$

where the evaluation bracket

$$\langle - | - \rangle \quad : \quad \mathscr{A}^{op} \times \mathscr{B} \quad \longrightarrow \quad Set$$

is defined as

$$\langle\, a \,|\, b \,\rangle \quad := \quad \mathscr{A}(\, a \,,\, Rb \,)$$

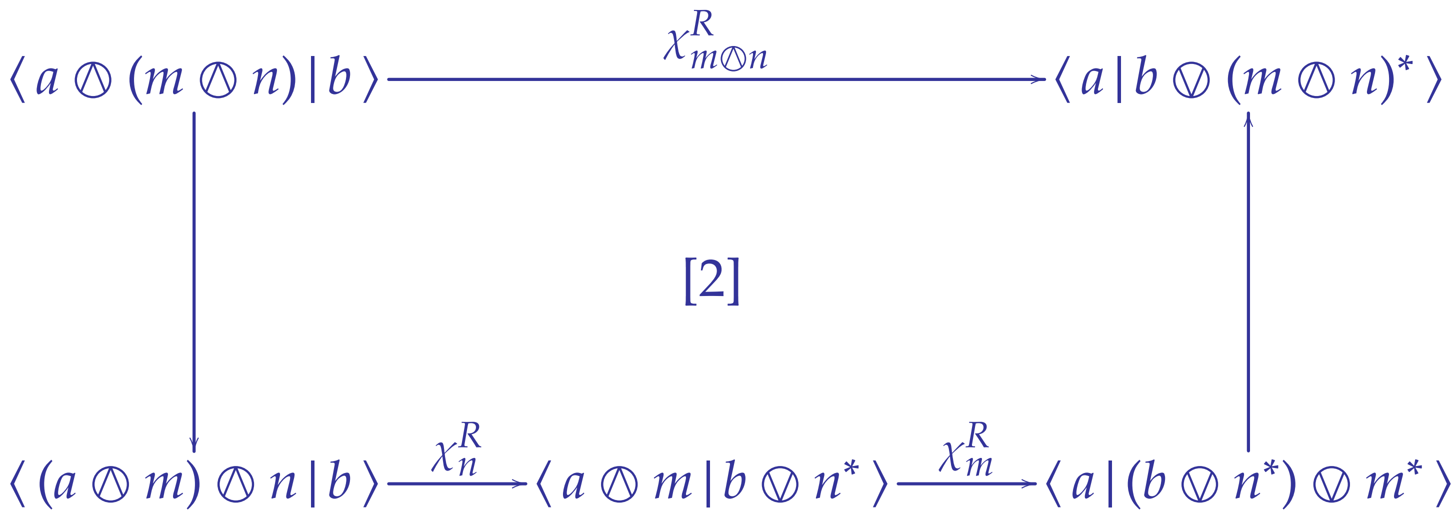
Dialogue chiralities

These are required to make the diagrams commute:

$$
\begin{array}{ccccc}
\langle\, (m \owedge n) \owedge a \,|\, b \,\rangle & & \xrightarrow{\chi^L_{m \owedge n}} & & \langle\, a \,|\, (m \owedge n)^* \ovee b \,\rangle \\
\Big\downarrow & & [1] & & \Big\uparrow \\
\langle\, m \owedge (n \owedge a) \,|\, b \,\rangle & \xrightarrow{\chi^L_m} & \langle\, n \owedge a \,|\, m^* \ovee b \,\rangle & \xrightarrow{\chi^L_n} & \langle\, a \,|\, n^* \ovee (m^* \ovee b) \,\rangle
\end{array}
$$

Dialogue chiralities

These are required to make the diagrams commute:



Dialogue chiralities

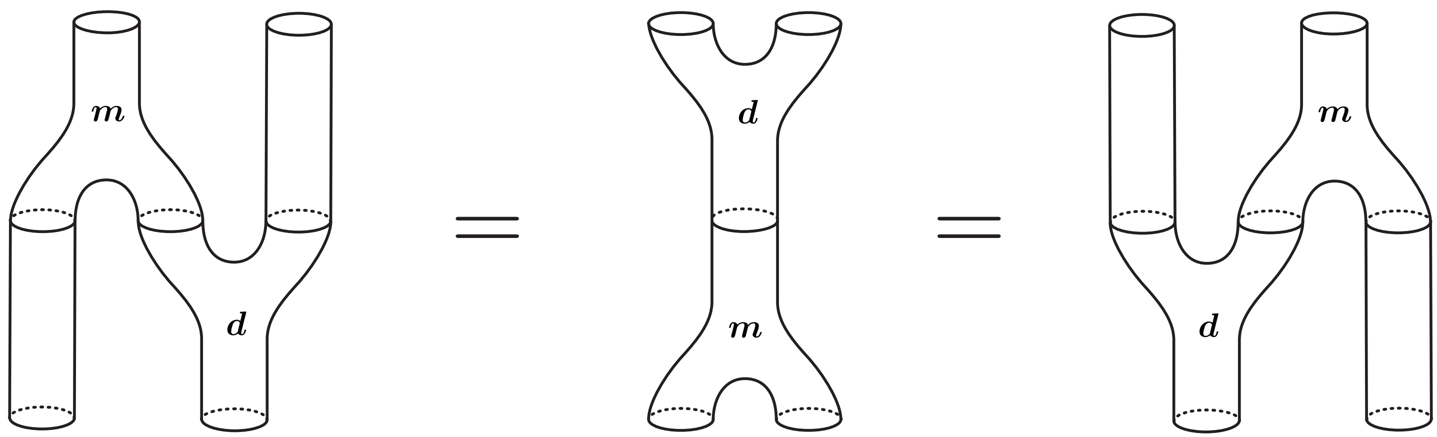
These are required to make the diagrams commute:

$$
\begin{array}{ccccc}
\langle (m \owedge a) \owedge n \mid b \rangle & \xrightarrow{\chi_n^R} & \langle m \owedge a \mid b \ovee n^* \rangle & \xrightarrow{\chi_m^L} & \langle a \mid m^* \ovee (b \ovee n^*) \rangle \\
\Big| & & [3] & & \Big| \\
\langle m \owedge (a \owedge n) \mid b \rangle & \xrightarrow{\chi_m^L} & \langle a \owedge n \mid m^* \ovee b \rangle & \xrightarrow{\chi_n^R} & \langle a \mid (m^* \ovee b) \ovee n^* \rangle
\end{array}
$$

