Dialogue categories and chiralities

Paul-André Melliès*

Abstract

In this paper, we consider a two-sided notion of dialogue category which we call dialogue chirality and which we formulate as an adjunction between a monoidal category $\mathscr{A}$ of proofs and a monoidal category $\mathscr{B}$ of counter-proofs equivalent to its opposite category $\mathscr{A}^{op(0,1)}$. The two-sided formulation of dialogue categories is compared to the original one-sided formulation by exhibiting a 2-dimensional equivalence between a 2-category of dialogue categories and a 2-category of dialogue chiralities. The resulting coherence theorem clarifies in what sense every dialogue chirality may be strictified to an equivalent dialogue category.

Forewords. This paper is part of an ongoing research program at the interface of logic, algebra and computer science, whose purpose is to investigate the interactive and game-theoretic nature of continuations in programming languages. The paper is guided by the insight that the interactive content of continuations is secretly concealed in their description in the language of categories and functors. This primary intuition leads us to the idea of formulating a dialogue category $\mathscr{C}$ as a dialogue chirality consisting of a category $\mathscr{A}$ of proofs (or programs) confronted with a category $\mathscr{B}$ of refutations (or environments). In order to justify this two-sided and properly symmetric formulation of dialogue categories, one needs to compare it with the original and traditional one-sided formulation. The purpose of the paper is precisely to develop this comparison in full detail, by exhibiting (§7.5, Thm. 3) a 2-dimensional equivalence between a pair of appropriately defined 2-categories of dialogue categories and of dialogue chiralities.

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Paul-André Melliès, CNRS, Laboratoire IRIF, UMR 8243, Université Paris Diderot, case 7014, 75205 Paris Cedex 13, France.
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1 Introduction

Deformation of algebraic structures. A strict monoidal category is defined as a category $\mathscr{C}$ equipped with a functor

$$\otimes \quad : \quad \mathscr{C} \times \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

and an object e satisfying the associativity and unity equations

$$(x \otimes y) \otimes z = x \otimes (y \otimes z) \qquad\qquad e \otimes x = x = x \otimes e \tag{1}$$

for all objects x, y, z of the category $\mathscr{C}$. It is well-known that a strict monoidal category may be alternatively defined as a monoid object in the cartesian category ***Cat*** of categories and functors. Hence, an interesting question is to characterize the algebraic structure inherited by a category $\mathscr{D}$ equivalent (in the categorical sense) to a strict monoidal category $\mathscr{C}$. Recall that by categorical equivalence, one means an adjunction

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{D} \arrow[l, bend left, "R"] \end{tikzcd}$$

whose unit and counit

$$\eta \quad : \quad Id \Rightarrow R \circ L \qquad\qquad\qquad \varepsilon \quad : \quad L \circ R \Rightarrow Id$$

are invertible. The answer to this question is provided by MacLane's coherence theorem, which states that a category $\mathscr{D}$ is equivalent to a strict monoidal category precisely when it is a monoidal category. By this, one means a category equipped with a functor $\otimes$ and with an object e together with three families of isomorphisms

$$(x \otimes y) \otimes z \xrightarrow{\alpha} x \otimes (y \otimes z) \qquad\qquad e \otimes x \xrightarrow{\lambda} x \xleftarrow{\rho} x \otimes e$$

natural in x, y, z and making the two diagrams

$$\begin{tikzcd}[column sep=small] & (w \otimes x) \otimes (y \otimes z) \arrow[dr, "\alpha"] & \\ ((w \otimes x) \otimes y) \otimes z \arrow[ur, "\alpha"] \arrow[d, "\alpha \otimes z"'] & & w \otimes (x \otimes (y \otimes z)) \\ (w \otimes (x \otimes y)) \otimes z \arrow[rr, "\alpha"] & & w \otimes ((x \otimes y) \otimes z) \arrow[u, "w \otimes \alpha"'] \end{tikzcd}$$

$$
\begin{tikzcd}
 & x \otimes y & \\
(x \otimes e) \otimes y \arrow[ur, "\rho \otimes y"] \arrow[rr, "\alpha"] & & x \otimes (e \otimes y) \arrow[ul, "x \otimes \lambda"']
\end{tikzcd}
$$

commute for all objects w, x, y, z of the category $\mathscr{D}$. It is instructive to analyze this result from a 2-categorical point of view. A monoidal category is the same thing as a pseudo-monoid in the cartesian 2-category ***Cat*** of categories, functors and natural transformations. Moreover, in any monoidal 2-category, every object $\mathscr{D}$ equivalent to a monoid object $\mathscr{C}$ inherits from $\mathscr{C}$ the structure of a pseudo-monoid object. This general result applied to the specific case of the cartesian 2-category ***Cat*** implies that every category $\mathscr{D}$ equivalent to a strict monoidal category $\mathscr{C}$ is a monoidal category. The converse property is not true in an arbitrary monoidal 2-category $\mathscr{W}$, since it is possible in general that a given pseudo-monoid object $\mathscr{D}$ is *not* equivalent to any monoid object $\mathscr{C}$ in the 2-category $\mathscr{W}$. However, the coherence theorem tells us that the converse property holds in the particular case when $\mathscr{W}$ is the cartesian 2-category ***Cat***. As a matter of fact, establishing this converse property is the difficult part of the coherence theorem: the theorem states that every pseudo-monoid object $\mathscr{D}$ in the 2-category ***Cat*** inherits its structure from an equivalence with a monoid object $\mathscr{C}$, that is, a strict monoidal category. At this stage, it is interesting to notice that a purely homotopic account of the coherence theorem is possible: the idea is to identify the theorem as an instance of the Boardman-Vogt W-construction of an algebraic theory modulo deformation. In this case, the deformation should be performed in the category ***Cat*** of categories equipped with the 'folk' model structure, whose weak equivalences are provided by the categorical equivalences, see for instance Berger and Moerdijk [6] or Weiss [46, Section 4.2]. This enables one to see the notion of monoidal category as a formal deformation of the notion of strict monoidal category.

Deformation of dual structures. The purpose of this article is to understand how the idea of formal deformation may be applied to dialogue categories and other notions of categories equipped with a duality. A dialogue category is defined as a monoidal category $\mathscr{C}$ equipped with an object $\perp$ together with two functors

$$
\begin{array}{ccc}
\mathscr{C}^{op} & \longrightarrow & \mathscr{C} \\
x & \mapsto & \perp \multimapinv x
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathscr{C}^{op} & \longrightarrow & \mathscr{C} \\
x & \mapsto & x \multimap \perp
\end{array}
$$

and two families of bijections

$$\mathscr{C}(x, \bot \multimapinv y) \quad \cong \quad \mathscr{C}(x \otimes y, \bot) \quad \cong \quad \mathscr{C}(y, x \multimap \bot)$$

natural in x and y. The notion of dialogue category is preserved by equivalence, in the sense that every category $\mathscr{D}$ equivalent to a dialogue category $\mathscr{C}$ is also a dialogue category. This implies that the idea of relaxing the notion of dialogue category by deformation is apparently meaningless... unless one applies a different and even stronger notion of deformation than categorical equivalence! A first step in that direction is to observe that any notion of self-dual category relates the category $\mathscr{C}$ to its opposite category $\mathscr{C}^{op}$. This leads us to the idea that one should think of the ambient 2-category $\boldsymbol{Cat}$ as an "involutive" 2-category equipped with a 2-functor

$$(-)^{op} \quad : \quad \boldsymbol{Cat} \quad \longrightarrow \quad \boldsymbol{Cat}^{op(2)} \tag{2}$$

which transports every category $\mathscr{C}$ to its opposite category $\mathscr{C}^{op}$. Here, the target 2-category $\boldsymbol{Cat}^{op(2)}$ is the 2-category $\boldsymbol{Cat}$ where the 2-cells have been reversed. This reversal reflects the fact that the 2-functor $(-)^{op}$ transports every natural transformation

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathscr{D} \end{tikzcd}$$

to a natural transformation in the opposite direction:

$$\begin{tikzcd} \mathscr{C}^{op} \arrow[r, bend left, "F^{op}"] \arrow[r, bend right, "G^{op}"'] \arrow[r, phantom, "\Uparrow\theta^{op}"] & \mathscr{D}^{op} \end{tikzcd}$$

Now, every dialogue category $\mathscr{C}$ is related to its opposite category $\mathscr{C}^{op}$ by an adjunction

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{C}^{op} \arrow[l, bend left, "R"] \end{tikzcd} \tag{3}$$

defined by the two functors

$$L(x) \quad = \quad \bot \multimapinv x \qquad \text{and} \qquad R(x) \quad = \quad x \multimap \bot$$

and by the families of bijections

$$\mathscr{C}^{op}(x \multimap \perp, y) \quad \cong \quad \mathscr{C}(x \otimes y, \perp) \quad \cong \quad \mathscr{C}(x, \perp \multimapinv y).$$

natural in x and y. This leads us to the main idea of the paper which is that the formal deformation of the dialogue category $\mathscr{C}$ should be *decorrelated* from the formal deformation of its opposite category $\mathscr{C}^{op}$. This means that we should study and characterize the pairs $(\mathscr{A}, \mathscr{B})$ of categories equivalent to a pair $(\mathscr{C}, \mathscr{C}^{op})$ consisting of a dialogue category $\mathscr{C}$ and of its opposite category $\mathscr{C}^{op}$. In other words, the deformation of a dialogue category $\mathscr{C}$ should not be performed inside the 2-category ***Cat***... but inside the larger 2-category $\boldsymbol{Cat} \times \boldsymbol{Cat}^{op(2)}$. We will see that this decorrelation of $\mathscr{C}$ and $\mathscr{C}^{op}$ provides an additional "degree of freedom" in the deformation process. This reveals hidden structures of dialogue categories, in the same way as traditional deformation by categorical equivalence does for strict monoidal categories. This decorrelated point of view also enables one to think of the two categories $\mathscr{C}$ and $\mathscr{C}^{op}$ in a symmetric and unbiased way, where the category $\mathscr{C}$ is not given priority over the category $\mathscr{C}^{op}$. Such a pair $(\mathscr{A}, \mathscr{B})$ is called a *chirality* because of the mirror-symmetry phenomena occurring between the two components $\mathscr{A}$ and $\mathscr{B}$.

Cartesian closed chiralities. The method is not limited to dialogue categories, as we illustrate below with cartesian closed categories. To that purpose, we define a *cartesian closed chirality* as a pair $(\mathscr{A}, \mathscr{B})$ where $\mathscr{A}$ is equivalent to a cartesian closed category $\mathscr{C}$ (and thus, is a cartesian closed category itself) and $\mathscr{B}$ is equivalent to its opposite category $\mathscr{C}^{op}$. A cartesian closed chirality $(\mathscr{A}, \mathscr{B})$ is easily characterized as a pair consisting of

- a category $\mathscr{A}$ with finite products noted $(a_1, a_2) \mapsto a_1 \wedge a_2$ for the binary products and **true** for the terminal object,
- a category $\mathscr{B}$ with finite sums noted $(b_1, b_2) \mapsto b_1 \vee b_2$ for the binary sums and **false** for the initial object,

equipped with:

- an equivalence of category between $\mathscr{A}$ and $\mathscr{B}^{op}$, which transports every object a of $\mathscr{A}$ to an object noted $\sim a$ of $\mathscr{B}$ and every object b of $\mathscr{B}$ to an object noted $\sim b$ of $\mathscr{A}$,
- a pseudo-action

$$\vee \quad : \quad \mathscr{B} \;\times\; \mathscr{A} \quad \longrightarrow \quad \mathscr{A} \tag{4}$$

of the monoidal category $(\mathscr{B}, \vee, \mathbf{false})$ on the category $\mathscr{A}$,

- a bijection
$$\mathscr{A}(\, a_1 \wedge a_2\, ,\, a_3\,) \quad \cong \quad \mathscr{A}(\, a_2\, ,\, (\sim a_1) \vee a_3\,) \tag{5}$$
natural in a_1, a_2 and a_3,

- a pseudo-action
$$\wedge \quad : \quad \mathscr{B} \quad \times \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B} \tag{6}$$
of the monoidal category $(\mathscr{A}, \wedge, \mathbf{true})$ on the category $\mathscr{B}$,

- a bijection
$$\mathscr{B}(\, b_1\, ,\, b_2 \vee b_3\,) \quad \cong \quad \mathscr{B}(\, b_1 \wedge (\sim b_3)\, ,\, b_2\,) \tag{7}$$
natural in b_1, b_2 and b_3.

In this unbiased and two-sided formulation of cartesian closed categories, the two pseudo-actions (4) and (6) are inherited from the functor

$$\Rightarrow \quad : \quad \mathscr{C}^{\,op} \quad \times \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C} \tag{8}$$

which transports every pair (x, y) of objects to the hom-object $x \Rightarrow y$. The two canonical isomorphisms

$$(x_1 \times x_2) \Rightarrow y \quad \cong \quad x_1 \Rightarrow (x_2 \Rightarrow y) \qquad\qquad 1 \Rightarrow x \quad \cong \quad x \tag{9}$$

of the cartesian closed category $\mathscr{C}$ are themselves translated as the two isomorphisms

$$(b_1 \vee b_2) \vee a \quad \cong \quad b_1 \vee (b_2 \vee a) \qquad\qquad \mathbf{false} \vee a \quad \cong \quad a$$

which make the operation $(b, a) \mapsto b \vee a$ a pseudo-action of $\mathscr{B}$ over $\mathscr{A}$. Symmetrically, and at the same time, the two canonical isomorphisms (9) are translated as the two isomorphisms

$$b \wedge (a_1 \wedge a_2) \quad \cong \quad (b \wedge a_1) \wedge a_2 \qquad\qquad b \wedge \mathbf{true} \quad \cong \quad b$$

which make the operation $(b, a) \mapsto b \wedge a$ a pseudo-action of $\mathscr{A}$ over $\mathscr{B}$. An unexpected outcome of the deformation of the category $\mathscr{C}$ into the chirality $(\mathscr{A}, \mathscr{B})$ is that the intuitionistic implication functor (8) factors as

$$\mathscr{A}^{\,op} \times \mathscr{A} \xrightarrow{\ \sim \times \mathscr{A}\ } \mathscr{B} \times \mathscr{A} \xrightarrow{\ \vee\ } \mathscr{A}$$

in just the same way as the implication $P \Rightarrow Q$ of two logical propositions factors in classical logic as the disjunction

$$P \Rightarrow Q \quad = \quad (\sim P) \vee Q \tag{10}$$

where $\sim P$ denotes the negation of the proposition P. This phenomenon is familiar and well-documented in monoidal categories equipped with a sufficiently strong notion of self-duality. Typically, the hom-object $x \multimap y = [x, y]$ may be defined as

- the object $x^* \otimes y$ in a ribbon category, where x^* is the right dual of x, see [43, 21, 22] for details,
- the object $x^* \wp y$ in a $*$-autonomous category, where $x \wp y$ is itself defined as $({}^*y \otimes {}^*x)^*$ where ${}^*x$ is the left dual of x, see [3, 30] for details.

However, it is the first time that the decomposition (10) is examined for cartesian closed categories. In that respect, our two-sided formulation of a cartesian closed category $\mathscr{C}$ reveals that the decomposition of implication $P \Rightarrow Q$ as $(\sim P) \vee Q$ is not limited to the familiar case of self-dual monoidal categories where the category $\mathscr{C}$ is equivalent to its opposite category $\mathscr{C}^{op}$. Since cartesian closed categories encode minimal intuitionistic logic, our approach etablishes the decomposition of implication (10) as a general principle of logic, valid both in intuitionistic logic and in classical logic. In particular, contrary to popular belief, what differentiates classical logic from intuitionistic logic is not the ability to decompose implication, into disjunction and negation, but the algebraic nature of the conjunction and of the disjunction connectives which define the logic: the connectives are actions of one side $(\mathscr{A}, \wedge, \mathbf{true})$ or $(\mathscr{B}, \vee, \mathbf{false})$ of the cartesian closed chirality in the case of intuitionistic logic, whereas they are tensor and cotensor products $\wedge = \otimes$ and $\vee = ⅋$ of a $*$-autonomous category $\mathscr{C}$ in the case of linear logic, and finite products $\wedge$ and finite sums $\vee$ of a boolean algebra in the case of classical logic.

It is worth mentioning that, in order to be complete, the characterization of cartesian closed chiralities requires in addition that the two coherence diagrams

$$\begin{tikzcd}
\mathscr{A}(a_1 \wedge a_2 \wedge a_3, a_4) \arrow[r] \arrow[d] & \mathscr{A}(a_3, \sim(a_1 \wedge a_2) \vee a_4) \arrow[d, "(*)"] \\
\mathscr{A}(a_2 \wedge a_3, \sim a_1 \vee a_4) \arrow[r] & \mathscr{A}(a_3, \sim a_2 \vee \sim a_1 \vee a_4)
\end{tikzcd} \tag{11}$$

$$\begin{tikzcd}
\mathscr{A}(\mathbf{true} \wedge a_1, a_2) \arrow[r] \arrow[d] & \mathscr{A}(a_1, \sim \mathbf{true} \vee a_2) \arrow[d, "(*)"] \\
\mathscr{A}(a_1, a_2) \arrow[r] & \mathscr{A}(a_1, \mathbf{false} \vee a_2)
\end{tikzcd} \tag{12}$$

commute, where the isomorphisms $(*)$ from $\sim(a_1 \wedge a_2)$ to $\sim a_2 \vee \sim a_1$ and from $\sim\mathbf{true}$ to $\mathbf{false}$ are deduced from the fact that the equivalence $\sim$ transports finite products of $\mathscr{A}$ into finite sums of $\mathscr{B}$. A similar pair of coherence diagrams is also required to commute for the pseudo-action (6).

Remark. We choose this specific formulation of cartesian closed chiralities in order to keep a perfect symmetry between the two sides $\mathscr{A}$ and $\mathscr{B}$ of the chirality. However, an easy calculation shows that in any cartesian closed chirality, there exists a natural family of isomorphisms $\sim(b \wedge a) \cong \sim a \vee \sim b$ which relates the two pseudo-actions (4) and (6). For that reason, it would be harmless to remove the pseudo-action (6) as well as the bijection (7) from our characterization of cartesian closed chiralities. Alternatively, one could remove the pseudo-action (4) as well as the bijection (5) and still have a proper characterization of cartesian closed chiralities. This point will be discussed in our section §9.2 on mixed chiralities.

A useful convention. Before going any further in our two-sided formulation of dialogue categories, we would like to introduce a useful convention. Since negation tends to reverse the orientation of the tensors, we find convenient to replace the opposite category $\mathscr{C}^{op(1)}$ by the category $\mathscr{C}^{op(0,1)}$ where the 0-dimensional cells (the objects) as well as the 1-dimensional cells (the morphisms) have been reversed. By "reversing the objects", we simply mean that the orientation of tensors is reversed in the following way:

$$x \otimes^{op(0,1)} y \quad := \quad y \otimes x.$$

The notation and terminology reflect the fact that a monoidal category $\mathscr{C}$ may be seen as a 2-category $\Sigma\mathscr{C}$ (more precisely, a bicategory) with one object, called its suspension. Now, the 1-cells of the suspension 2-category $\Sigma\mathscr{C}$ are the 0-cells of the category $\mathscr{C}$. Hence, reversing the 0-cells in $\mathscr{C}$ means reversing the 1-cells in $\Sigma\mathscr{C}$, or equivalently, reversing the orientation of the tensor product in the category $\mathscr{C}$. One benefit of this convention on 0-cells is that the expected equation holds:

$$(\Sigma\,\mathscr{C})^{op(1,2)} \quad = \quad \Sigma\,(\mathscr{C}^{op(0,1)})$$

where $(\Sigma\mathscr{C})^{op(1,2)}$ is the 2-category $\Sigma\mathscr{C}$ where the orientation of the 1-cells and of the 2-cells have been reversed.

Dialogue chiralities. As already mentioned, the main purpose of the article is to characterize the pairs $(\mathscr{A},\mathscr{B})$ obtained by deforming a dialogue category $\mathscr{C}$ into a category $\mathscr{A}$, and by deforming at the same time but independently its opposite category $\mathscr{C}^{op}$ into a category $\mathscr{B}$. Every pair $(\mathscr{A},\mathscr{B})$ obtained in this way is called a *dialogue chirality*. A preliminary observation is that in every dialogue chirality:

- the category $\mathscr{A}$ inherits a tensor product $\owedge$ and a unit **true**, reflecting the tensor product $\otimes$ and the unit e of the category $\mathscr{C}$,
- the category $\mathscr{B}$ inherits a tensor product $\ovee$ and a unit **false** from the very same monoidal structure, but considered this time in the opposite category $\mathscr{C}^{op(0,1)}$ where the orientation of objects and morphisms have been reversed.

The equality in the category $\mathscr{C}$ induces a monoidal equivalence

$$\begin{tikzcd}(\mathscr{A},\owedge,\mathbf{true}) \arrow[r, bend left, "(-)^*"] \arrow[r, phantom, "\text{monoidal equivalence}"] & (\mathscr{B},\ovee,\mathbf{false})^{op(0,1)} \arrow[l, bend left, "{}^*(-)"]\end{tikzcd} \tag{13}$$

which transports every object a of the category $\mathscr{A}$ into the corresponding object a^* of the category $\mathscr{B}$, and symmetrically, every object b of the category $\mathscr{B}$ into the corresponding object ${}^*b$ of the category $\mathscr{A}$. By monoidal equivalence, one means that the functors $(-)^*$ and ${}^*(-)$ are equipped with natural isomorphisms

$$\begin{array}{rclcrcl} (a_1 \owedge a_2)^* & \cong & {a_2}^* \ovee {a_1}^* & \qquad & \mathbf{true}^* & \cong & \mathbf{false} \\ {}^*(b_1 \ovee b_2) & \cong & {}^*b_2 \owedge {}^*b_1 & & {}^*\mathbf{false} & \cong & \mathbf{true} \end{array}$$

making the expected coherence diagrams commute. It should be stressed that our notations are directly inspired by logic, just as in the case of the notations (negation, conjunction and disjunction) used for cartesian closed chiralities. The idea is that the functors $(a \mapsto a^*)$ and $(b \mapsto {}^*b)$ are involutive forms of negation transporting the objects of $\mathscr{A}$ into the objects of $\mathscr{B}$ and conversely. Accordingly, the tensor product $\oslash$ is interpreted in the category $\mathscr{A}$ as a conjunction with its unit thus noted **true**, whereas the tensor product $\varobslash$ is interpreted in the category $\mathscr{B}$ as a disjunction with its unit thus noted **false**.

Another important observation on dialogue chiralities is that the two categories $\mathscr{A}$ and $\mathscr{B}$ are related by an adjunction

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, bend left, "R"] \end{tikzcd} \tag{14}$$

inherited from the original adjunction (3) between the categories $\mathscr{C}$ and $\mathscr{C}^{op}$. This adjunction enables one to construct the functor

$$\langle - | - \rangle \quad : \quad \mathscr{A}^{op} \times \mathscr{B} \quad \longrightarrow \quad \boldsymbol{Set}$$

also called distributor or $\mathscr{A}\mathscr{B}$-module, defined as

$$\langle a | b \rangle \quad = \quad \mathscr{A}(a, R\, b). \tag{15}$$

For aesthetic reasons, we will consider in §5 the more general notion of *dispute chirality* where the pair of adjoint functors $L \dashv R$ is replaced by the distributor $\langle - | - \rangle$. A dialogue chirality will be then identified in §6 as a dispute chirality where the distributor $\langle - | - \rangle$ is generated by an adjunction $L \dashv R$ in the sense of Equation (15). In particular, the coherence theorem (Theorem 3) established at the end of §7 states that every dialogue chirality may be strictified to a dialogue category.

Deformations of dialogue categories. One motivation for deforming strict monoidal categories into monoidal categories is to encompass natural examples arising in algebra and in topology. Typically, a cartesian category like ***Set*** is monoidal, but not strict monoidal, because the two sets $X \times (Y \times Z)$ and $(X \times Y) \times Z$ are isomorphic, but not equal. One feels the need for a similar deformation of dialogue categories related to their duality structure in order to understand better their relationship to $*$-autonomous categories. Let us explain why. As we have just explained, every dialogue category $\mathscr{C}$ may be seen as a dialogue chirality $(\mathscr{C}, \mathscr{C}^{op})$ where the two functors $(-)^*$ and ${}^*(-)$ are defined as the identity on the category $\mathscr{C}$. By convention, we call "strict" every dialogue chirality $(\mathscr{C},\mathscr{C}^{op})$ obtained in this way. Note that for every strict dialogue chirality $(\mathscr{A},\mathscr{B})$, one has the equality $\mathscr{B}=\mathscr{A}^{op}$ with the adjunction

$$\begin{tikzcd} \mathscr{A}=\mathscr{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{C}^{op}=\mathscr{B} \arrow[l, bend left, "R"] \end{tikzcd}$$

defined as the adjoint pair $L \dashv R$ between the two negation functors

$$L : x \mapsto \bot \multimapinv x : \mathscr{A} \to \mathscr{B} \qquad\qquad R : x \mapsto x \multimap \bot : \mathscr{B} \to \mathscr{A}.$$

The construction applies to dialogue categories in general, and in particular to $*$-autonomous categories. However, the shift from dialogue categories to dialogue chiralities enables us to think of $*$-autonomous categories in another fundamentally different way. We call a dialogue chirality $(\mathscr{A},\mathscr{B})$ "self-dual" when the two sides $\mathscr{A}$ and $\mathscr{B}$ are equal to the same category $\mathscr{C}$, and when the two functors L and R are identity functors:

$$\begin{tikzcd} \mathscr{A}=\mathscr{C} \arrow[r, bend left, "L=id"] \arrow[r, phantom, "\perp"] & \mathscr{C}=\mathscr{B} \arrow[l, bend left, "R=id"] \end{tikzcd}$$

Every $*$-autonomous category $\mathscr{C}$ may be seen as such a self-dual dialogue chirality $(\mathscr{C},\mathscr{C})$ where $\mathscr{A}=\mathscr{C}$ and $\mathscr{B}=\mathscr{C}$. The two operations $a \mapsto a^*$ and $b \mapsto {}^*b$ are provided in that case by the *involutive negations* of the $*$-autonomous category. Note that the $*$-autonomous category $\mathscr{B}=\mathscr{C}$ on the righthand-side of the adjunction is *equivalent* but not equal in general to the opposite $\mathscr{A}^{op}=\mathscr{C}^{op}$ of the category $\mathscr{A}=\mathscr{C}$ on the lefthand side of the adjunction. In that respect, the dialogue chirality $(\mathscr{C},\mathscr{C})$ induced by a $*$-autonomous category is self-dual but not strict in general. However, as we will see in the course of the paper, the self-dual dialogue chirality $(\mathscr{C},\mathscr{C})$ is equivalent (in the sense of dialogue chiralities) to the strict dialogue chirality $(\mathscr{C},\mathscr{C}^{op})$ associated to the $*$-autonomous category $\mathscr{C}$. As such, the self-dual dialogue chirality $(\mathscr{C},\mathscr{C})$ may be seen as a "deformation" of its strict counterpart $(\mathscr{C},\mathscr{C}^{op})$.

This example of $*$-autonomous categories lies at the heart of our work on tensorial logic and dialogue categories. It illustrates our claim that the notion of dialogue chirality plays a similar role for dialogue categories as the notion of monoidal category for strict monoidal categories. In particular, shifting from dialogue categories to dialogue chiralities enables us to capture new examples of interest like these self-dual dialogue chiralities $(\mathscr{C},\mathscr{C})$ associated to a $*$-autonomous category $\mathscr{C}$. The discussion may be summarized in a table:

strict notions	**notions up to deformation**
strict monoidal categories	monoidal categories
cartesian closed categories	cartesian closed chiralities
dialogue categories	dialogue chiralities

Polarities and symmetrization of logic. One purpose of this work is to provide a categorical explanation for the notion of polarity in logic. The notion emerged in the early 1990s in the work by Andreoli on focalization in proof search [1] and in the work by Girard on the semantics of classical logic [14]. The notion of polarity then became prominent in the linear logic circles, in particular after the definition of ludics [15] and of polarized linear logic [29]. The basic principle of polarization is to distinguish two classes of formulas, called *positive* and *negative*, and to apply logical connectives only when the formulas are of the appropriate polarity. Because of its origins in Girard's work [14], the notion of polarity is often believed to be intrinsically connected to classical logic. The conception is misleading and one purpose of the present work is precisely to clarify this issue by observing the situation through the prism of higher dimensional algebra. One benefit of our 2- and 3-categorical approach is to explain in what sense polarities are entirely independent of the intuitionistic or classical nature of the underlying logic. It appears that the effect of polarities is not to alter the logic but to provide a symmetric and two-sided *point of view* on it, where the original category $\mathscr{C}$ of denotations is replaced by a pair of categories $(\mathscr{A},\mathscr{B})$ consisting of a "positive" category $\mathscr{A}$ of proofs (or programs) and of a "negative" category $\mathscr{B}$ of counter-proofs (or counter-programs). Understood in this way, the idea of polarity is sufficiently general to work for any reasonable notion of category $\mathscr{C}$ with structure, as already illustrated by our chiral reformulation of cartesian closed categories. In retrospect, the polarity table introduced by Girard after his discovery of the interpretation of classical logic (LC) in *correlation spaces* [14] is not intrinsic since it mainly reflects the structure of a specific dialogue chirality $(\mathscr{A},\mathscr{B})$ induced by the dialogue category $\mathscr{C}$ of correlation spaces:

$+\oplus+=+$	$+\otimes+=+$	$!(-)=+$
$-\&-=-$	$-\invamp-=-$	$?(+)=-$

In this dialogue chirality, the category $\mathscr{A}$ of positive correlation spaces or commutative $\otimes$-comonoids has cartesian products $\otimes$ and finite sums $\oplus$ while the category $\mathscr{B}$ of negative correlation spaces or commutative $\parr$-monoids has cartesian products & and finite sums $\parr$. The negation or shift functors L and R are then defined by the exponential modalities

$$L = ? \ : \ \mathscr{A} \longrightarrow \mathscr{B} \qquad\qquad R = ! \ : \ \mathscr{B} \longrightarrow \mathscr{A}.$$

Since the polarity table reflects the structure of a dialogue category $\mathscr{D}$ with finite sums, it makes sense to rewrite it according to our notation for dialogue chiralities. One obtains in this way the polarity table

$+ \oplus + = +$	$+ \owedge + = +$	$R(-) = +$
$- \,\&\, - = -$	$- \ovee - = -$	$L(+) = -$

As expected, this polarity table is different from the polarity table associated to a monoidal closed category $\mathscr{C}$ with finite sums:

$+ \oplus + = +$	$+ \owedge + = +$	$+ \owedge - = -$
$- \,\&\, - = -$	$- \ovee - = -$	$- \ovee + = +$

Indeed, in just the same way as for cartesian closed chiralities, the polarity table of a monoidal closed chirality $(\mathscr{A}, \mathscr{B})$ includes a pseudo-action

$$\ovee \quad : \quad \mathscr{A} \times \mathscr{B} \quad \longrightarrow \quad \mathscr{A}$$

of the "negative" category $\mathscr{B}$ on the "positive" category $\mathscr{A}$ as well as a pseudo-action

$$\owedge \quad : \quad \mathscr{A} \times \mathscr{B} \quad \longrightarrow \quad \mathscr{B}$$

of the "positive" category $\mathscr{A}$ on the "negative" category $\mathscr{B}$. This alternative polarity table is interesting for its own sake and different from the original polarity table introduced by Girard. The polarity table also happens to describe several computational situations of interest, with a close connection to the notion of stack in programming languages, see §9 for a discussion.

A microcosm principle for duality. One last motivation for this work is to establish a "microcosm principle" for dialogue categories and similar notions of categories with dualities. Indeed, a distinctive property of dialogue categories is that the two negation functors $A \mapsto A \multimap \perp$ and $A \mapsto \perp \multimapinv A$ are contravariant. For this reason, as already mentioned, the two functors cannot be expressed in the 2-category ***Cat*** without mentioning the self-duality 2-functor $\mathscr{C} \mapsto \mathscr{C}^{op}$. This phenomenon is similar to the fact that one needs the monoidal structure of ***Cat*** provided by finite products of categories $\mathscr{C}, \mathscr{D} \mapsto \mathscr{C} \times \mathscr{D}$ in order to define the very notion of monoidal category ; and that, more generally, one needs a monoidal category in order to define a monoid object in it. This "microcosm principle" for monoidal categories has been recognized and extensively studied on n-dimensional categories equipped with various monoidal (or algebraic) structures, in particular by Baez and Dolan [2]. One point of the article is that the microcosm principle is not limited to monoidal structures, and that it also regulates the definition of dialogue categories and other algebraic structures equipped with a duality. In particular, there exists an operation $\mathscr{C} \mapsto \mathscr{C}^{op(k)}$ which transforms every n-dimensional category $\mathscr{C}$ into the n-dimensional category $\mathscr{C}^{op(k)}$ where the directions of the k-dimensional cells has been formally reversed, for $k \leq n$. An interesting question is thus to understand what are the dualities required at higher dimensions in order to define the dual structures at lower dimensions. One purpose of the article is to investigate this microcosm principle in the special case of dialogue categories, and to clarify along the way how the traditional dualities of logic based on negation are incorporated inside the first ladders (dimensions 1, 2 and 3) of higher dimensional algebra.

Related works. As already mentioned, our purpose in the present work is to provide a 2-categorical foundation to the notion of polarity introduced by Girard in his work on classical logic (LC) and on correlation spaces [14]. The connection between LC and continuation passing style (CPS) translations of classical logic into intuitionistic logic was recognized and investigated for the first time by Murthy [38] after the seminal work by Griffin [16]. The semantic study of a new CPS translation inspired by a connection between an early manuscript by Lafont [27] and Krivine's account of Gödel's translation [25] was independently developed by Lafont, Reus and Streicher [28]. The duality between call-by-name and call-by-value translations was then observed by Reus and Streicher in subsequent work [41]. The duality between the LKQ and LKT proof systems for classical logic emerged at about the same time in the work by Danos, Joinet and Schellinx [10] whose purpose was to replay Girard's work on classical logic in the framework of linear logic, and to clarify the relationship between LC and Parigot's $\lambda\mu$-calculus [39].