Chiralities as Frobenius monoids

Frobenius monoids

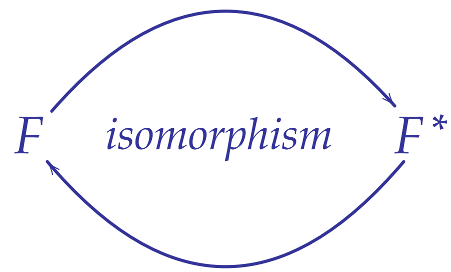
A Frobenius monoid F is a monoid and a comonoid satisfying



A deep relationship with $*$-autonomous categories discovered by Brian Day and Ross Street.

Frobenius monoids are self-dual

An isomorphism between the Frobenius monoid F and its dual F^*



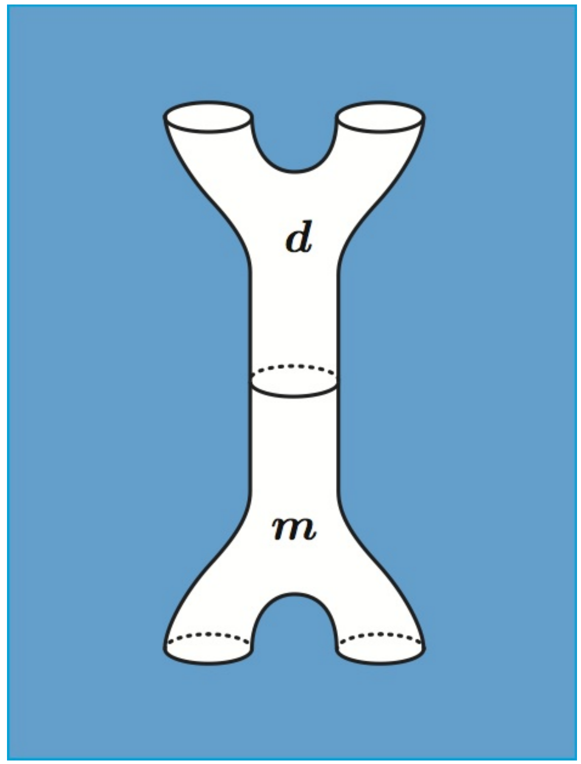
induced by a non-degenerate 2-form

$$\langle -, - \rangle \quad : \quad F \otimes F \quad \longrightarrow \quad I$$

satisfying the equality:

$$\langle x \cdot y , z \rangle \quad = \quad \langle x , y \cdot z \rangle$$

The symmetry of Frobenius algebras



Monoid speaks to comonoid who speaks to monoid who speaks to...

A symmetric presentation of Frobenius algebras

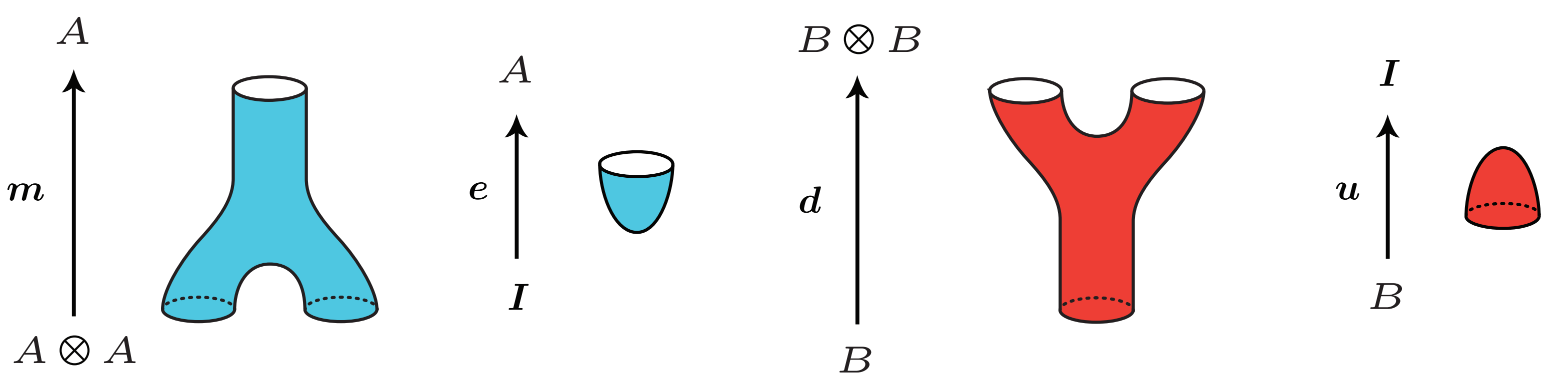
Key idea. Separate the monoid part

$$m : A \otimes A \longrightarrow A \qquad\qquad e : A \otimes A \longrightarrow A$$

from the comonoid part

$$m : B \longrightarrow B \otimes B \qquad\qquad d : B \longrightarrow I$$

in a Frobenius algebra:

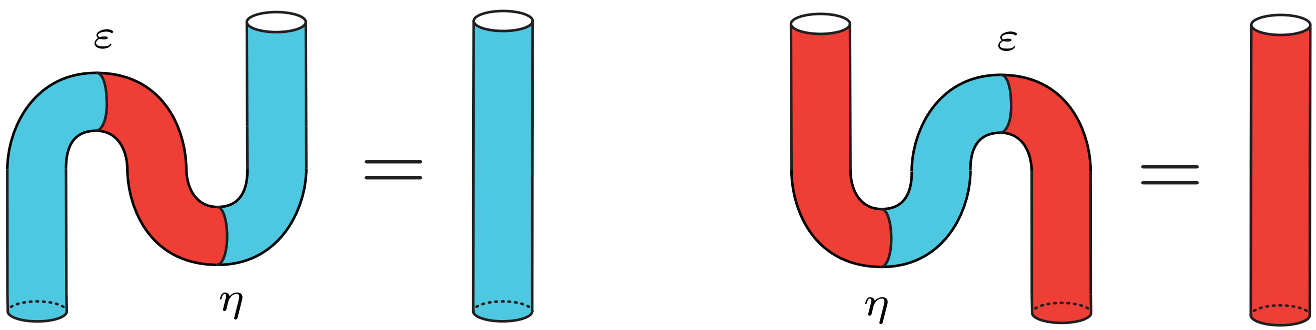


A symmetric presentation of Frobenius algebras

Then, relate A and B by a dual pair

$$\eta \; : \; I \longrightarrow B \otimes A \qquad\qquad \varepsilon \; : \; A \otimes B \longrightarrow I$$

in the sense that:

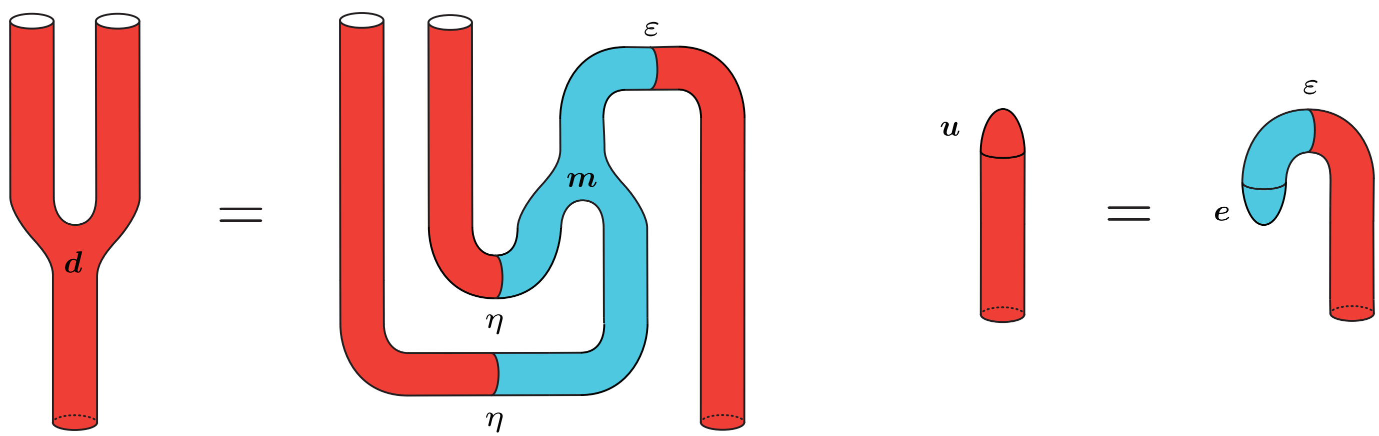


A symmetric presentation of Frobenius algebras

Require moreover that the dual pair

$$(A, m, e) \quad \dashv \quad (B, d, u)$$

relates the algebra structure to the coalgebra structure, in the sense that:

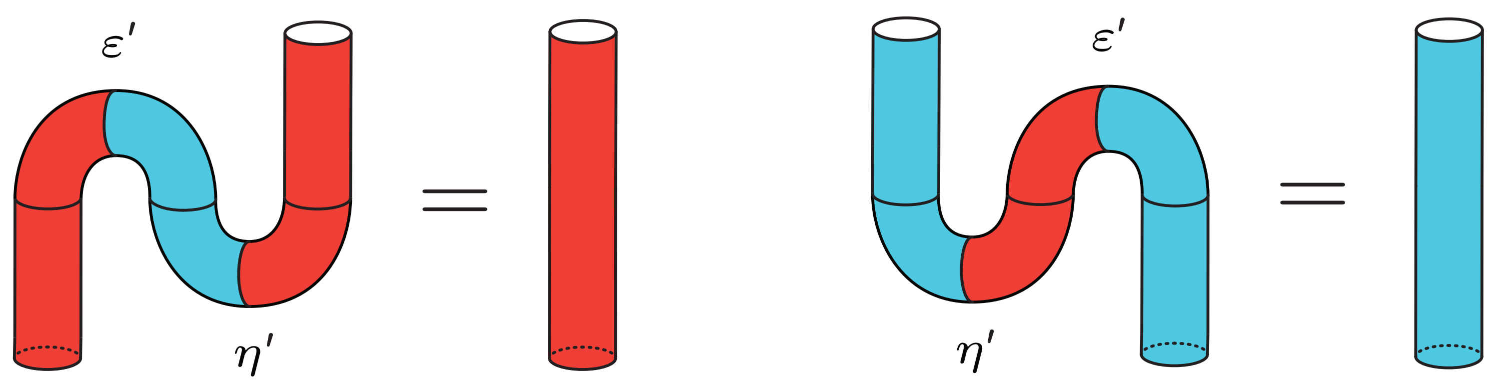


Symmetrically

Relate B and A by a dual pair

$$\eta' \ : \ I \longrightarrow B \otimes A \qquad\qquad \varepsilon' \ : \ A \otimes B \longrightarrow I$$

this meaning that the equations below hold:

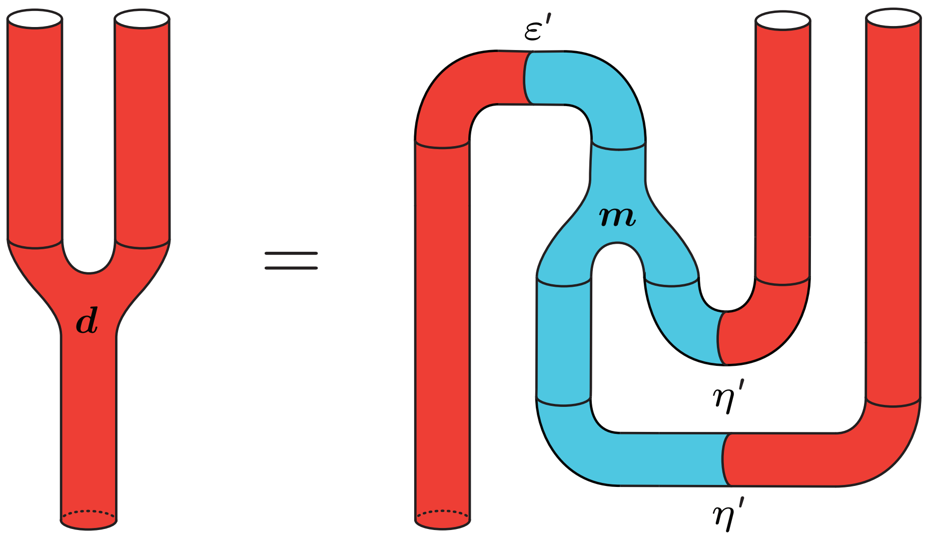


Symmetrically

and ask that the dual pair

$$A \dashv B$$

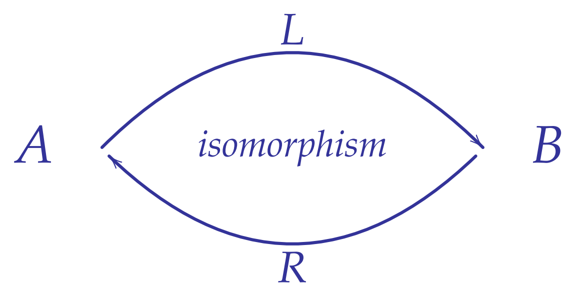
relates the coalgebra structure to the algebra structure, in the sense that:



An alternative formulation

Key observation:

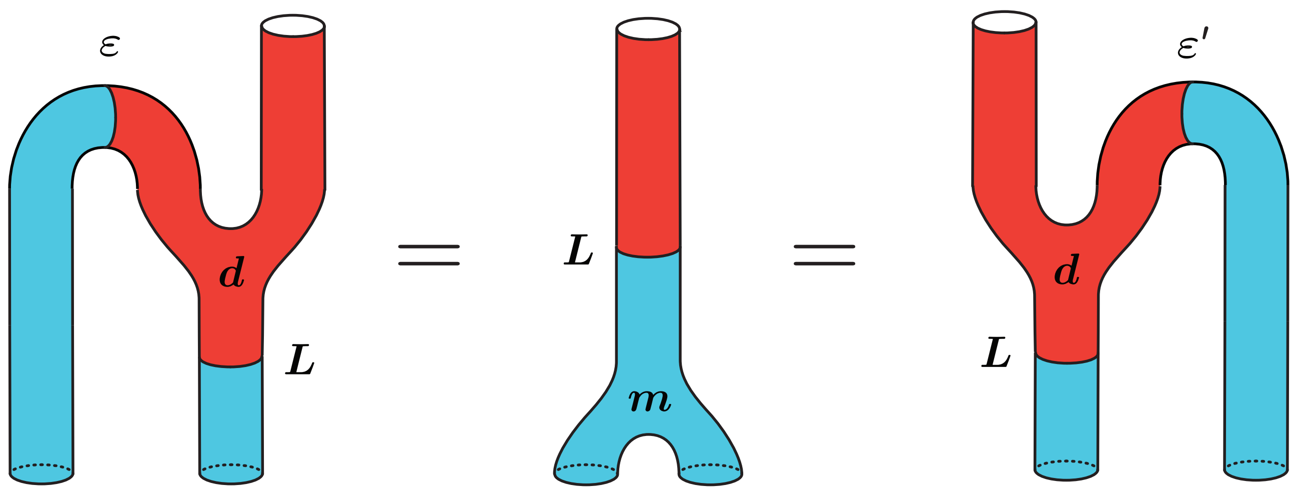
A Frobenius monoid is the same thing as such a pair (A, B) equipped with



between the underlying spaces A and B and...

Frobenius monoids

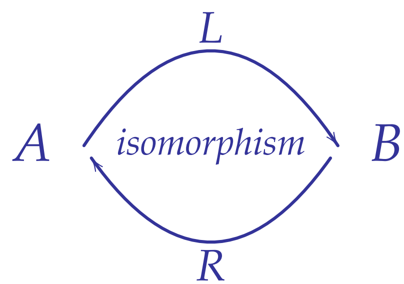
... satisfying the two equalities below:



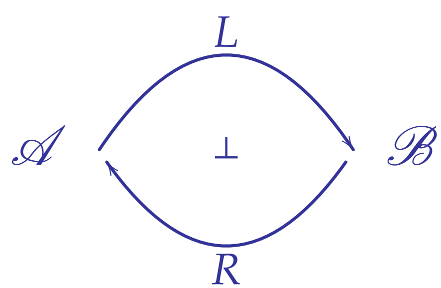
Reminiscent of currification in the λ-calculus...