Our definition of dialogue chirality is based on the fact that negation induces an adjunction between the category $\mathscr{C}$ and its opposite category $\mathscr{C}^{op}$. This fact was observed by A. Kock in his study of dualities in monoidal categories [24]. It was then rediscovered and promoted by Thielecke [42] in his study of continuations prompted by the early observation by Filinski [12] of a duality between the call-by-name and call-by-value evaluation mechanisms. Inspired by the completeness theorem established by Hofmann and Streicher [20] for the continuation models of the $\lambda\mu$-calculus, Selinger introduced the notion of control category, and formulated this duality between call-by-name and call-by-value as a duality between control and co-control categories [40]. Much work has been devoted in the late 1990s by Curien and Herbelin [8, 9] and more recently by Munch [37] to develop programming languages and abstract machines based on this duality. The reader is advised to read the nice account of this line of work by Wadler [44, 45]. Note also that similar ideas were recently developed by Carraro, Ehrhard and Salibra in a calculus of stacks [11].

One distinctive feature of our work on tensorial logic [32, 33] compared to Girard's original work on LC [14] and the subsequent work by Laurent on polarized linear logic [29] is that we decorrelate the exponential modality $A \mapsto\, !A$ from the tensorial negation $A \mapsto \neg A$. To that purpose, we focus on a system of *linear* rather than *intuitionistic* continuations, in a spirit closer to ludics [15]. In doing so, we carry on a line of work on linear continuations initiated and developed by Berdine, O'Hearn, Reddy and Thielecke [4, 5] as well as M. Hasegawa [18, 19] who considered linear continuations both in a call-by-value and in a call-by-name scenario.

The idea of describing a category of dialogue games $\mathscr{C}$ as a pair of opposite categories $\mathscr{A} = \mathscr{C}$ and $\mathscr{B} = \mathscr{C}^{op}$ related by an adjunction $L \dashv R$ emerged in our work when we were studying the categorical properties of asynchronous games [31]. The notion of dialogue chirality $(\mathscr{A}, \mathscr{B})$ became then an essential ingredient of our connection between dialogue categories, dialogue games and string diagrams [33]. A similar line of research on polarized categories and game semantics was independently taken by Cockett and Seely [7] with somewhat different purposes. The interested reader will find in the recent work by Munch [36] a development of the categorical framework described twelve years ago in [31] in connection to the work by Führmann on the computational λ-calculus [13].

Plan of the article. Before analyzing the specific case of dialogue categories and dialogue chiralities, we find clarifying to study the simpler case of categories and chiralities. We thus establish in §2 a coherence theorem (Thm. 1) for categories and chiralities, formulated as a biequivalence of 2-categories ***Cat*** and ***Chir***. This preliminary coherence theorem provides us with the organizing and recurrent pattern of the article. We carry on in this 2-categorical spirit and establish in §3 a similar coherence theorem (Thm. 2) between monoidal categories and monoidal chiralities. The remainder of the paper is then devoted to an adaptation of these two coherence theorems to the more sophisticated case of dialogue categories. We prepare the work by defining the 2-category ***DiaCat*** of dialogue categories in §4, the 2-category ***DisChir*** of dispute chiralities in §5 and the 2-category ***DiaChir*** of dialogue chiralities in §6. The next section §7 is entirely devoted to the construction of a biequivalence between the 2-categories ***DiaCat*** of dialogue categories and ***DiaChir*** of dialogue chiralities. This leads us to our main theorem (Thm. 3) stated at the end of §7. We then do some reverse engineering in §8 and explicate the notion of dispute category corresponding to the notion of dispute chirality. We conclude the article in §9 with a series of side remarks on the notion of dialogue chirality.

2 The basic case: categories and chiralities

The main result of the article (Thm. 3) is established at the end of §7. The theorem states that the notions of "dialogue category" and of "dialogue chirality" are equivalent in an appropriate 2-categorical sense. The proof is not particularly difficult in itself, but it requires a great care in the definitions, and thus spans along the sections §4 – §7 of the paper. Instead of going directly into the proof of this coherence theorem, we find convenient to establish first a similar coherence theorem for the much simpler case of categories and chiralities. The argument will be then adapted in §3 to the case of monoidal categories and monoidal chiralities.

Definition 1 (chirality) *A chirality is defined as a pair* $(\mathscr{A}, \mathscr{B})$ *of categories equipped with an equivalence of categories:*

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "(-)^*"] \arrow[r, phantom, "\textit{equivalence}"] & \mathscr{B}^{op} \arrow[l, bend left, "{}^*(-)"] \end{tikzcd}$$

As in the case of dialogue chiralities discussed in the introduction, one defines

Definition 2 (strict chirality) *A chirality* $(\mathscr{A},\mathscr{B})$ *is called strict when* $\mathscr{B}$ *is equal to* $\mathscr{A}^{op}$ *and when the two functors* $(-)^*$ *and* ${}^*(-)$ *are equal to the identity functor on the category* $\mathscr{A}$*.*

Note that there is an obvious one-to-one relationship between categories and strict chiralities, where every category $\mathscr{C}$ is associated to the strict chirality $(\mathscr{C},\mathscr{C}^{op})$. Hence, a rudimentary coherence theorem for chiralities would state that every chirality $(\mathscr{A},\mathscr{B})$ is equivalent to the strict chirality $(\mathscr{A},\mathscr{A}^{op})$ in the 2-category $\boldsymbol{Cat} \times \boldsymbol{Cat}^{op(2)}$. This assertion is true but essentially straightforward, and not particularly useful for the applications we have in mind. As a matter of fact, our main interest in this work is to understand what notions of 1-cell and 2-cell between chiralities should replace the familiar notions of functor and natural transformation between categories. In that respect, it makes little sense to consider chiralities as specific objects of the 2-category $\boldsymbol{Cat} \times \boldsymbol{Cat}^{op(2)}$ since the equivalence between $\mathscr{A}$ and $\mathscr{B}^{op}$ disappears when one considers the chirality $(\mathscr{A},\mathscr{B})$ as an object of the 2-category $\boldsymbol{Cat} \times \boldsymbol{Cat}^{op(2)}$. Consequently, the notions of 1-cell and 2-cell are too liberal in the 2-category $\boldsymbol{Cat} \times \boldsymbol{Cat}^{op(2)}$ since a 1-cell

$$F = (F_\bullet, F_\circ) \quad : \quad (\mathscr{A}_1,\mathscr{B}_1) \to (\mathscr{A}_2,\mathscr{B}_2)$$

is defined there as a pair of entirely decorrelated functors $F_\bullet : \mathscr{A}_1 \to \mathscr{A}_2$ and $F_\circ : \mathscr{B}_1 \to \mathscr{B}_2$, and similarly for the notion of 2-cell. This discussion leads us to construct the 2-category $\boldsymbol{Chir}$ of chiralities, chirality functors and chirality natural transformations, defined as follows.

The 1-dimensional cells. A chirality functor

$$(\mathscr{A}_1,\mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2,\mathscr{B}_2)$$

is defined as a triple $(F_\bullet, F_\circ, \widetilde{F})$ consisting of two functors

$$F_\bullet \quad : \quad \mathscr{A}_1 \quad \longrightarrow \quad \mathscr{A}_2 \qquad\qquad F_\circ \quad : \quad \mathscr{B}_1 \quad \longrightarrow \quad \mathscr{B}_2$$

and a natural isomorphism

$$\widetilde{F} \quad : \quad (-)^* \circ F_\bullet \;\Rightarrow\; F_\circ^{op} \circ (-)^*$$

depicted as

$$\begin{tikzcd}
\mathscr{A}_1 \arrow[r, "F_\bullet"] \arrow[d, "(-)^*"'] \arrow[dr, phantom, "\widetilde{F}"] & \mathscr{A}_2 \arrow[d, "(-)^*"] \\
\mathscr{B}_1^{op} \arrow[r, "F_\circ^{op}"'] & \mathscr{B}_2^{op}
\end{tikzcd}$$

Note that an alternative and unbiased formulation of the same notion of chirality functor would be to equip the pair of functors $(F_\bullet, F_\circ)$ with a pair of natural isomorphisms

$$(-)^* \circ F_\bullet \Rightarrow F_\circ^{op} \circ (-)^* \qquad\qquad F_\bullet \circ {}^*(-) \Rightarrow {}^*(-) \circ F_\circ^{op}$$

together with a coherence diagram ensuring that the second natural isomorphism coincides with the mate of the first one, in the sense of Kelly and Street [23]. The two definitions are equivalent, and we thus pick the simplest formulation.

The 2-dimensional cells. A chirality natural transformation

$$\theta \quad : \quad F \quad \Rightarrow \quad G \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is defined as a pair of natural transformations

$$\begin{tikzcd} \mathscr{A}_1 \arrow[r, bend left, "F_\bullet"] \arrow[r, bend right, "G_\bullet"'] \arrow[r, phantom, "\Downarrow \theta_\bullet"] & \mathscr{A}_2 \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathscr{B}_1 \arrow[r, bend left, "F_\circ"] \arrow[r, bend right, "G_\circ"'] \arrow[r, phantom, "\Uparrow \theta_\circ"] & \mathscr{B}_2 \end{tikzcd}$$

satisfying the equality below:

$$\begin{tikzcd} \mathscr{A}_1 \arrow[r, bend left, "F_\bullet"] \arrow[r, bend right, "G_\bullet"'] \arrow[r, phantom, "\Downarrow \theta_\bullet"] \arrow[d, "(-)^*"'] \arrow[dr, phantom, "\widetilde{G}"] & \mathscr{A}_2 \arrow[d, "(-)^*"] \\ \mathscr{B}_1^{op} \arrow[r, bend right, "G_\circ^{op}"'] & \mathscr{B}_2^{op} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathscr{A}_1 \arrow[r, bend left, "F_\bullet"] \arrow[d, "(-)^*"'] \arrow[dr, phantom, "\widetilde{F}"] & \mathscr{A}_2 \arrow[d, "(-)^*"] \\ \mathscr{B}_1^{op} \arrow[r, bend left, "F_\circ^{op}"] \arrow[r, bend right, "G_\circ^{op}"'] \arrow[r, phantom, "\Downarrow \theta_\circ^{op}"] & \mathscr{B}_2^{op} \end{tikzcd} \tag{16}$$

This defines a 2-category ***Chir*** with chiralities as objects, chirality functors as 1-cells and chirality natural transformations as 2-cells, with the expected composition and identity laws.

Remark. Every functor $F : \mathscr{C} \to \mathscr{D}$ induces a chirality functor

$$(F_\bullet, F_\circ, \widetilde{F}) \quad : \quad (\mathscr{C}, \mathscr{C}^{op}) \quad \longrightarrow \quad (\mathscr{D}, \mathscr{D}^{op}) \tag{17}$$

where the natural transformation $\widetilde{F}$ is equal to the identity id_F and where $F_\bullet = F$ and $F_\circ = F^{op}$. This translation defines a one-to-one relationship between functors F and chirality functors $(F_\bullet, F_\circ, \widetilde{F})$ where $\widetilde{F}$ is equal to the identity. Moreover, every natural transformation

$$\theta \quad : \quad F \quad \Rightarrow \quad G \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{D}$$

induces a chirality natural transformation

$$(\theta_\bullet, \theta_\circ) \quad : \quad (F, F^{op}, id_F) \quad \Rightarrow \quad (G, G^{op}, id_G) \quad : \quad (\mathscr{C}, \mathscr{C}^{op}) \quad \longrightarrow \quad (\mathscr{D}, \mathscr{D}^{op})$$

and once again this relation is one-to-one. This means that the 2-category ***Cat*** is isomorphic to the sub-2-category of ***Chir*** consisting of strict chiralities $(\mathscr{C}, \mathscr{C}^{op})$ and chirality functors $F = (F_\bullet, F_\circ, id)$ with a trivial natural isomorphism $\widetilde{F} = id$ between them, leaving the chirality natural transformations unconstrained. On the other hand, the natural transformation $\widetilde{F}$ is not required to be the identity in the definition of a chirality functor $(F_\bullet, F_\circ, \widetilde{F})$. Consequently, there are in general more chirality functors of the form (17) than functors $F : \mathscr{C} \to \mathscr{D}$. This implies that the functor ***Cat*** $\to$ ***Chir*** obtained by translating categories into strict chiralities does not define an equivalence of categories, since such an equivalence would be fully faithful. This observation justifies to climb one step of the n-dimensional ladder and to move to 2-categories where we can establish that the translation defines a *biequivalence* between the 2-categories ***Cat*** and ***Chir***.

A biequivalence of 2-categories. In order to establish the biequivalence between ***Cat*** and ***Chir***, we construct a pair of 2-functors

$$\mathcal{F} \; : \; \boldsymbol{Cat} \; \longrightarrow \; \boldsymbol{Chir} \qquad\qquad \mathcal{G} \; : \; \boldsymbol{Chir} \; \longrightarrow \; \boldsymbol{Cat}$$

in the following way. The 2-functor $\mathcal{F}$ transports

- every category $\mathscr{C}$ to the strict chirality $(\mathscr{C}, \mathscr{C}^{op})$ with ${}^*(-) = (-)^*$ defined as the identity functor on $\mathscr{C}$,
- every functor F to the chirality functor (F, F^{op}, id_F) ,
- every natural transformation θ to the chirality natural transformation (θ, θ^{op}) ,

while the 2-functor $\mathcal{G}$ transports

- every chirality $(\mathscr{A},\mathscr{B})$ to the category $\mathscr{A}$,
- every chirality functor $F = (F_\bullet, F_\circ, \widetilde{F})$ to the underlying functor $F_\bullet$,
- every chirality natural transformation $\theta = (\theta_\bullet, \theta_\circ)$ to the natural transformation $\theta_\bullet$.

This leads us to the following coherence theorem:

Theorem 1 (coherence theorem) *The pair of 2-functors $\mathcal{F}$ and $\mathcal{G}$ defines a biequivalence between the 2-categories **Cat** and **Chir**.*

Proof. The composite 2-functor $\mathcal{G} \circ \mathcal{F}$ is equal to the identity on the 2-category ***Cat***. In order to establish the coherence property, it is thus sufficient to construct a pair of pseudo-natural transformations

$$\Phi \;:\; Id \longrightarrow \mathcal{F} \circ \mathcal{G} \qquad\qquad \Psi \;:\; \mathcal{F} \circ \mathcal{G} \longrightarrow Id$$

between the identity 2-functor on ***Chir*** and the 2-functor $\mathcal{F} \circ \mathcal{G}$, and to show that their components $\Phi_{(\mathscr{A},\mathscr{B})}$ and $\Psi_{(\mathscr{A},\mathscr{B})}$ define together an equivalence in the 2-category ***Chir***. The pseudo-natural transformation Φ is defined as follows. To every chirality $(\mathscr{A},\mathscr{B})$, one associates the 1-dimensional cell

$$\Phi_{(\mathscr{A},\mathscr{B})} \quad : \quad (\mathscr{A},\mathscr{B}) \quad \longrightarrow \quad (\mathscr{A},\mathscr{A}^{op})$$

defined as the chirality functor consisting of the two functors

$$(\Phi_{(\mathscr{A},\mathscr{B})})_\bullet \quad : \quad \mathscr{A} \quad \xrightarrow{id} \quad \mathscr{A} \qquad\qquad (\Phi_{(\mathscr{A},\mathscr{B})})_\circ \quad : \quad \mathscr{B} \quad \xrightarrow{(^*(-))^{op}} \quad \mathscr{A}^{op}$$

equipped with the natural isomorphism

$$\widetilde{\Phi_{(\mathscr{A},\mathscr{B})}} \quad = \quad \begin{tikzcd} \mathscr{A} \arrow[rr, "id"] \arrow[d, "(-)^*"'] & & \mathscr{A} \arrow[d, "id"] \\ \mathscr{B}^{op} \arrow[rr, "^*(-)"'] \arrow[urr, phantom, "\Swarrow\eta"] & & (\mathscr{A}^{op})^{op} \end{tikzcd}$$

defined as the unit η of the equivalence $(-)^* \dashv {}^*(-)$ between $\mathscr{A}$ and $\mathscr{B}^{op}$. Then, to every 1-dimensional cell $F : (\mathscr{A}_1,\mathscr{B}_1) \longrightarrow (\mathscr{A}_2,\mathscr{B}_2)$ one associates the chirality natural transformation

$$\Phi_F \quad : \quad \mathcal{F}\mathcal{G}(F) \circ \Phi_{(\mathscr{A}_1,\mathscr{B}_1)} \quad \Rightarrow \quad \Phi_{(\mathscr{A}_2,\mathscr{B}_2)} \circ F \quad : \quad (\mathscr{A}_1,\mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2,\mathscr{A}_2^{op})$$

defined as the pair of natural isomorphisms

$$
(\Phi_F)_\bullet \quad = \quad
\begin{tikzcd}
\mathscr{A}_1 \arrow[r, bend left=50, "F_\bullet"{name=U}] \arrow[r, bend right=50, "F_\bullet"'{name=D}] & \mathscr{A}_2
\arrow[from=U, to=D, Rightarrow, "id"]
\end{tikzcd}
$$

$$
(\Phi_F)_\circ \quad = \quad
\begin{tikzcd}
 & \mathscr{A}_1^{op} \arrow[r, "(F_\bullet)^{op}"] \arrow[dr, "((-)^*)^{op}"] \arrow[d, phantom, "\varepsilon^{op}\Uparrow"] & \mathscr{A}_2^{op} \arrow[r, "id"] \arrow[dr, "((-)^*)^{op}"] \arrow[d, phantom, "\Uparrow \widetilde{F}^{op}"] & \mathscr{A}_2^{op} \arrow[d, phantom, "\Uparrow \eta^{op}"] \\
\mathscr{B}_1 \arrow[ur, "(^*(-))^{op}"] \arrow[r, "id"'] & \mathscr{B}_1 \arrow[r, "F_\circ"'] & \mathscr{B}_2 \arrow[ur, "(^*(-))^{op}"'] &
\end{tikzcd}
$$

where ε denotes the counit of the equivalence between $\mathscr{A}_1$ and $\mathscr{B}_1^{op}$ while η denotes the unit of the equivalence between $\mathscr{A}_2$ and $\mathscr{B}_2^{op}$. One checks that the pair of natural isomorphisms satisfy the equation (16) and thus define a chirality natural transformation. Moreover, since the chirality natural transformation Φ_F consists of two reversible natural transformations, it defines a reversible 2-cell in the 2-category ***Chir***. Then, it is not difficult to show that Φ defines a pseudo-natural transformation, because

- the 2-cell $\Phi_{G\circ F}$ associated to the composite of two 1-cells F and G pasted along the 0-cell $(\mathscr{A}, \mathscr{B})$ coincides with the composite of the 2-cells Φ_G and Φ_F pasted along the 1-cell $\Phi_{(\mathscr{A},\mathscr{B})}$,
- the 2-cell $\Phi_{id} : \Phi_{(\mathscr{A},\mathscr{B})} \Rightarrow \Phi_{(\mathscr{A},\mathscr{B})}$ associated to the identity 1-cell $id : (\mathscr{A}, \mathscr{B}) \to (\mathscr{A}, \mathscr{B})$ coincides with the identity 2-cell on the 1-cell $\Phi_{(\mathscr{A},\mathscr{B})} : (\mathscr{A}, \mathscr{B}) \to (\mathscr{A}, \mathscr{A}^{op})$,
- for every 2-cell $\theta : F \Rightarrow G$, the 2-cell Φ_F pasted to the 2-cell $\mathcal{F}\mathcal{G}(\theta)$ along the 1-cell $\mathcal{F}\mathcal{G}(F)$ is equal to the 2-cell Φ_G pasted to the 2-cell θ along the 1-cell G. Note that establishing this last property requires the coherence diagram (16).

The pseudo-natural transformation Ψ is defined as follows. To every chirality $(\mathscr{A}, \mathscr{B})$, one associates the 1-cell

$$\Psi_{(\mathscr{A},\mathscr{B})} \quad : \quad (\mathscr{A}, \mathscr{A}^{op}) \quad \longrightarrow \quad (\mathscr{A}, \mathscr{B})$$

defined as the chirality functor consisting of the two functors

$$(\Psi_{(\mathscr{A},\mathscr{B})})_\bullet \quad : \quad \mathscr{A} \quad \xrightarrow{id} \quad \mathscr{A} \qquad\qquad (\Psi_{(\mathscr{A},\mathscr{B})})_\circ \quad : \quad \mathscr{A}^{op} \quad \xrightarrow{((-)^*)^{op}} \quad \mathscr{B}$$

equipped with the natural isomorphism

$$\widetilde{\Psi_{(\mathscr{A},\mathscr{B})}} \quad = \quad \begin{tikzcd} \mathscr{A} \arrow[r, "id"] \arrow[d, "id"'] \arrow[dr, phantom, "\swarrow id"] & \mathscr{A} \arrow[d, "(-)^*"] \\ (\mathscr{A}^{op})^{op} \arrow[r, "(-)^*"'] & \mathscr{B}^{op} \end{tikzcd}$$

To every 1-dimensional cell $F : (\mathscr{A}_1, \mathscr{B}_1) \longrightarrow (\mathscr{A}_2, \mathscr{B}_2)$ one associates the reversible 2-cell

$$\Psi_F \quad : \quad F \circ \Psi_{(\mathscr{A}_1,\mathscr{B}_1)} \quad \Rightarrow \quad \Psi_{(\mathscr{A}_2,\mathscr{B}_2)} \circ \mathcal{F}\mathcal{G}(F)$$

defined as the chirality natural transformations consisting of the two natural isomorphisms

$$(\Psi_F)_\bullet \quad = \quad \begin{tikzcd} \mathscr{A}_1 \arrow[r, bend left, "F_\bullet"] \arrow[r, bend right, "F_\bullet"'] \arrow[r, phantom, "\Downarrow id"] & \mathscr{A}_2 \end{tikzcd} \qquad (\Psi_F)_\circ \quad = \quad \begin{tikzcd} & \mathscr{B}_1 \arrow[dr, "F_\circ"] \arrow[d, phantom, "\Uparrow (\widetilde{F}^{-1})^{op}"] & \\ \mathscr{A}_1^{op} \arrow[ur, "((-)^*)^{op}"] \arrow[dr, "F_\bullet^{op}"'] & {} & \mathscr{B}_2 \\ & \mathscr{A}_2^{op} \arrow[ur, "((-)^*)^{op}"'] & \end{tikzcd}$$

It is not difficult to check that Ψ defines a pseudo-natural transformation in the same way as for Φ. Once this property has been established, there simply remains to show that the pair $\Phi_{(\mathscr{A},\mathscr{B})}$ and $\Psi_{(\mathscr{A},\mathscr{B})}$ defines an equivalence in the 2-category ***Chir***. The proof of this last statement is essentially immediate. This concludes the proof of the coherence theorem for chiralities.

Remark. It is worth mentioning that there exists a simpler proof that the 2-categories ***Cat*** and ***Chir*** are equivalent. First, one proves that the 2-functor $\mathcal{F}$ is a local equivalence, in other words, that every functor

$$\mathcal{F}(\mathscr{C},\mathscr{D}) \quad : \quad \boldsymbol{Cat}(\mathscr{C},\mathscr{D}) \quad \longrightarrow \quad \boldsymbol{Chir}(\mathcal{F}\mathscr{C},\mathcal{F}\mathscr{D})$$

is an equivalence of categories. Then, one proves that every chirality $(\mathscr{A},\mathscr{B})$ is equivalent in ***Chir*** to a chirality of the form $\mathcal{F}\mathscr{C} = (\mathscr{C},\mathscr{C}^{op})$. Both facts are easy to establish, and they imply together that the 2-functor $\mathcal{F}$ is a biequivalence, see [17] for details. This alternative proof works but it is less explicit, since it does not exhibit the 2-functor $\mathcal{G}$ nor the pseudo-natural transformations Φ and Ψ as in the proof of Theorem 1.

Strictification. Theorem 1 is inspired by a similar coherence result for monoidal categories, where one constructs a biequivalence between

- the 2-category of strict monoidal categories, strict monoidal functors and monoidal natural transformations, and
- the 2-category of monoidal categories, monoidal functors and monoidal natural transformations.

From that point of view, the 1-cell $\Phi_{(\mathscr{A},\mathscr{B})}$ should be understood as the operation of *strictifying* the chirality $(\mathscr{A},\mathscr{B})$ into the strict chirality $(\mathscr{A},\mathscr{A}^{op})$. As already mentioned, the category of 1-cells and 2-cells between two chiralities $(\mathscr{A}_1,\mathscr{B}_1)$ and $(\mathscr{A}_2,\mathscr{B}_2)$ is equivalent but in general not isomorphic to the category of functors between the categories $\mathscr{A}_1$ and $\mathscr{A}_2$. As a matter of fact, the functor $\mathcal{F}$ is faithful, but not full ; conversely, the functor $\mathcal{G}$ is full, but not faithful.

3 Monoidal categories and chiralities

Now that we have established in §2 a coherence theorem (Thm. 1) for categories and chiralities, we move to the notion of *monoidal chirality* which provides a two-sided and symmetric formulation of the notion of monoidal category.

Definition 3 *A monoidal chirality is defined as a pair of monoidal categories*

$$(\mathscr{A},\varowedge,\mathbf{true}) \qquad\qquad (\mathscr{B},\varovee,\mathbf{false})$$

equipped with a monoidal equivalence

$$\begin{tikzcd} \mathscr{A} \arrow[rr, bend left, "(-)^*"] \arrow[rr, phantom, "\substack{\textit{monoidal}\\ \textit{equivalence}}"] && \mathscr{B}^{op(0,1)} \arrow[ll, bend left, "{}^*(-)"] \end{tikzcd} .$$

In order to establish the equivalence of this notion with the familiar notion of monoidal category, we adapt the proof of Theorem 1 to this situation. We thus proceed as in §2 and define the 2-category ***MonCat*** with monoidal categories as 0-dimensional cells, lax monoidal functors as 1-dimensional cells and monoidal transformations as 2-dimensional cells. Recall that a lax monoidal functor

$$(F,m) \quad : \quad (\mathscr{C},\otimes,e) \quad \longrightarrow \quad (\mathscr{D},\otimes,e)$$

between monoidal categories is a functor $F : \mathscr{C} \to \mathscr{D}$ equipped with morphisms

$$m_{x,y} : Fx \otimes Fy \to F(x \otimes y) \qquad\qquad m_e : e \to F(e)$$

natural in x, y and satisfying the expected coherence diagrams. A monoidal natural transformation

$$\theta \quad : \quad (F, m) \;\Rightarrow\; (G, n) \quad : \quad (\mathscr{C}, \otimes, e) \quad \longrightarrow \quad (\mathscr{D}, \otimes, e)$$

between lax monoidal functors is a natural transformation

$$\theta \quad : \quad F \;\Rightarrow\; G \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{D}$$

making the diagrams commute:

$$\begin{tikzcd}
Fx \otimes Fy \arrow[r, "m_{x,y}"] \arrow[d, "\theta_x \otimes \theta_y"'] & F(x \otimes y) \arrow[d, "\theta_{x \otimes y}"] \\
Gx \otimes Gy \arrow[r, "n_{x,y}"] & G(x \otimes y)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & F(e) \arrow[dd, "\theta_e"] \\
e \arrow[ur, "m_e"] \arrow[dr, "n_e"'] & \\
 & G(e)
\end{tikzcd}$$

for all objects x, y of the category $\mathscr{C}$. This defines the 2-category ***MonCat***. One defines in the same way the 2-category ***OpMonCat*** with monoidal categories as 0-dimensional cells, oplax monoidal functors as 1-dimensional cells and monoidal transformations as 2-dimensional cells. Recall that an oplax monoidal functor

$$(F, m) \quad : \quad (\mathscr{C}, \otimes, e) \quad \longrightarrow \quad (\mathscr{D}, \otimes, e)$$

between monoidal categories is defined as a lax monoidal functor except for the orientation of the coercion morphisms defined as families of morphisms

$$m_{x,y} : F(x \otimes y) \to Fx \otimes Fy \qquad\qquad m_e : F(e) \to e$$

natural in x, y and satisfying the expected coherence diagrams. In order to adapt the equational argument of Theorem 1 to the case of monoidal categories and chiralities, we observe that the operation

$$\mathscr{C} \;\mapsto\; \mathscr{C}^{\,op(0,1)}$$

of taking the opposite of a monoidal category defines a pair of isomorphisms between the 2-categories:

$$\begin{tikzcd}
\boldsymbol{MonCat} \arrow[r, bend left, "(-)^{op(0,1)}"] \arrow[r, phantom, "\text{isomorphism}"] & \boldsymbol{OpMonCat}^{op(2)} \arrow[l, bend left, "(-)^{op(0,1)}"]
\end{tikzcd} . \tag{18}$$

This basic observation enables us to define the 2-category ***MonChir*** with monoidal chiralities as 0-dimensional cells, and the following notions of 1-cells and 2-cells.

The 1-dimensional cells. A 1-dimensional cell in ***MonChir***

$$F \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is defined as a triple $(F_\bullet, F_\circ, \widetilde{F})$ consisting of a lax monoidal functor

$$F_\bullet \quad : \quad (\mathscr{A}_1, \owedge_1, \mathbf{true}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \owedge_2, \mathbf{true}_2)$$

an oplax monoidal functor

$$F_\circ \quad : \quad (\mathscr{B}_1, \ovee_1, \mathbf{false}_1) \quad \longrightarrow \quad (\mathscr{B}_2, \ovee_2, \mathbf{false}_2)$$

and a monoidal natural isomorphism

$$\begin{tikzcd}
\mathscr{A}_1 \arrow[rr, "F_\bullet"] \arrow[d, "(-)^*"'] \arrow[drr, phantom, "\widetilde{F}"] & & \mathscr{A}_2 \arrow[d, "(-)^*"] \\
\mathscr{B}_1^{op(0,1)} \arrow[rr, "F_\circ^{op(0,1)}"'] & & \mathscr{B}_2^{op(0,1)}
\end{tikzcd}$$

The 2-dimensional cells. A 2-dimensional cell in ***MonChir***

$$\theta \quad : \quad F \Rightarrow G \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is defined as a pair $(\theta_\bullet, \theta_\circ)$ of monoidal natural transformations

$$\begin{tikzcd}
\mathscr{A}_1 \arrow[r, bend left, "F_\bullet"] \arrow[r, bend right, "G_\bullet"'] \arrow[r, phantom, "\Downarrow\theta_\bullet"] & \mathscr{A}_2
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathscr{B}_1 \arrow[r, bend left, "F_\circ"] \arrow[r, bend right, "G_\circ"'] \arrow[r, phantom, "\Uparrow\theta_\circ"] & \mathscr{B}_2
\end{tikzcd}$$

satisfying the equality below:

$$\begin{tikzcd}
\mathscr{A}_1 \arrow[rr, bend left, "F_\bullet"] \arrow[rr, bend right, "G_\bullet"'] \arrow[rr, phantom, "\Downarrow\theta_\bullet"] \arrow[d, "(-)^*"'] \arrow[drr, phantom, "\widetilde{G}"] & & \mathscr{A}_2 \arrow[d, "(-)^*"] \\
\mathscr{B}_1^{op(0,1)} \arrow[rr, "G_\circ^{op(0,1)}"'] & & \mathscr{B}_2^{op(0,1)}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathscr{A}_1 \arrow[rr, bend left, "F_\bullet"] \arrow[d, "(-)^*"'] \arrow[drr, phantom, "\widetilde{F}"] & & \mathscr{A}_2 \arrow[d, "(-)^*"] \\
\mathscr{B}_1^{op} \arrow[rr, bend left, "F_\circ^{op(0,1)}"] \arrow[rr, bend right, "G_\circ^{op(0,1)}"'] \arrow[rr, phantom, "\Downarrow\theta_\circ^{op(0,1)}"] & & \mathscr{B}_2^{op(0,1)}
\end{tikzcd}
\tag{19}$$

The 1-dimensional and 2-dimensional cells are then composed by pasting the underlying lax monoidal functors and natural transformations in the same way as in the case of the 2-category ***Chir***. Put together, these data define the announced 2-category ***MonChir*** of monoidal chiralities.

A biequivalence of 2-categories. In order to establish the biequivalence between ***MonCat*** and ***MonChir***, we proceed as in §2 except that the original involution (2) between categories is replaced by the involution (18) between monoidal categories. More specifically, we construct a pair of 2-functors

$$\mathcal{F} \;:\; \boldsymbol{MonCat} \longrightarrow \boldsymbol{MonChir} \qquad \mathcal{G} \;:\; \boldsymbol{MonChir} \longrightarrow \boldsymbol{MonCat}$$

where the 2-functor $\mathcal{F}$ transports

- every monoidal category $(\mathscr{C}, \otimes, e)$ to the monoidal chirality $(\mathscr{C}, \mathscr{C}^{\,op(0,1)})$ where ${}^*(-) = (-)^*$ are defined as the identity functor on $(\mathscr{C}, \otimes, e)$,
- every lax monoidal functor F to the 1-dimensional cell $(F, F^{op(0,1)}, id_F)$,
- every monoidal natural transformation θ to the 2-dimensional cell (θ, θ^{op}) ,

and where the 2-functor $\mathcal{G}$ transports

- every monoidal chirality $(\mathscr{A}, \mathscr{B})$ to the monoidal category $\mathscr{A}$,
- every 1-dimensional cell $F = (F_\bullet, F_\circ, \widetilde{F})$ to the lax monoidal functor $F_\bullet$,
- every 2-dimensional cell $\theta = (\theta_\bullet, \theta_\circ)$ to the monoidal natural transformation $\theta_\bullet$.

In the same way as for categories and chiralities, this leads us to the following coherence theorem for monoidal categories and chiralities:

Theorem 2 (coherence theorem) *The pair of 2-functors $\mathcal{F}$ and $\mathcal{G}$ defines a biequivalence between the 2-categories **MonCat** and **MonChir**.*

Note that the argument works in the same way for other notions of category with structure, like braided or symmetric monoidal categories.

4 Dialogue categories

We have just established in the previous sections a coherence theorem for categories and chiralities (§2, Thm. 1) followed by a similar coherence theorem for monoidal categories and chiralities (§3, Thm. 2). In the remainder of the article, we adapt these two inaugural theorems to the more sophisticated case of dialogue categories. To that purpose, we follow the same pattern as in §2 and §3 and start by constructing a 2-category ***DiaCat*** of dialogue categories, dialogue functors and dialogue natural transformations.

4.1 Definition

We start by recalling the definition of a dialogue category.

Definition 4 (tensorial pole) *A tensorial pole in a monoidal category $\mathscr{C}$ is an object $\perp$ equipped with a representation*

$$\varphi_{x,y} \quad : \quad \mathscr{C}(x \otimes y, \perp) \quad \cong \quad \mathscr{C}(y, x \multimap \perp)$$

of the presheaf functor

$$y \mapsto \mathscr{C}(x \otimes y, \perp) \quad : \quad \mathscr{C}^{op} \longrightarrow \boldsymbol{Set}$$

for each object x, and with a representation

$$\psi_{x,y} \quad : \quad \mathscr{C}(x \otimes y, \perp) \quad \cong \quad \mathscr{C}(x, \perp \multimapinv y)$$

of the presheaf functor

$$x \mapsto \mathscr{C}(x \otimes y, \perp) \quad : \quad \mathscr{C}^{op} \longrightarrow \boldsymbol{Set}$$

for each object y.