Not far from the connection, but...

Idea: the « self-duality » of Frobenius monoids



is replaced by an **adjunction** in dialogue chiralities:



Key objection: the category $\mathscr{B} \cong \mathscr{A}^{op}$ is not dual to the category $\mathscr{A}$.

Categorical bimodules

A bimodule

$$M \quad : \quad \mathscr{A} \quad ⇸ \quad \mathscr{B}$$

between categories $\mathscr{A}$ and $\mathscr{B}$ is defined as a functor

$$M \quad : \quad \mathscr{A}^{op} \times \mathscr{B} \quad \longrightarrow \quad \mathbf{Set}$$

Composition of two bimodules

$$\mathscr{A} \xrightarrow{\;M\;} \mathscr{B} \xrightarrow{\;N\;} \mathscr{C}$$

is defined by the coend formula:

$$M \circledast N \quad : \quad (a,c) \quad \mapsto \quad \int^{b \in \mathscr{B}} M(a,b) \times N(b,c)$$

A well-known 2-categorical miracle

Fact. Every category $\mathscr{C}$ comes with a biexact pairing

$$\mathscr{C} \quad \dashv \quad \mathscr{C}^{op}$$

defined as the bimodule

$$\text{hom} \quad : \quad (x, y) \quad \mapsto \quad \mathscr{A}(x, y) \quad : \quad \mathscr{C}^{op} \times \mathscr{C} \quad \longrightarrow \quad \textbf{Set}$$

in the bicategory **BiMod** of categorical bimodules.

The opposite category $\mathscr{C}^{op}$ becomes <u>dual</u> to the category $\mathscr{C}$

Biexact pairing

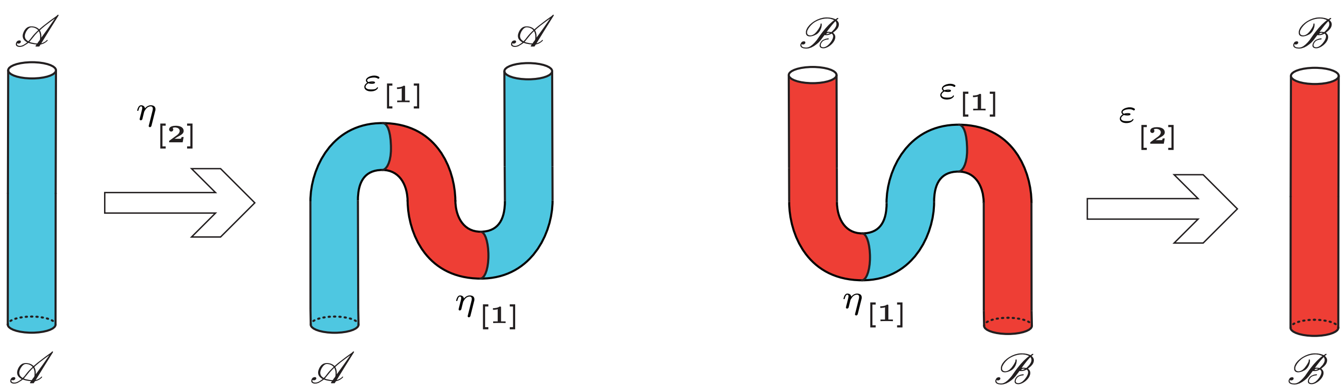
Definition. A biexact pairing

$$\mathscr{A} \dashv \mathscr{B}$$

in a monoidal bicategory is a pair of 1-dimensional cells

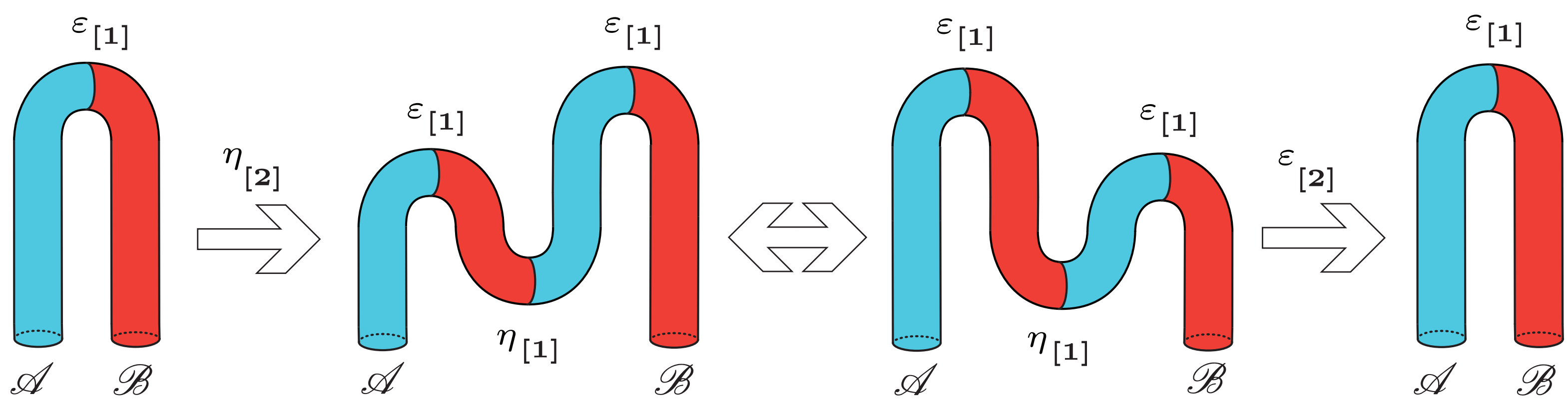
$$\eta_{[1]} : \mathscr{A} \otimes \mathscr{B} \longrightarrow I \qquad \varepsilon_{[1]} : I \longrightarrow \mathscr{B} \otimes \mathscr{A}$$