Definition 5 (dialogue category) *A dialogue category is a monoidal category equipped with a tensorial pole.*

Terminology: the objects $x \multimap \perp$ and $\perp \multimapinv x$ in a dialogue category $\mathscr{C}$ are called the tensorial negations of the object x.

4.2 The 2-category *DiaCat* of dialogue categories

We construct the 2-category ***DiaCat*** with dialogue categories as 0-cells, dialogue functors as 1-cells and dialogue natural transformations as 2-cells.

The 1-dimensional cells. A dialogue functor

$$(F, \perp_F) \quad : \quad (\mathscr{C}, \perp_{\mathscr{C}}) \quad \longrightarrow \quad (\mathscr{D}, \perp_{\mathscr{D}})$$

between dialogue categories is defined as a lax monoidal functor

$$F \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{D}$$

equipped with a morphism

$$\perp_F \quad : \quad F(\perp_{\mathscr{C}}) \quad \longrightarrow \quad \perp_{\mathscr{D}}.$$

The 2-dimensional cells. A dialogue natural transformation

$$\theta \quad : \quad (F, \perp_F) \quad \Rightarrow \quad (G, \perp_G)$$

is defined as a monoidal natural transformation

$$\theta \quad : \quad F \quad \Rightarrow \quad G$$

making the diagram

$$\begin{tikzcd}
F(\perp_{\mathscr{C}}) \arrow[dd, "\theta_{\perp_{\mathscr{C}}}"'] \arrow[dr, "\perp_F"] & \\
& \perp_{\mathscr{D}} \\
G(\perp_{\mathscr{C}}) \arrow[ur, "\perp_G"'] &
\end{tikzcd} \tag{20}$$

commute.

Remark. The reader will notice that, somewhat surprisingly, neither the negations $(x \mapsto x \multimap \perp)$ and $(x \mapsto \perp \multimapinv x)$ nor the natural bijections φ and ψ appear in the definition of a dialogue functor and of a dialogue natural transformation. This is unnecessary because a canonical map

$$F(\perp_{\mathscr{C}} \multimapinv x) \quad \longrightarrow \quad \perp_{\mathscr{D}} \multimapinv F(x)$$

can be deduced from the composite map

$$F(\perp_{\mathscr{C}} \multimapinv x) \otimes Fx \quad \longrightarrow \quad F((\perp_{\mathscr{C}} \multimapinv x) \otimes x) \quad \longrightarrow \quad F(\perp_{\mathscr{C}}) \quad \longrightarrow \quad \perp_{\mathscr{D}}$$

and similarly for the canonical map

$$F(x \multimap \perp_{\mathscr{C}}) \quad \longrightarrow \quad F(x) \multimap \perp_{\mathscr{D}}.$$

Moreover, the two resulting maps make the expected coherence diagrams

$$\begin{tikzcd}
\mathscr{C}(x \otimes y, \bot_{\mathscr{C}}) \arrow[rr, "\varphi_{x,y}"] \arrow[d, "F"'] & & \mathscr{C}(y, x \multimap \bot_{\mathscr{C}}) \arrow[d, "F"] \\
\mathscr{D}(F(x \otimes y), F(\bot_{\mathscr{C}})) \arrow[d, "monoidality\, of\, F"'] & & \mathscr{D}(Fy, F(x \multimap \bot_{\mathscr{C}})) \arrow[d, "(*)"] \\
\mathscr{D}(Fx \otimes Fy, \bot_{\mathscr{D}}) \arrow[rr, "\phi_{F(x),F(y)}"] & & \mathscr{D}(Fy, Fx \multimap \bot_{\mathscr{D}})
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{C}(x \otimes y, \bot_{\mathscr{C}}) \arrow[rr, "\psi_{x,y}"] \arrow[d, "F"'] & & \mathscr{C}(x, \bot_{\mathscr{C}} \multimapinv y) \arrow[d, "F"] \\
\mathscr{D}(F(x \otimes y), F(\bot_{\mathscr{C}})) \arrow[d, "monoidality\, of\, F"'] & & \mathscr{D}(Fx, F(\bot_{\mathscr{C}} \multimapinv y)) \arrow[d] \\
\mathscr{D}(Fx \otimes Fy, \bot_{\mathscr{D}}) \arrow[rr, "\psi_{F(x),F(y)}"] & & \mathscr{D}(Fx, \bot_{\mathscr{D}} \multimapinv Fy)
\end{tikzcd}$$

commute for all objects x, y of the category $\mathscr{C}$. Typically, the commutation of the first diagram is established by replacing the map $(*)$ by its definition as the unique morphism making the diagram

$$\begin{tikzcd}
\mathscr{D}(Fy, F(x \multimap \bot_{\mathscr{C}})) \arrow[rr, "(*)"] \arrow[d, "Fx\otimes-"'] & & \mathscr{D}(Fy, Fx \multimap \bot_{\mathscr{D}}) \\
\mathscr{D}(Fx \otimes Fy, Fx \otimes F(x \multimap \bot_{\mathscr{C}})) \arrow[d, "monoidality\, of\, F"'] & & \mathscr{D}(Fx \otimes Fy, \bot_{\mathscr{D}}) \arrow[u, "\phi_{Fx,Fy}"'] \\
\mathscr{D}(Fx \otimes Fy, F(x \otimes (x \multimap \bot_{\mathscr{C}})) \arrow[rr, "F(eval)"] & & \mathscr{D}(Fx \otimes Fy, F(\bot_{\mathscr{C}})) \arrow[u, "\bot_F"']
\end{tikzcd}$$

commutes in ***Set*** for all objects x, y of the category $\mathscr{C}$.

4.3 An adjunction between negation and itself

In every dialogue category, the family of objects $(x \multimap \bot)_{x \in obj(\mathscr{C})}$ defines a functor

$$x \;\mapsto\; (x \multimap \bot) \quad : \quad \mathscr{C}^{op} \quad \longrightarrow \quad \mathscr{C}$$

uniquely determined by the requirement that the bijection $\varphi_{x,y}$ is natural in x and y. This property is established by a simple argument based on the Yoneda lemma. Similarly, the family of objects $(\perp \multimapinv y)_{y \in obj(\mathscr{C})}$ defines a functor

$$y \;\mapsto\; (\perp \multimapinv y) \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}^{op}$$

uniquely determined by the requirement that the bijection $\psi_{x,y}$ is natural in x and y. Moreover, the two functors

$$L(x) \quad = \quad \perp \multimapinv x \qquad \text{and} \qquad R(x) \quad = \quad x \multimap \perp$$

are related by an adjunction

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{C}^{op} \arrow[l, bend left, "R"] \end{tikzcd} \tag{21}$$

induced by the series of natural bijections

$$\begin{array}{rcll} \mathscr{C}(y, x \multimap \perp) & \cong & \mathscr{C}(x \otimes y, \perp) & \text{defined by } \varphi_{x,y} \\ & \cong & \mathscr{C}(x, \perp \multimapinv y) & \text{defined by } \psi_{x,y} \\ & = & \mathscr{C}^{op}(\perp \multimapinv y, x) & \text{by definition of } \mathscr{C}^{op}. \end{array}$$

Remark. The choice of the negation $(x \mapsto \perp \multimapinv x)$ rather than the negation $(x \mapsto x \multimap \perp)$ as left adjoint functor is somewhat arbitrary, because the 2-functor $(-)^{op}$ transports the adjunction (21) into its companion adjunction

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "R^{op}"] \arrow[r, phantom, "\perp"] & \mathscr{C}^{op} \arrow[l, bend left, "L^{op}"] \end{tikzcd}$$

where the role of the two functors $(x \mapsto \perp \multimapinv x)$ and $(x \mapsto x \multimap \perp)$ have been interchanged. This second adjunction is witnessed by the series of natural bijections

$$\begin{array}{rcll} \mathscr{C}(x, \perp \multimapinv y) & \cong & \mathscr{C}(x \otimes y, \perp) & \text{defined by } \psi_{x,y} \\ & \cong & \mathscr{C}(y, x \multimap \perp) & \text{defined by } \varphi_{x,y} \\ & = & \mathscr{C}^{op}(x \multimap \perp, y) & \text{by definition of } \mathscr{C}^{op}. \end{array}$$

This notion of companionship between dialogue categories and dialogue chiralities will be further discussed in §8.

5 Dispute chiralities

In this section, we formulate a notion of dispute chirality in §5.1 and construct a 2-category ***DisChir*** of dispute chiralities in §5.2. The notion of dispute chirality is introduced here to prepare the notion of dialogue category formulated in the next section §4. We believe however that the notion is interesting for its own sake, as a simpler and more primitive variant of the notion of dialogue chirality. We will also see in §8 that it corresponds to a notion of dispute category in the same way as the notion of dialogue chirality corresponds to the notion of dialogue category.

5.1 Definition

Definition 6 (dispute chirality) *A dispute chirality is defined as a pair of monoidal categories*

$$(\mathscr{A}, \oslash, \mathbf{true}) \qquad\qquad (\mathscr{B}, \varobar, \mathbf{false})$$

equipped with a monoidal equivalence

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "(-)^*"] \arrow[r, phantom, "\substack{\text{monoidal}\\ \text{equivalence}}"] & \mathscr{B}^{op(0,1)} \arrow[l, bend left, "{}^*(-)"] \end{tikzcd}$$

with a distributor, or categorical bimodule:

$$\langle - | - \rangle \quad : \quad \mathscr{A}^{op} \times \mathscr{B} \quad \longrightarrow \quad \textbf{\textit{Set}}$$

and with a family of bijections

$$\chi_{m,a,b} \quad : \quad \langle\, m \oslash a \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, a \,|\, m^* \varobar b \,\rangle$$

natural in a and b. The family χ is moreover required to make the diagram below commute:

$$\begin{tikzcd} \langle\, (m \oslash n) \oslash a \,|\, b \,\rangle \arrow[rr, "\chi_{m \oslash n}"] \arrow[d, "associativity"'] & & \langle\, a \,|\, (m \oslash n)^* \varobar b \,\rangle \\ \langle\, m \oslash (n \oslash a) \,|\, b \,\rangle \arrow[r, "\chi_m"] & \langle\, n \oslash a \,|\, m^* \varobar b \,\rangle \arrow[r, "\chi_n"] & \langle\, a \,|\, n^* \varobar (m^* \varobar b) \,\rangle \arrow[u, "\substack{associativity\\ monoidality\ of\ negation}"'] \end{tikzcd} \tag{22}$$

Remark. The coherence diagram below provides a nullary counterpart to Diagram (22) and the careful reader may thus find unexpected not to see it mentioned in our definition of a dispute chirality.

$$
\begin{tikzcd}
\langle\, \mathbf{true} \owedge a \,|\, b \,\rangle \arrow[rr, "associativity"] \arrow[d, "\chi_{\mathbf{true}}"'] & & \langle\, a \,|\, b \,\rangle \arrow[d, "associativity"] \\
\langle\, a \,|\, \mathbf{true}^{*} \ovee b \,\rangle & & \langle\, a \,|\, \mathbf{false} \ovee b \,\rangle \arrow[ll, "\substack{monoidality \\ of\ negation}"']
\end{tikzcd}
$$

The reason is that the diagram always commutes in a dispute chirality: this fact is established by instantiating the coherence diagram (22) at $m = n = \mathbf{true}$ and by applying the naturality of the bijection χ and the coherence properties of the monoidal categories $\mathscr{A}$ and $\mathscr{B}$.

5.2 The 2-category *DisChir* of dispute chiralities

Now that the notion of dispute chirality has been introduced in §5.1, we are ready to define the 2-category ***DisChir*** with dispute chiralities as 0-cells, and the following notions of 1-cell and 2-cell between them:

The 1-dimensional cells. A 1-dimensional cell in ***DisChir***

$$F \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is defined as a quadruple $(F_{\bullet}, F_{\circ}, \widetilde{F}, \overline{F})$ consisting of

- a lax monoidal functor $F_{\bullet} : \mathscr{A}_1 \longrightarrow \mathscr{A}_2$,
- an oplax monoidal functor $F_{\circ} : \mathscr{B}_1 \longrightarrow \mathscr{B}_2$
- a monoidal natural isomorphism $\widetilde{F} : (-)^{*} \circ F_{\bullet} \Rightarrow (F_{\circ})^{op(0,1)} \circ (-)^{*}$

together with a natural transformation

$$
\begin{tikzcd}
& \boldsymbol{Set} & \\
\mathscr{A}_1^{op} \times \mathscr{B}_1 \arrow[ur, "\langle - | - \rangle_1"] \arrow[rr, "F_{\bullet}^{op} \times F_{\circ}"'] \arrow[rr, phantom, shift left=6ex, "\overset{\Rightarrow}{\overline{F}}"] & & \mathscr{A}_2^{op} \times \mathscr{B}_2 \arrow[ul, "\langle - | - \rangle_2"']
\end{tikzcd}
$$

making the diagram

$$
\begin{tikzcd}
\langle\, m \oslash a \,|\, b \,\rangle \arrow[rr, "\chi_m"] \arrow[d, "\overline{F}"'] & & \langle\, a \,|\, m^* \obslash b \,\rangle \arrow[d, "\overline{F}"] \\
\langle\, F_\bullet(m \oslash a) \,|\, F_\circ(b) \,\rangle \arrow[dd, "monoidality\ of\ F_\bullet"'] & & \langle\, F_\bullet(a) \,|\, F_\circ(m^* \obslash b) \,\rangle \arrow[d, "monoidality\ of\ F_\circ"] \\
& & \langle\, F_\bullet(a) \,|\, F_\circ(m^*) \obslash F_\circ(b) \,\rangle \arrow[d, "\widetilde{F}"] \\
\langle\, F_\bullet(m) \oslash F_\bullet(a) \,|\, F_\circ(b) \,\rangle \arrow[rr, "\chi_{F_\bullet(m)}"] & & \langle\, F_\bullet(a) \,|\, F_\bullet(m)^* \obslash F_\circ(b) \,\rangle
\end{tikzcd}
\tag{23}
$$

commute for all objects a, m in $\mathscr{A}$ and b in $\mathscr{B}$.

The 2-dimensional cells. A 2-dimensional cell in ***DisChir***

$$\theta \quad : \quad F \Rightarrow G \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is defined as a pair $(\theta_\bullet, \theta_\circ)$ of monoidal natural transformations

$$\theta_\bullet : F_\bullet \Rightarrow G_\bullet : \mathscr{A}_1 \to \mathscr{A}_2 \qquad\qquad \theta_\circ : G_\circ \Rightarrow F_\circ : \mathscr{B}_1 \to \mathscr{B}_2$$

making the diagram (19) as well as the diagram below commute:

$$
\begin{tikzcd}[column sep=small]
& \textbf{\textit{Set}} \arrow[d, phantom, "\overset{\overline{F}}{\Rightarrow}"] & \\
\mathscr{A}_1^{op} \times \mathscr{B}_1 \arrow[ur, "\langle - | - \rangle_1"] \arrow[rr, bend right, "F_\bullet^{op} \times F_\circ"'] & & \mathscr{A}_2^{op} \times \mathscr{B}_2 \arrow[ul, "\langle - | - \rangle_2"']
\end{tikzcd}
\quad = \quad
\begin{tikzcd}[column sep=small]
& \textbf{\textit{Set}} \arrow[d, phantom, "\overset{\overline{G}}{\Rightarrow}"] & \\
\mathscr{A}_1^{op} \times \mathscr{B}_1 \arrow[ur, "\langle - | - \rangle_1"] \arrow[rr, bend left, "G_\bullet^{op} \times G_\circ"] \arrow[rr, bend right, "F_\bullet^{op} \times F_\circ"'] \arrow[rr, phantom, "\Downarrow \theta_\bullet^{op} \times \theta_\circ"] & & \mathscr{A}_2^{op} \times \mathscr{B}_2 \arrow[ul, "\langle - | - \rangle_2"']
\end{tikzcd}
$$

The 1-dimensional and 2-dimensional cells are then composed by pasting the underlying functors and natural transformations in the same way as for ***Chir*** in §2 and for ***MonChir*** in §3. Put together, these data define the announced 2-category ***DisChir*** of dispute chiralities.

Remark. The coherence diagram (23) in the definition of a 1-dimensional cell ensures that the natural transformation $\overline{F}$ may be recovered from the induced natural transformation

$$\langle\, a \,|\, \mathbf{false} \,\rangle_1 \xrightarrow{\overline{F}} \langle\, F_\bullet(a) \,|\, F_\circ(\mathbf{false}) \,\rangle_2 \xrightarrow{\textit{monoidality}} \langle\, F_\bullet(a) \,|\, \mathbf{false} \,\rangle_2$$

together with the combined data of the lax monoidal functor $F_\bullet$, the monoidal natural transformation $\widetilde{F}$ and the natural bijection χ.

6 Dialogue chiralities

Although the notion of dispute chirality is nice and primitive, we prefer to focus in the present paper on the notion of dialogue chirality which provides a two-sided and properly symmetric formulation of the notion of dialogue category. The purpose of the section is to introduce the notion and to construct the 2-category ***DiaChir*** of dialogue chiralities.

Definition 7 (dialogue chirality) *A dialogue chirality is a pair of monoidal categories*

$$(\mathscr{A}, \oslash, \mathbf{true}) \qquad\qquad (\mathscr{B}, \obslash, \mathbf{false})$$

equipped with a monoidal equivalence

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "(-)^*"] \arrow[r, phantom, "\textit{monoidal equivalence}"] & \mathscr{B}^{op(0,1)} \arrow[l, bend left, "{}^*(-)"] \end{tikzcd}$$

with an adjunction

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, bend left, "R"] \end{tikzcd}$$

and with a family of bijections

$$\chi_{m,a,b} \quad : \quad \langle\, m \oslash a \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, a \,|\, m^* \obslash b \,\rangle$$

natural in a and b, where $\langle\, a \,|\, b \,\rangle$ is defined as

$$\langle\, a \,|\, b \,\rangle \quad = \quad \mathscr{A}(\, a \,,\, R\, b\,).$$

The family χ is moreover required to make the diagram below

$$\begin{tikzcd}
\langle\, (m \oslash n) \oslash a \,|\, b \,\rangle \arrow[rr, "\chi_{m \oslash n}"] \arrow[d, "associativity"'] & & \langle\, a \,|\, (m \oslash n)^{*} \obslash b \,\rangle \\
\langle\, m \oslash (n \oslash a) \,|\, b \,\rangle \arrow[r, "\chi_m"] & \langle\, n \oslash a \,|\, m^{*} \obslash b \,\rangle \arrow[r, "\chi_n"] & \langle\, a \,|\, n^{*} \obslash (m^{*} \obslash b) \,\rangle \arrow[u, "\substack{associativity \\ monoidality\ of\ negation}"']
\end{tikzcd} \tag{24}$$

commute.

One way to think of a dialogue chirality is to understand it as a particular kind of dispute chirality $(\mathscr{A}, \mathscr{B})$ whose evaluation bracket $\langle\, - \,|\, - \,\rangle$ is induced by an adjunction $L \dashv R$. The picture is essentially correct. One should be careful however that the adjunction $L \dashv R$ is an additional structure rather than an additional property of the dispute chirality. Interestingly, we will establish in Proposition 8 (see below) that the 2-category ***DiaChir*** is obtained by translating in the language of dialogue chiralities the definitions of 1-cells and 2-cells between dispute chiralities defined in the 2-category ***DisChir***.

The 1-dimensional cells. A 1-dimensional cell in ***DiaChir***

$$F \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is defined as a quadruple $F = (F_\bullet, F_\circ, \widetilde{F}, \overline{F})$ consisting of

- a lax monoidal functor $F_\bullet : \mathscr{A}_1 \longrightarrow \mathscr{A}_2$,
- an oplax monoidal functor $F_\circ : \mathscr{B}_1 \longrightarrow \mathscr{B}_2$
- a monoidal natural isomorphism $\widetilde{F} : (-)^* \circ F_\bullet \Rightarrow (F_\circ)^{op(0,1)} \circ (-)^*$

together with a natural transformation:

$$\begin{tikzcd}
\mathscr{A}_1 \arrow[r, "F_\bullet"] & \mathscr{A}_2 \\
\mathscr{B}_1 \arrow[u, "R"] \arrow[r, "F_\circ"'] \arrow[ur, phantom, "\overline{F}"] & \mathscr{B}_2 \arrow[u, "R"']
\end{tikzcd}$$

making the diagram

$$
\begin{tikzcd}
\mathscr{A}_1(m \oslash a, R\,b) \arrow[rr, "\chi_m"] \arrow[d, "F_\bullet"'] & & \mathscr{A}_1(a, R(m^* \ovee b)) \arrow[d, "F_\bullet"] \\
\mathscr{A}_2(F_\bullet(m \oslash a), F_\bullet\, R\,b) \arrow[d, "\overline{F}"'] & & \mathscr{A}_2(F_\bullet(a), F_\bullet\, R\,(m^* \ovee b)) \arrow[d, "\overline{F}"] \\
\mathscr{A}_2(F_\bullet(m \oslash a), R\,F_\circ(b)) \arrow[dd, "monoidality\,of\,F_\bullet"'] & & \mathscr{A}_2(F_\bullet(a), R\,F_\circ(m^* \ovee b)) \arrow[d, "monoidality\,of\,F_\circ"] \\
& & \mathscr{A}_2(F_\bullet(a), R\,(F_\circ(m^*) \ovee F_\circ(b))) \arrow[d, "\widetilde{F}"] \\
\mathscr{A}_2(F_\bullet(m) \oslash F_\bullet(a), R\,F_\circ(b)) \arrow[rr, "\chi_{F_\bullet(m)}"] & & \mathscr{A}_2(F_\bullet(a), R\,(F_\bullet(m)^* \ovee F_\circ(b)))
\end{tikzcd}
\tag{25}
$$

commute for all objects a, m in $\mathscr{A}_1$ and b in $\mathscr{B}_1$.

The 2-dimensional cells. A 2-dimensional cell in ***DiaChir***

$$\theta \quad : \quad F \Rightarrow G \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is defined as a pair $(\theta_\bullet, \theta_\circ)$ of monoidal natural transformations

$$\theta_\bullet : F_\bullet \Rightarrow G_\bullet : \mathscr{A}_1 \to \mathscr{A}_2 \qquad\qquad \theta_\circ : G_\circ \Rightarrow F_\circ : \mathscr{B}_1 \to \mathscr{B}_2$$

making the diagram (19) as well as the diagram (26) below commute:

$$
\begin{tikzcd}
\mathscr{A}_1 \arrow[r, "F_\bullet"] & \mathscr{A}_2 \\
\mathscr{B}_1 \arrow[u, "R"] \arrow[r, "F_\circ"'] \arrow[ur, phantom, "\overline{F}"] & \mathscr{B}_2 \arrow[u, "R"']
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathscr{A}_1 \arrow[r, bend left=50, "F_\bullet"] \arrow[r, "G_\bullet"'] \arrow[r, phantom, bend left=25, "\Downarrow\theta_\bullet"] & \mathscr{A}_2 \\
\mathscr{B}_1 \arrow[u, "R"] \arrow[r, "G_\circ"] \arrow[r, bend right=50, "F_\circ"'] \arrow[r, phantom, bend right=25, "\Downarrow\theta_\circ"] \arrow[ur, phantom, "\overline{G}"] & \mathscr{B}_2 \arrow[u, "R"']
\end{tikzcd}
\tag{26}
$$

The following proposition is essentially straightforward:

Proposition 8 *The operation of forgetting the adjunction $L \dashv R$ in a dialogue chirality defines a 2-functor*

$$U \quad : \quad \textbf{\textit{DiaChir}} \quad \longrightarrow \quad \textbf{\textit{DisChir}}$$

which is fully faithful in the sense that the hom-functors

$$\textbf{\textit{DiaChir}}\,((\mathscr{A}_1,\mathscr{B}_1),(\mathscr{A}_2,\mathscr{B}_2)) \quad\longrightarrow\quad \textbf{\textit{DisChir}}\,(U(\mathscr{A}_1,\mathscr{B}_1),U(\mathscr{A}_2,\mathscr{B}_2))$$

are categorical isomorphisms for all dialogue chiralities $(\mathscr{A}_1,\mathscr{B}_1)$ *and* $(\mathscr{A}_2,\mathscr{B}_2)$.

7 The coherence theorem

In this section, we construct a 2-dimensional equivalence between the 2-category ***DiaCat*** of dialogue categories and the 2-category ***DiaChir*** of dialogue chiralities. Among other properties, this result implies that every dialogue chirality $(\mathscr{A},\mathscr{B})$ is equivalent in the 2-category ***DiaChir*** to the strict dialogue chirality $(\mathscr{C},\mathscr{C}^{op})$ associated to a dialogue category $\mathscr{C}$. This establishes the coherence theorem claimed in the introduction, as well as a recipe to strictify any dialogue chirality into a dialogue category.

7.1 From dialogue categories to dialogue chiralities

We start by constructing a 2-functor

$$\mathcal{F} \quad : \quad \textbf{\textit{DiaCat}} \quad\longrightarrow\quad \textbf{\textit{DiaChir}}$$

from the 2-category ***DiaCat*** of dialogue categories to the 2-category ***DiaChir*** of dialogue chiralities.

The 0-dimensional cells. To every dialogue category $\mathscr{C}$, the 2-functor $\mathcal{F}$ associates the dialogue chirality defined as

$$(\mathscr{A},\oslash,\textbf{true}) \;:=\; (\mathscr{C},\otimes,e) \qquad\qquad (\mathscr{B},\odot,\textbf{false}) \;:=\; (\mathscr{C},\otimes,e)^{op(0,1)}$$

where the monoidal equivalence between $\mathscr{A}$ and $\mathscr{B}^{op(0,1)}$ is trivially defined as the identity functor on the monoidal category $\mathscr{C}$, since

$$\mathscr{A} \quad=\quad \mathscr{C} \quad=\quad \mathscr{B}^{op(0,1)}.$$

The two adjoint functors L and R are defined as

$$L : x \mapsto \bot \multimapinv x \qquad\qquad R : x \mapsto x \multimap \bot$$

with the adjunction $L \dashv R$ witnessed by the series of bijections

$$\begin{array}{rcl} \mathscr{A}(x, R(y)) & = & \mathscr{C}(x, y \multimap \bot) \\ & \cong & \mathscr{C}(y \otimes x, \bot) \\ & \cong & \mathscr{C}(y, \bot \multimapinv x) \\ & = & \mathscr{B}(L(x), y) \end{array}$$

natural in x and y. Finally, the natural bijection $\chi_{m,x,y}$ is defined as the composite

$$\begin{tikzcd} \mathscr{C}(m \otimes x, y \multimap \bot) \arrow[d, "\varphi^{-1}_{x \otimes m, y}"'] & & \mathscr{C}(x, (y \otimes m) \multimap \bot) \\ \mathscr{C}(y \otimes (m \otimes x), \bot) \arrow[rr, "associativity"] & & \mathscr{C}((y \otimes m) \otimes x, \bot) \arrow[u, "\varphi_{y \otimes m, x}"'] \end{tikzcd}$$

It is not difficult to show that this definition makes the family χ satisfy the Equation (24) required in the definition of a dialogue chirality.

The 1-dimensional cells. To every dialogue functor

$$(F, \bot_F) \quad : \quad (\mathscr{C}, \bot_{\mathscr{C}}) \quad \longrightarrow \quad (\mathscr{D}, \bot_{\mathscr{D}})$$

the 2-functor $\mathcal{F}$ associates the 1-dimensional cell $\mathcal{F}(F)$ defined as the quadruple consisting of the lax monoidal functor

$$\mathcal{F}(F)_{\bullet} \ : \ \mathscr{C} \ \xrightarrow{F} \ \mathscr{D}$$

the oplax monoidal functor

$$\mathcal{F}(F)_{\circ} \ : \ \mathscr{C}^{op(0,1)} \ \xrightarrow{F^{op(0,1)}} \ \mathscr{D}^{op(0,1)}$$

the monoidal isomorphism $\widetilde{\mathcal{F}(F)}$ defined as the identity on the functor F, and the natural transformation

$$\overline{\mathcal{F}(F)} \quad : \quad R \circ F \quad \longrightarrow \quad F \circ R$$

whose components

$$F(x \multimap \bot_{\mathscr{C}}) \quad \longrightarrow \quad F(x) \multimap \bot_{\mathscr{D}}$$

is associated by currification $\varphi_{F(x), F(x \multimap \bot_{\mathscr{C}})}$ to the morphism

$$F(x) \otimes F(x \multimap \bot_{\mathscr{C}}) \ \longrightarrow \ F(x \otimes (x \multimap \bot_{\mathscr{C}})) \ \longrightarrow \ F(\bot_{\mathscr{C}}) \ \longrightarrow \ \bot_{\mathscr{D}}.$$

The monoidality of the functor F implies that this definition of the quadruple $\mathcal{F}(F)$ satisfies the Equation (25) required of a 1-cell between dialogue chiralities.

The 2-dimensional cells. To every dialogue natural transformation

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\theta"] & \mathscr{D} \end{tikzcd}$$

the 2-functor $\mathcal{F}$ associates the 2-dimensional cell $\mathcal{F}(\theta)$ defined as the pair of monoidal natural transformations

$$\mathcal{F}(\theta)_{\bullet} \quad = \quad \begin{tikzcd} \mathscr{A}_1 = \mathscr{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\theta"] & \mathscr{D} = \mathscr{A}_2 \end{tikzcd}$$

$$\mathcal{F}(\theta)_{\circ} \quad = \quad \begin{tikzcd} \mathscr{B}_1 = \mathscr{C}^{op(0,1)} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] \arrow[from=D, to=U, Rightarrow, "\theta^{op(0,1)}"] & \mathscr{D}^{op(0,1)} = \mathscr{B}_2 \end{tikzcd}$$

In order to establish that this pair of monoidal natural transformations define a 2-dimensional cell in the 2-category ***DiaChir***, one needs to show that the two equations (19) and (26) are satisfied: this is essentially immediate for equation (19) and this follows from the naturality of χ for equation (26).

7.2 From dialogue chiralities to dialogue categories

Now that the 2-functor $\mathcal{F}$ has been constructed, we go in the reverse direction, and define a 2-functor

$$\mathcal{G} \quad : \quad \textbf{\textit{DiaChir}} \quad \longrightarrow \quad \textbf{\textit{DiaCat}}$$

from the 2-category of dialogue chiralities to the 2-category of dialogue categories.

The 0-dimensional cells. The 2-functor transports every dialogue chirality $(\mathscr{A}, \mathscr{B})$ to the dialogue category defined as

$$(\mathscr{C}, \otimes, e) \quad := \quad (\mathscr{A}, \owedge, \mathbf{true})$$

equipped with the tensorial pole

$$\bot \quad := \quad R(\mathbf{false})$$

together with the functors:

$$\perp \multimapinv x \;=\; {}^*(L(x)) \qquad\qquad x \multimap \perp \;=\; R(x^*).$$

The natural bijections φ and ψ are defined by composing the series of natural bijections

$$\begin{array}{rcll}
\mathscr{C}(x \otimes y, \perp) & = & \mathscr{A}(x \oslash y, R(\mathbf{false})) & \text{by definition of } \mathscr{C} \text{ and of } \perp, \\
& \cong & \mathscr{A}(y, R(x^* \otimes \mathbf{false})) & \text{by applying } \chi_{x,y,\mathbf{false}}, \\
& \cong & \mathscr{A}(y, R(x^*)) & \text{by applying the unit law in } \mathscr{B}, \\
& \cong & \mathscr{B}(L(y), x^*) & \text{by the adjunction } L \dashv R, \\
& \cong & \mathscr{A}(x, {}^*(L(y))) & \text{by the adjunction } (-)^* \dashv {}^*(-), \\
& = & \mathscr{C}(x, {}^*(L(y))) & \text{by definition of } \mathscr{C}.
\end{array}$$

$$\begin{array}{rcll}
\mathscr{C}(x \otimes y, \perp) & = & \mathscr{A}(x \oslash y, R(\mathbf{false})) & \text{by definition of } \mathscr{C} \text{ and of } \perp, \\
& \cong & \mathscr{A}(y, R(x^* \otimes \mathbf{false})) & \text{by applying } \chi_{x,y,\mathbf{false}}, \\
& \cong & \mathscr{A}(y, R(x^*)) & \text{by applying the unit law in } \mathscr{B}, \\
& = & \mathscr{C}(y, R(x^*)) & \text{by definition of } \mathscr{C}.
\end{array}$$

The 1-dimensional cells. Every 1-dimensional cell

$$F = (F_\bullet, F_\circ, \widetilde{F}, \overline{F}) \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is transported to the dialogue functor $(F_\bullet, \perp_F)$ consisting of the functor

$$F_\bullet \quad : \quad \mathscr{A}_1 \quad \longrightarrow \quad \mathscr{A}_2.$$

and the morphism

$$\perp_F \quad : \quad F_\bullet(\perp_{\mathscr{A}_1}) \quad \longrightarrow \quad \perp_{\mathscr{A}_2}$$

defined as the composite

$$F_\bullet \circ R\,(\mathbf{false}) \xrightarrow{\overline{F}_{\mathbf{false}}} R \circ F_\circ\,(\mathbf{false}) \xrightarrow{\textit{monoidality}} R\,(\mathbf{false})$$

The 2-dimensional cells. Every 2-dimensional cell $\theta = (\theta_\bullet, \theta_\circ)$ is transported to the dialogue natural transformation $\theta_\bullet$. One easily checks that the monoidal natural transformation $\theta_\bullet$ makes Diagram (20) commute.