together with a pair of invertible 2-dimensional cells



Biexact pairing

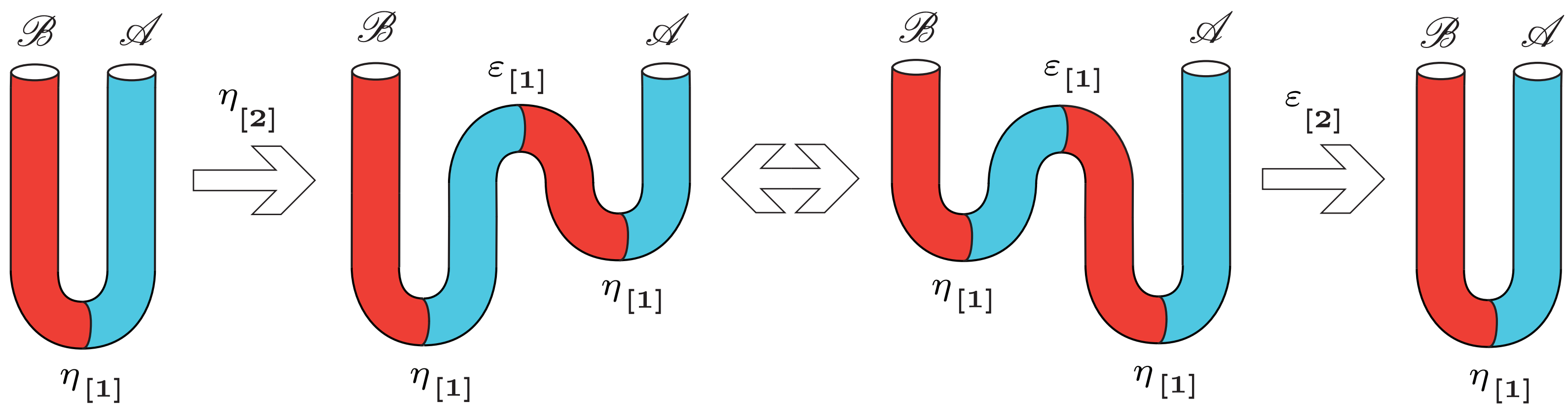
such that the composite 2-dimensional cell



coincides with the identity on the 1-dimensional cell $\varepsilon_{[1]}$,

Biexact pairing

and symmetrically, such that the composite 2-dimensional cell



coincides with the identity on the 1-dimensional cell $\eta_{[1]}$.

Amphimonoid

In any symmetric monoidal bicategory like **BiMod**...

Definition. An amphimonoid is a pseudomonoid

$$(\mathscr{A}, \owedge, \mathrm{true})$$

and a pseudocomonoid

$$(\mathscr{B}, \ovee, \mathrm{false})$$

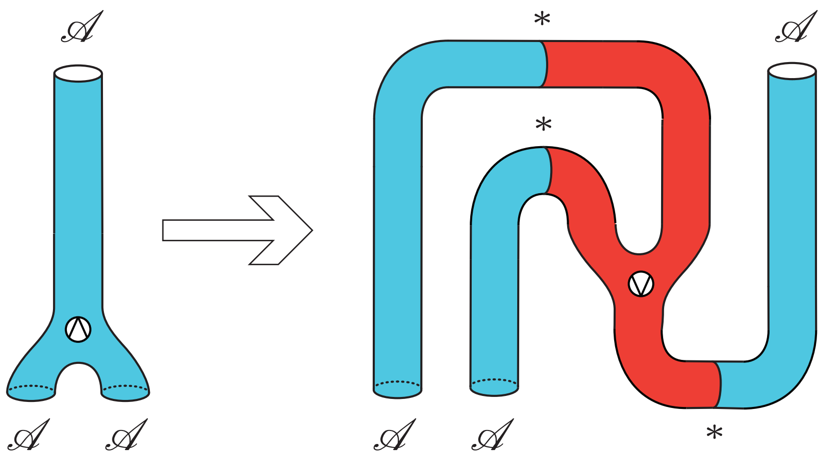
equipped with a biexact pairing

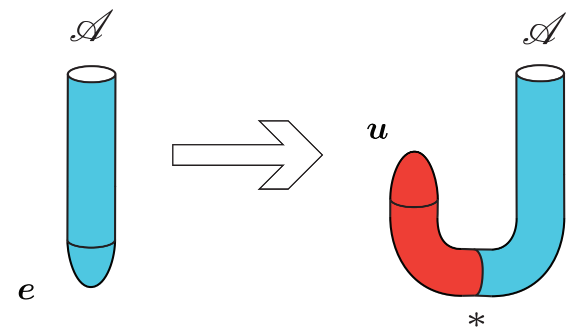
$$\mathscr{A} \dashv \mathscr{B}$$

Bialgebraic counterpart to the notion of chirality

Amphimonoid

together with a pair of invertible 2-dimensional cells



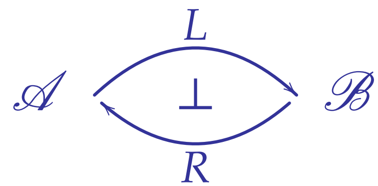


defining a pseudomonoid equivalence.