7.3 The pseudo-natural transformation Φ

As in the introductory case of categories and chiralities investigated in §2, the composite 2-functor

$$\textbf{\textit{DiaCat}} \xrightarrow{\mathcal{F}} \textbf{\textit{DiaChir}} \xrightarrow{\mathcal{G}} \textbf{\textit{DiaCat}}$$

coincides with the identity on the 2-category ***DiaCat*** of dialogue categories. In order to establish that the 2-categories ***DiaCat*** and ***DiaChir*** are biequivalent, we proceed in the same way as in the proof of Theorem 1: we construct a pair of pseudo-natural transformations

$$\Phi \;:\; Id \longrightarrow \mathcal{F}\circ\mathcal{G} \qquad\qquad \Psi \;:\; \mathcal{F}\circ\mathcal{G} \longrightarrow Id$$

between the identity 2-functor on ***Chir*** and the 2-functor $\mathcal{F}\circ\mathcal{G}$, and we show that their components $\Phi_{(\mathscr{A},\mathscr{B})}$ and $\Psi_{(\mathscr{A},\mathscr{B})}$ define an equivalence in the 2-category ***DiaChir***, for every dialogue chirality $(\mathscr{A},\mathscr{B})$. In order to achieve this purpose, it is important to describe very precisely the dialogue chirality $(\mathscr{A},\mathscr{A}^{op(0,1)})$ obtained by applying the 2-functor $\mathcal{F}\circ\mathcal{G}$ to a given dialogue chirality $(\mathscr{A},\mathscr{B})$. First of all, the dialogue chirality $(\mathscr{A},\mathscr{A}^{op(0,1)})$ is equipped with the trivial monoidal equivalence:

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "id"] \arrow[r, phantom, "\substack{monoidal\\ equivalence}"] & (\mathscr{A}^{op(0,1)})^{op(0,1)} \arrow[l, bend left, "id"] \end{tikzcd}$$

and with the adjunction obtained by composing the two adjunctions

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, bend left, "R"] \arrow[r, bend left, "({}^{*}(-))^{op(0,1)}"] \arrow[r, phantom, "\perp"] & \mathscr{A}^{op(0,1)} \arrow[l, bend left, "((-)^{*})^{op(0,1)}"] \end{tikzcd}$$

From this follows that

$$\langle\, a_1 \,|\, a_2 \,\rangle_{(\mathscr{A},\mathscr{A}^{op(0,1)})} \quad=\quad \mathscr{A}(a_1, R\,(a_2{}^{*})) \quad=\quad \langle\, a_1 \,|\, a_2{}^{*} \,\rangle_{(\mathscr{A},\mathscr{B})}$$

Moreover, the natural transformation $\chi_{(\mathscr{A},\mathscr{A}^{op(0,1)})}$ at instance (m,a,b) is defined as the composite function

$$\begin{tikzcd} \langle\, m \oslash a_1 \,|\, a_2{}^{*} \,\rangle \arrow[d] & \langle\, a_1 \,|\, (a_2 \oslash m)^{*} \,\rangle \\ \langle\, m \oslash a_1 \,|\, a_2{}^{*} \obslash \mathbf{false} \,\rangle \arrow[d, "(\chi_{(\mathscr{A},\mathscr{B})})^{-1}"'] & \langle\, a_1 \,|\, (a_2 \oslash m)^{*} \obslash \mathbf{false} \,\rangle \arrow[u] \\ \langle\, a_2 \oslash (m \oslash a_1) \,|\, \mathbf{false} \,\rangle \arrow[r] & \langle\, (a_2 \oslash m) \oslash a_1 \,|\, \mathbf{false} \,\rangle \arrow[u, "\chi_{(\mathscr{A},\mathscr{B})}"'] \end{tikzcd}$$

By using the fact that the dialogue chirality $(\mathscr{A}, \mathscr{B})$ satisfies the coherence diagram (24), one establishes that the diagram below commutes:

$$\begin{tikzcd}[column sep=small]
& \langle\, m \oslash a_1 \,|\, {a_2}^* \,\rangle \arrow[dl, "\chi_{(\mathscr{A},\mathscr{B})}"'] \arrow[dr, "\chi_{(\mathscr{A},\mathscr{A}^{op(0,1)})}"] & \\
\langle\, a_1 \,|\, m^* \oslash {a_2}^* \,\rangle \arrow[rr, "monoidality"] & & \langle\, a_1 \,|\, (a_2 \oslash m)^* \,\rangle
\end{tikzcd} \tag{27}$$

We are now ready to define the pseudo-natural transformation Φ.

The 1-dimensional cells $\Phi_{(\mathscr{A},\mathscr{B})}$. To every dialogue chirality $(\mathscr{A}, \mathscr{B})$ one associates the 1-dimensional cell

$$\Phi_{(\mathscr{A},\mathscr{B})} \quad : \quad (\mathscr{A}, \mathscr{B}) \quad \longrightarrow \quad (\mathscr{A}, \mathscr{A}^{\,op(0,1)})$$

defined as the pair of lax and oplax monoidal functors

$$(\Phi_{(\mathscr{A},\mathscr{B})})_\bullet \;:\; \mathscr{A} \xrightarrow{\;id\;} \mathscr{A} \qquad\qquad (\Phi_{(\mathscr{A},\mathscr{B})})_\circ \;:\; \mathscr{B} \xrightarrow{\;({}^*(-))^{op(0,1)}\;} \mathscr{A}^{\,op(0,1)}$$

together with the monoidal natural isomorphism

$$\widetilde{\Phi_{(\mathscr{A},\mathscr{B})}} \quad = \quad \begin{tikzcd}
\mathscr{A} \arrow[r, "id"] \arrow[d, "(-)^*"'] \arrow[dr, phantom, "\eta"] & \mathscr{A} \arrow[d, "id"] \\
\mathscr{B}^{\,op(0,1)} \arrow[r, "{}^*(-)"'] & (\mathscr{A}^{\,op(0,1)})^{op(0,1)}
\end{tikzcd}$$

and the natural transformation

$$\overline{\Phi_{(\mathscr{A},\mathscr{B})}} \quad = \quad \begin{tikzcd}
\mathscr{A} \arrow[rr, "id"] & & \mathscr{A} \\
 & & \mathscr{B} \arrow[u, "R"'] \\
\mathscr{B} \arrow[uu, "R"] \arrow[rr, "({}^*(-))^{op(0,1)}"'] \arrow[uurr, phantom, "\varepsilon^{\,op(0,1)}"] & & \mathscr{A}^{\,op(0,1)} \arrow[u, "((-)^*)^{op(0,1)}"']
\end{tikzcd}$$

where η and ε denote the unit and the counit of the adjunction $(-)^* \dashv {}^*(-)$. Using the fact that the diagram (27) commutes, one readily checks that this definition of $\Phi_{(\mathscr{A},\mathscr{B})}$ makes the diagram (25) commute, and thus provides a valid definition of a 1-cell in the 2-category ***DiaChir***.

The 2-dimensional cells Φ_F. To every 1-dimensional cell in ***DiaChir***

$$F \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

one associates the 2-cell in ***DiaChir***

$$\Phi_F \quad : \quad \Phi_{(\mathscr{A}_2,\mathscr{B}_2)} \circ F \quad \Rightarrow \quad \mathcal{F}\mathcal{G}(F) \circ \Phi_{(\mathscr{A}_1,\mathscr{B}_1)}$$

defined as the pair of monoidal natural transformations

$$(\Phi_F)_\bullet \quad = \quad \begin{tikzcd} \mathscr{A}_1 \arrow[r, bend left, "F_\bullet"{name=U}] \arrow[r, bend right, "F_\bullet"'{name=D}] & \mathscr{A}_2 \arrow[from=U, to=D, Rightarrow, "id"] \end{tikzcd}$$

$$(\Phi_F)_\circ \quad = \quad \begin{tikzcd} & \mathscr{A}_1^{op} \arrow[rr, "(F_\bullet)^{op}"] \arrow[dr, "((-)^*)^{op}"] \arrow[d, phantom, "\varepsilon^{op}\Uparrow"] & & \mathscr{A}_2^{op} \arrow[rr, "id"] \arrow[dr, "((-)^*)^{op}"] \arrow[d, phantom, "\Uparrow\widetilde{F}^{op}"] & & \mathscr{A}_2^{op} \arrow[dl, phantom, "\Uparrow\eta^{op}"] \\ \mathscr{B}_1 \arrow[ur, "(^*(-))^{op}"] \arrow[rr, "id"'] & & \mathscr{B}_1 \arrow[rr, "F_\circ"'] & & \mathscr{B}_2 \arrow[ur, "(^*(-))^{op}"'] & \end{tikzcd}$$

Note that the 2-cell Φ_F is defined in the same way as the corresponding 2-cell Φ_F in the proof of Theorem 1 in §2. One checks that the definition makes the diagrams (19) and (26) commute, and thus defines a 2-cell in the 2-category ***DiaChir***. Moreover, the family Φ defines a pseudo-natural transformation.

7.4 The pseudo-natural transformation Ψ

The 1-dimensional cells $\Psi_{(\mathscr{A},\mathscr{B})}$. To every dialogue chirality $(\mathscr{A},\mathscr{B})$, one associates the 1-dimensional cell $\Psi_{(\mathscr{A},\mathscr{B})}$ defined as the pair of functors

$$(\Psi_{(\mathscr{A},\mathscr{B})})_\bullet \ : \ \mathscr{A} \xrightarrow{id} \mathscr{A} \qquad\qquad (\Psi_{(\mathscr{A},\mathscr{B})})_\circ \ : \ \mathscr{A}^{op(0,1)} \xrightarrow{((-)^*)^{op(0,1)}} \mathscr{B}$$

equipped with the trivial monoidal natural isomorphism

$$\widetilde{\Psi_{(\mathscr{A},\mathscr{B})}} \quad = \quad \begin{tikzcd} \mathscr{A} \arrow[rr, "id"] \arrow[d, "id"'] & \arrow[d, phantom, "id\ \Swarrow"] & \mathscr{A} \arrow[d, "(-)^*"] \\ (\mathscr{A}^{op(0,1)})^{op(0,1)} \arrow[rr, "(-)^*"'] & & \mathscr{B}^{op(0,1)} \end{tikzcd}$$

and with the trivial natural transformation

$$\overline{\Psi_{(\mathscr{A},\mathscr{B})}} \quad = \quad \begin{tikzcd} \mathscr{A} \arrow[rr, "id"] & & \mathscr{A} \\ \mathscr{B} \arrow[u, "R"] & \arrow[r, phantom, "\Searrow id"] & \\ \mathscr{A}^{op(0,1)} \arrow[u, "((-)^*)^{op(0,1)}"] \arrow[rr, "((-)^*)^{op(0,1)}"'] & & \mathscr{B} \arrow[uu, "R"'] \end{tikzcd}$$

Just as in the case of $\Phi_{(\mathscr{A},\mathscr{B})}$, one establishes that this definition of $\Psi_{(\mathscr{A},\mathscr{B})}$ makes the diagram (25) commute using the fact that the diagram (27) commutes. As such, $\Psi_{(\mathscr{A},\mathscr{B})}$ provides a valid definition of a 1-dimensional cell in the 2-category ***DiaChir***.

The 2-dimensional cells Ψ_F. To every 1-dimensional cell F in ***DiaChir***

$$F \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

one associates the reversible 2-cell in ***DiaChir***

$$\Psi_F \quad : \quad F \circ \Psi_{(\mathscr{A}_1,\mathscr{B}_1)} \quad \Rightarrow \quad \Psi_{(\mathscr{A}_2,\mathscr{B}_2)} \circ \mathcal{F}\mathcal{G}(F)$$

defined as the pair of monoidal natural transformations

$$(\Psi_F)_\bullet = \begin{tikzcd} \mathscr{A}_1 \arrow[r, bend left, "F_\bullet"] \arrow[r, bend right, "F_\bullet"'] \arrow[r, phantom, "\Downarrow id"] & \mathscr{A}_2 \end{tikzcd} \qquad (\Psi_F)_\circ = \begin{tikzcd} & \mathscr{B}_1 \arrow[dr, "F_\circ"] & \\ \mathscr{A}_1^{op(0,1)} \arrow[ur, "((-)^*)^{op(0,1)}"] \arrow[dr, "F_\bullet^{op(0,1)}"'] & \Uparrow (\widetilde{F}^{-1})^{op(0,1)} & \mathscr{B}_2 \\ & \mathscr{A}_2^{op(0,1)} \arrow[ur, "((-)^*)^{op(0,1)}"'] & \end{tikzcd}$$

One easily checks that the natural transformations $(\Psi_F)_\circ$ and $(\Psi_F)_\bullet$ satisfy the equation (19) of §5.2 as well as the equation (26) of §6. It is also not difficult to see that Ψ defines a pseudo-natural transformation, in the same way as in the proof of Theorem 1 in §2.

7.5 Main theorem

At this stage of the proof, there only remains to show that for every dialogue chirality $(\mathscr{A},\mathscr{B})$, the pair of 1-cells $\Phi_{(\mathscr{A},\mathscr{B})}$ and $\Psi_{(\mathscr{A},\mathscr{B})}$ defines an equivalence in the 2-category ***DiaChir***. This statement is essentially immediate to establish. This leads us to the main result of the article:

Theorem 3 (Coherence theorem) *The pair of 2-functors $\mathcal{F}$ and $\mathcal{G}$ defines a biequivalence between the 2-categories **DiaCat** and **DiaChir**.*

This coherence theorem is important in our work because it enables us to replace dialogue categories by dialogue chiralities whenever an unbiased and properly two-sided description of proofs and counter-proofs appears necessary. The alternative formulation of dialogue categories is an essential ingredient in our 2-categorical reconstruction of game semantics in the language of string diagrams [33] as well as in our description of dialogue categories as a lax notion of Frobenius algebras [34].

8 Back to dispute chiralities and categories

Now that the main theorem of the paper (Thm. 3 in §7.5) has been established, it makes sense to look backwards to our definition of a dispute chirality. We describe in §8.1 the one-sided notion of *dispute category* corresponding to the two-sided and symmetric notion of dispute chirality introduced in §5. We then formulate in §8.2 and §8.3 what we call the companion of a dialogue chirality and of a dispute chirality.

8.1 Dispute categories

A dispute category is defined as a pair $(\mathscr{C}, \bot_{\mathscr{C}})$ consisting of a monoidal category $\mathscr{C}$ together with a presheaf functor

$$\bot_{\mathscr{C}} \quad : \quad \mathscr{C}^{op} \quad \longrightarrow \quad \boldsymbol{Set}.$$

The 2-category ***DisCat*** is then defined as the 2-category with dispute categories as objects, dispute functors as 1-cells, and dispute natural transformations as 2-cells.

The 1-dimensional cells. A dispute functor

$$(F, \bot_F) \quad : \quad (\mathscr{C}, \bot_{\mathscr{C}}) \quad \longrightarrow \quad (\mathscr{D}, \bot_{\mathscr{D}})$$

between dispute categories is defined as a lax monoidal functor

$$F \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{D}$$

equipped with a natural transformation

$$\bot_F \quad : \quad \bot_{\mathscr{C}} \quad \Longrightarrow \quad \bot_{\mathscr{D}} \circ F^{op}.$$

The 2-dimensional cells. A dispute natural transformation

$$\theta \quad : \quad (F, \perp_F) \quad \Rightarrow \quad (G, \perp_G)$$

is defined as a monoidal natural transformation

$$\theta \quad : \quad F \quad \Rightarrow \quad G$$

satisfying the equality:

$$\begin{tikzcd}
 & \boldsymbol{Set} & \\
 & \arrow[u, phantom, "\overset{\perp_F}{\Rightarrow}"] & \\
\mathscr{C}^{op} \arrow[uur, "\perp_{\mathscr{C}}"] \arrow[rr, bend right, "F^{op}"'] & & \mathscr{D}^{op} \arrow[uul, "\perp_{\mathscr{D}}"']
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & \boldsymbol{Set} & \\
 & \arrow[u, phantom, "\overset{\perp_G}{\Rightarrow}"] & \\
\mathscr{C}^{op} \arrow[uur, "\perp_{\mathscr{D}}"] \arrow[rr, bend left, "G^{op}"] \arrow[rr, bend right, "F^{op}"'] \arrow[rr, phantom, "\Downarrow \theta^{op}"] & & \mathscr{D}^{op} \arrow[uul, "\perp_{\mathscr{D}}"']
\end{tikzcd}$$

Every dispute category $(\mathscr{C}, \perp_{\mathscr{C}})$ induces a dispute chirality $(\mathscr{A}, \mathscr{B})$ defined as the pair of monoidal categories

$$(\mathscr{A}, \owedge, \mathbf{true}) := (\mathscr{C}, \otimes, e) \qquad\qquad (\mathscr{B}, \ovee, \mathbf{false}) := \mathscr{C}^{op(0,1)}$$

with monoidal equivalence ${}^*(-)$ and $(-)^*$ defined as the identity between the monoidal category $\mathscr{A}$ and itself, and with distributor

$$\langle - | - \rangle \quad : \quad \mathscr{A}^{op} \times \mathscr{B} \quad \longrightarrow \quad \boldsymbol{Set}$$

defined as the composite

$$\mathscr{C}^{op} \times \mathscr{C}^{op} \quad \xrightarrow{\otimes^{op}} \quad \mathscr{C}^{op} \quad \xrightarrow{\perp_{\mathscr{C}}} \quad \boldsymbol{Set}.$$

We let the reader check that the resulting 2-functor $\boldsymbol{DisCat} \to \boldsymbol{DisChir}$ induces a biequivalence between the 2-category of dispute categories and the 2-category of dispute chiralities. This biequivalence is moreover consistent with the biequivalence established earlier (Thm. 3 in §7.5) between the 2-category $\boldsymbol{DiaCat}$ of dialogue categories and the 2-category $\boldsymbol{DiaChir}$ of dialogue chiralities. This justifies to consider that the two notions of dispute category and of dispute chirality are equivalent.