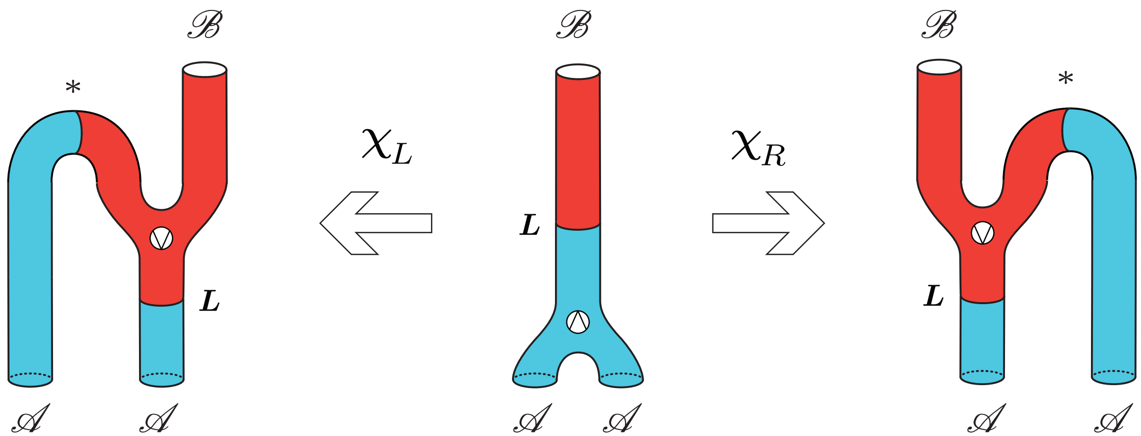
Bialgebraic counterpart to the notion of monoidal chirality

Frobenius amphimonoid

Definition. An amphimonoid together with an adjunction



and two invertible 2-dimensional cells:



Bialgebraic counterpart to the notion of dialogue chirality

Frobenius amphimonoid

The 1-dimensional cell

$$L \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}$$

may be understood as defining a bracket

$$\langle\, a \,|\, b \,\rangle$$

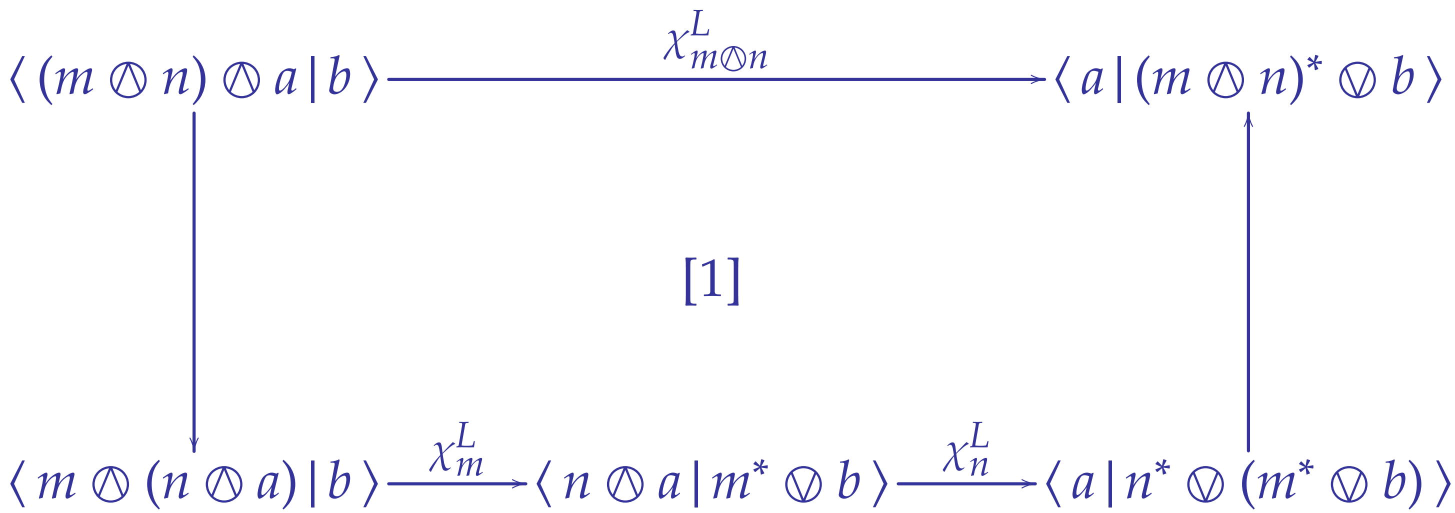
between the objects $\mathscr{A}$ and $\mathscr{B}$ of the bicategory $\mathscr{V}$.

Each side of the equation implements currification:

$$\chi_L : \langle\, a_1 \owedge a_2 \,|\, b \,\rangle \Rightarrow \langle\, a_2 \,|\, a_1^* \ovee b \,\rangle \qquad \chi_R : \langle\, a_1 \owedge a_2 \,|\, b \,\rangle \Rightarrow \langle\, a_1 \,|\, b \ovee a_2^* \,\rangle$$

Frobenius amphimonoid

These are required to make the diagrams commute:



Frobenius amphimonoid

These are required to make the diagrams commute:

$$
\begin{array}{ccccc}
\langle\, a \owedge (m \owedge n) \,|\, b \,\rangle & & \xrightarrow{\chi^R_{m \owedge n}} & & \langle\, a \,|\, b \ovee (m \owedge n)^* \,\rangle \\
\downarrow & & [2] & & \uparrow \\
\langle\, (a \owedge m) \owedge n \,|\, b \,\rangle & \xrightarrow{\chi^R_n} & \langle\, a \owedge m \,|\, b \ovee n^* \,\rangle & \xrightarrow{\chi^R_m} & \langle\, a \,|\, (b \ovee n^*) \ovee m^* \,\rangle
\end{array}
$$

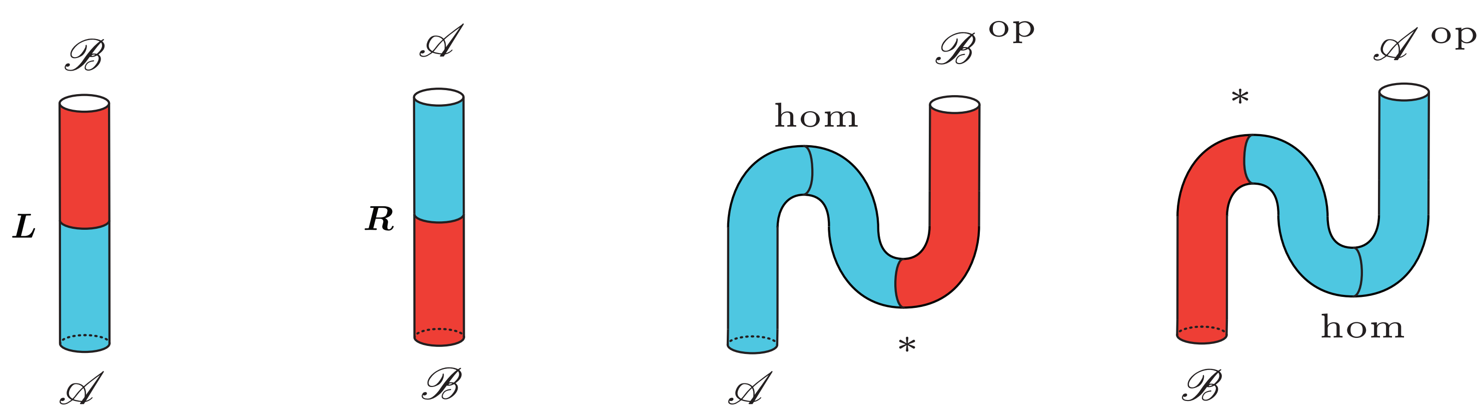
Frobenius amphimonoid

These are required to make the diagrams commute:

$$
\begin{array}{ccccc}
\langle\, (m \owedge a) \owedge n \,|\, b \,\rangle & \xrightarrow{\chi^R_n} & \langle\, m \owedge a \,|\, b \ovee n^* \,\rangle & \xrightarrow{\chi^L_m} & \langle\, a \,|\, m^* \ovee (b \ovee n^*) \,\rangle \\
\Big\| & & [3] & & \Big\| \\
\langle\, m \owedge (a \owedge n) \,|\, b \,\rangle & \xrightarrow{\chi^L_m} & \langle\, a \owedge n \,|\, m^* \ovee b \,\rangle & \xrightarrow{\chi^R_n} & \langle\, a \,|\, (m^* \ovee b) \ovee n^* \,\rangle
\end{array}
$$

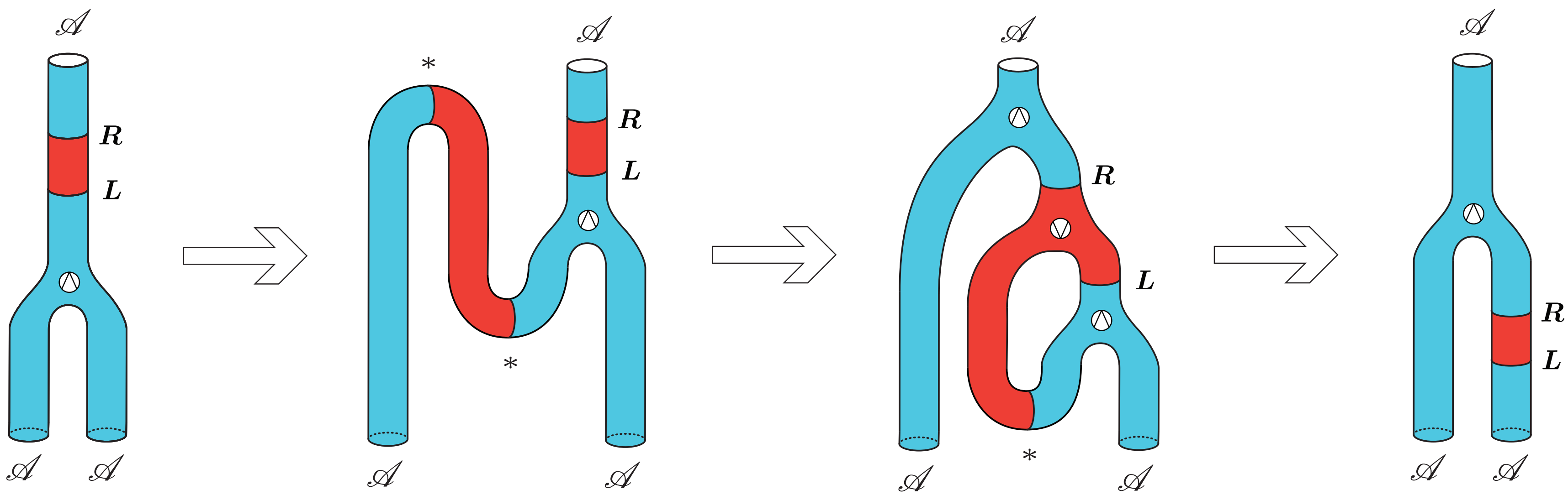
Correspondence theorem

Theorem. A helical chirality is the same thing as a Frobenius amphimonoid in the bicategory **BiMod** whose 1-dimensional cells



are representable, that is, induced by functors.

Tensorial strength formulated in cobordism



$$a_1 \owedge RL(a_2) \quad \vdash \quad RL(a_1 \owedge a_2)$$

$$\mathscr{A}(RL(a_1 \owedge a_2), a) \quad \longrightarrow \quad \mathscr{A}(a_1 \owedge RL(a_2), a)$$

Thank you ！