8.2 The companion of a dialogue chirality

We have seen in §4.3 that every dialogue category $\mathscr{C}$ defines an adjunction between the left adjoint functor

$$L \quad : \quad x \mapsto (\bot \multimapinv x) \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}^{op(0,1)}$$

and the right adjoint functor

$$R \quad : \quad x \mapsto (x \multimap \bot) \quad : \quad \mathscr{C}^{op(0,1)} \quad \longrightarrow \quad \mathscr{C}.$$

A subtle point about dialogue categories already mentioned in §4.3 is that another choice of adjunction is possible, given by the functor

$$R^{op(1)} \quad : \quad x \mapsto (x \multimap \bot) \quad : \quad \mathscr{C}^{op(0)} \quad \longrightarrow \quad \mathscr{C}^{op(1)}$$

left adjoint to the functor

$$L^{op(1)} \quad : \quad x \mapsto (\bot \multimapinv x) \quad : \quad \mathscr{C}^{op(1)} \quad \longrightarrow \quad \mathscr{C}^{op(0)}$$

A simple way to understand this alternative choice between the pair of adjunctions $L \dashv R$ and $R^{op(1)} \dashv L^{op(1)}$ is to notice that every dialogue category $\mathscr{C}$ comes together with another dialogue category $\mathcal{H}(\mathscr{C}) = \mathscr{C}^{op(0)}$ which we find convenient to call its *companion* dialogue category. As a matter of fact, it is not difficult to see that the operation $\mathcal{H}$ defines a 2-functor

$$\mathcal{H} = (-)^{op(0)} \quad : \quad \boldsymbol{DiaCat} \quad \longrightarrow \quad \boldsymbol{DiaCat}.$$

This together with the coherence theorem (§7.5, Thm. 3) ensures that there exists a corresponding bifunctor (it is in fact a 2-functor)

$$\mathcal{H}' \quad : \quad \boldsymbol{DiaChir} \quad \longrightarrow \quad \boldsymbol{DiaChir}$$

which transports every dialogue chirality $(\mathscr{A}, \mathscr{B})$ to what we call its *companion* dialogue chirality. We find instructive to describe this 2-functor here because it sheds light on the secret nature of dialogue categories and chiralities. The 2-functor is defined as follows:

$$\mathcal{H} \quad : \quad (\mathscr{A}, \mathscr{B}) \quad \mapsto \quad (\mathscr{A}^{op(0)}, \mathscr{B}^{op(0)})$$

where the dialogue chirality $(\mathscr{A}^{op(0)}, \mathscr{B}^{op(0)})$ has its adjunction $\overline{L} \dashv \overline{R}$ defined as the composite of the three adjunctions below:

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left, "(-)^*"] \arrow[r, phantom, "\perp"] & \mathscr{B}^{op} \arrow[l, bend left, "{}^*(-)"] \arrow[r, bend left, "R^{op}"] \arrow[r, phantom, "\perp"] & \mathscr{A}^{op} \arrow[l, bend left, "L^{op}"] \arrow[r, bend left, "(-)^*"] \arrow[r, phantom, "\perp"] & \mathscr{B}. \arrow[l, bend left, "{}^*(-)"]
\end{tikzcd}$$

The right adjoint functor is thus defined as $\overline{R} = {}^*(-) \circ L^{op} \circ {}^*(-)$. The natural bijection

$$\overline{\chi}_{m,a,b} \quad : \quad \mathscr{A}(m \otimes^{op} a, \overline{R}b) \quad \longrightarrow \quad \mathscr{A}(a, \overline{R}(m^* \oslash^{op} b))$$

which implements currification in the companion dialogue chirality $\mathcal{H}'(\mathscr{A}, \mathscr{B})$ is defined by the series of bijections

$$\begin{array}{rclc}
\mathscr{A}(m \otimes^{op} a, {}^*(L({}^*b))) & \cong & \mathscr{A}(a \otimes m, {}^*(L({}^*b))) & \text{by definition of } \otimes^{op}, \\
& \cong & \mathscr{B}(L({}^*b), (a \otimes m)^*) & \text{by equivalence } (-)^* \dashv {}^*(-), \\
& \cong & \mathscr{B}(L({}^*b), m^* \oslash a^*) & \text{by monoidality,} \\
& \cong & \mathscr{A}({}^*b, R(m^* \oslash a^*)) & \text{by adjunction } L \dashv R, \\
& \cong & \mathscr{A}(m \otimes {}^*b, R(a^*)) & \text{by currification } \chi_m, \\
& \cong & \mathscr{A}({}^*(m^*) \otimes {}^*b, R(a^*)) & \text{by equivalence } (-)^* \dashv {}^*(-), \\
& \cong & \mathscr{A}({}^*(b \oslash m^*), R(a^*)) & \text{monoidality of the equivalence,} \\
& \cong & \mathscr{A}({}^*(m^* \oslash^{op} b), R(a^*)) & \text{by definition of } \oslash^{op}, \\
& \cong & \mathscr{B}(L({}^*(m^* \oslash^{op} b)), a^*) & \text{by adjunction } L \dashv R, \\
& \cong & \mathscr{A}(a, {}^*(L({}^*(m^* \oslash^{op} b)))) & \text{by equivalence } (-)^* \dashv {}^*(-),
\end{array}$$

natural in the objects m, a in the category $\mathscr{A}$ and in the object b in the category $\mathscr{B}$. Note that the natural bijection $\overline{\chi}$ may be also written:

$$\overline{\chi}_{m,a,b} \quad : \quad \mathscr{A}(a \otimes m, \overline{R}b) \quad \longrightarrow \quad \mathscr{A}(a, \overline{R}(b \oslash m^*))$$

For that reason, the companion dialogue chirality may be seen as the original dialogue chirality $(\mathscr{A}, \mathscr{B})$ where currification on the left has been replaced by currification on the right, at the price of changing R into $\overline{R}$. It is also interesting to notice that the canonical isomorphism which lives in any dialogue category $\mathscr{C}$

$$e \multimap \bot \quad \cong \quad \bot \multimapinv e$$

is reflected in every dialogue chirality $(\mathscr{A}, \mathscr{B})$ by a canonical isomorphism

$$R(\textbf{false}) \quad \cong \quad \overline{R}(\textbf{false}). \tag{28}$$

This isomorphism may be derived from the Yoneda lemma, together with the series of bijections natural in a:

$$\begin{array}{rclc}
\mathscr{A}(a, R(\textbf{false})) & \cong & \mathscr{A}(a \otimes \textbf{true}, R(\textbf{false})) & \text{by the unit law,} \\
& \cong & \mathscr{A}(\textbf{true}, R(a^* \oslash \textbf{false})) & \text{by currification } \chi_a, \\
& \cong & \mathscr{A}(\textbf{true}, R(a^*)) & \text{by the unit law,} \\
& \cong & \mathscr{B}(L(\textbf{true}), a^*) & \text{by the adjunction } L \dashv R, \\
& \cong & \mathscr{A}(a, {}^*L(\textbf{true})) & \text{by the equivalence } (-)^* \dashv {}^*(-), \\
& \cong & \mathscr{A}(a, \overline{R}(\textbf{false})) & \text{by monoidality of equivalence.}
\end{array}$$

8.3 The companion of a dispute chirality

The previous construction on dialogue categories and chiralities generalizes to dispute categories and chiralities. The reason is that there exists a 2-functor

$$\mathcal{H} \quad : \quad \boldsymbol{DisCat} \quad \longrightarrow \quad \boldsymbol{DisCat}$$

which transports a dispute category $(\mathscr{C}, \perp_{\mathscr{C}})$ to the dispute category $(\mathscr{C}^{\,op(0)}, \perp_{\mathscr{C}})$ where the orientation of tensors has been reversed. As in the case of dialogue categories, the coherence theorem induces the existence of a bifunctor (in fact a 2-functor)

$$\mathcal{H}' \quad : \quad \boldsymbol{DisChir} \quad \longrightarrow \quad \boldsymbol{DisChir}$$

which transports every dispute chirality $(\mathscr{A}, \mathscr{B})$ to a companion dispute chirality $(\mathscr{A}^{\,op(0)}, \mathscr{B}^{\,op(0)})$ equipped with the monoidal equivalence

$$\begin{tikzcd}
\mathscr{A}^{\,op(0)} \arrow[rr, bend left, "(-)^{\circledast}"] \arrow[rr, phantom, "\substack{\textit{monoidal}\\ \textit{equivalence}}"] & & \mathscr{B}^{\,op(1)} \arrow[ll, bend left, "{}^{\circledast}(-)"]
\end{tikzcd}$$

obtained by taking the opposite of degree 0 of the original monoidal equivalence ${}^{*}(-) \dashv (-)^{*}$ between the monoidal categories $\mathscr{A}$ and $\mathscr{B}^{\,op(0,1)}$. As expected, the dispute chirality is equipped with the distributor

$$\langle\!\langle - \,|\, - \rangle\!\rangle \quad : \quad \mathscr{A}^{op} \times \mathscr{B} \quad \longrightarrow \quad \boldsymbol{Set}$$

defined as the composite

$$\mathscr{A}^{\,op} \times \mathscr{B} \xrightarrow{(-)^{*} \times {}^{*}(-)} \mathscr{B} \times \mathscr{A}^{\,op} \xrightarrow{\quad permute \quad} \mathscr{A}^{\,op} \times \mathscr{B} \xrightarrow{\quad \langle - | - \rangle \quad} \boldsymbol{Set}$$

together with the natural bijection

$$\overline{\chi}_{m,a,b} \quad : \quad \langle\!\langle\, m \owedge^{op} a \,|\, b \,\rangle\!\rangle \quad \longrightarrow \quad \langle\!\langle\, a \,|\, m^{*} \ovee^{op} b \,\rangle\!\rangle$$

defined as the unique function making the diagram below commute:

$$\begin{tikzcd}
\langle\!\langle\, m \owedge^{op} a \,|\, b \,\rangle\!\rangle \arrow[r, "\overline{\chi}_{m,a,b}"] \arrow[d, dash, "definition"'] & \langle\!\langle\, a \,|\, m^{*} \ovee^{op} b \,\rangle\!\rangle \arrow[d, dash, "definition"] \\
\langle\!\langle\, a \owedge m \,|\, b \,\rangle\!\rangle \arrow[d, dash, "definition"'] & \langle\!\langle\, a \,|\, b \ovee m^{*} \,\rangle\!\rangle \arrow[d, dash, "definition"] \\
\langle\, {}^{*}b \,|\, (a \owedge m)^{*} \,\rangle \arrow[d, dash, "monoidality"'] & \langle\, {}^{*}(b \ovee m^{*}) \,|\, a^{*} \,\rangle \arrow[d, dash, "monoidality"] \\
\langle\, {}^{*}b \,|\, m^{*} \ovee a^{*} \,\rangle & \langle\, m \owedge {}^{*}b \,|\, a^{*} \,\rangle \arrow[l, "\chi_{m,{}^{*}b,a^{*}}"]
\end{tikzcd}$$

One easily checks that $\overline{\chi}_m$ satisfies the two coherence axioms required of a dispute chirality. In the same way as in the previous §8.2, the isomorphism (28) is reflected by a family of isomorphisms

$$\langle\, a \,|\, \mathbf{false} \,\rangle \quad \cong \quad \langle\!\langle\, a \,|\, \mathbf{false} \,\rangle\!\rangle$$

natural in a. The family of isomorphisms is defined as the composite:

$$\begin{array}{rclc}
\langle\!\langle\, a \,|\, \mathbf{false} \,\rangle\!\rangle & = & \langle\, {}^*\mathbf{false} \,|\, a^* \,\rangle & \text{by definition of } \langle\!\langle\, a \,|\, b \,\rangle\!\rangle \\
& \cong & \langle\, \mathbf{true} \,|\, a^* \,\rangle & \text{by monoidality of } {}^*(-), \\
& \cong & \langle\, \mathbf{true} \,|\, a^* \oslash \mathbf{false} \,\rangle & \text{by associativity,} \\
& \cong & \langle\, a \oslash \mathbf{true} \,|\, \mathbf{false} \,\rangle & \text{by applying the isomorphism } \chi_a, \\
& \cong & \langle\, a \,|\, \mathbf{false} \,\rangle & \text{by associativity.}
\end{array}$$

Moreover, in the particular case of a dispute chirality induced from a dialogue chirality $(\mathscr{A}, \mathscr{B})$, the series of natural bijections

$$\begin{array}{rcll}
\langle\!\langle\, a \,|\, b \,\rangle\!\rangle & = & \langle\, {}^*b \,|\, a^* \,\rangle & \text{by definition of } \langle\!\langle\, - \,|\, - \,\rangle\!\rangle \\
& \cong & \mathscr{A}({}^*b, R(a^*)) & \text{by definition of a special distributor} \\
& \cong & \mathscr{B}(L({}^*b), a^*) & \text{by adjunction } L \dashv R \\
& \cong & \mathscr{A}(a, {}^*(L({}^*b))) & \text{by adjunction } {}^*(-) \dashv (-)^*
\end{array}$$

establishes that the notion of companionship for dispute chiralities coincides with the notion of companionship for dialogue chiralities in §8.2.

9 Final remarks

We conclude with a series of remarks and variations on the notions of dialogue category and of dialogue chirality studied in the article.

9.1 Dialogue categories

The notion of dialogue category considered in this article was chosen for its simplicity. It is designed to provide an elementary and tractable notion of a category with a tensorial negation. In particular, it is likely that a more general and satisfactory notion of dialogue category should be equipped with a functor

$$\begin{array}{ccc}
\mathscr{C}^{op} \times \mathscr{C}^{op} & \longrightarrow & \mathscr{C} \\
(x, y) & \mapsto & x \multimap \perp \multimapinv y
\end{array}$$

and a family of bijections

$$\mathscr{C}(x \otimes y \otimes z, \perp) \quad \cong \quad \mathscr{C}(y, x \multimap \perp \multimapinv z)$$

natural in x, y and z. We establish in a companion paper that this is the case when the underlying monoidal category $\mathscr{C}$ is balanced or symmetric, and more generally when the tensorial pole $\perp$ is pivotal, see [35] for details. This is also the case when the monoidal category $\mathscr{C}$ is biclosed, that is, when for every object x of the category $\mathscr{C}$, each of the two endofunctors

$$y \;\mapsto\; x \otimes y \qquad\qquad y \;\mapsto\; y \otimes x$$

has a right adjoint noted $y \mapsto x \multimap y$ and $y \mapsto y \multimapinv x$ respectively. In that case, there exists a canonical isomorphism

$$(x \multimap \perp) \multimapinv y \quad \cong \quad x \multimap (\perp \multimapinv y)$$

natural in x and y, and the object $x \multimap \perp \multimapinv y$ may be thus defined as one of these two objects in the category $\mathscr{C}$.

9.2 Mixed chiralities

Although this aspect is not really explored in the article, it is possible to relax the requirement that the category $\mathscr{B}$ is equivalent to the category $\mathscr{A}^{op}$ in the two-sided description of any category $\mathscr{C}$ with duality. This relaxation is often useful, and enables one to consider situations where the category $\mathscr{B}$ of counter-proofs is not equivalent to the opposite of the category $\mathscr{A}$ of proofs. In that case, we speak of a *mixed chirality* $(\mathscr{A}, \mathscr{B})$ in order to differentiate that kind of chiralities from the notion of *pure chirality* introduced in the present article. A typical illustration of such a mixed chirality is provided by the notion of *exponential ideal* $\mathscr{B}$ in a monoidal category $\mathscr{A}$. By definition, a *closed* monoidal chirality is a monoidal chirality $(\mathscr{A}, \mathscr{B})$ in the sense of §3 equipped moreover with a pair of pseudo-actions

$$\begin{array}{ccccc} \ovee & : & \mathscr{A} \times \mathscr{B} & \longrightarrow & \mathscr{A} \\ \owedge & : & \mathscr{A} \times \mathscr{B} & \longrightarrow & \mathscr{B} \end{array} \tag{29}$$

related by an isomorphism of pseudo-action

$$^{*}(\, a \owedge b \,) \quad \cong \quad {}^{*}b \ovee a^{*}$$

together with a bijection

$$\mathscr{A}(a_1 \owedge a_2, {}^{*}b) \quad \cong \quad \mathscr{A}(a_1, {}^{*}(a_2 \owedge b)) \tag{30}$$

natural in a_1, a_2 and b and making the diagrams below

$$\begin{tikzcd}
\mathscr{A}((a_1 \oslash a_2) \oslash a_3, {}^*b) \arrow[r] \arrow[dd] & \mathscr{A}(a_1 \oslash a_2, {}^*(a_3 \oslash b)) \arrow[d] \\
 & \mathscr{A}(a_1, {}^*(a_2 \oslash (a_3 \oslash b))) \arrow[d] \\
\mathscr{A}(a_1 \oslash (a_2 \oslash a_3), {}^*b) \arrow[r] & \mathscr{A}(a_1, {}^*((a_2 \oslash a_3) \oslash b))
\end{tikzcd} \tag{31}$$

$$\begin{tikzcd}
\mathscr{A}(a \oslash \mathbf{true}, {}^*b) \arrow[rr] \arrow[dr] & & \mathscr{A}(a, {}^*(\mathbf{true} \oslash b)) \arrow[dl] \\
 & \mathscr{A}(a, {}^*b) &
\end{tikzcd} \tag{32}$$

commute for all objects a, a_1, a_2, a_3 of the category $\mathscr{A}$ and all objects b of the category $\mathscr{B}$. This definition of monoidal closed chirality characterizes the chiralities $(\mathscr{A}, \mathscr{B})$ where the category $\mathscr{A}$ is monoidal closed. Now, the familiar definition of a monoidal category $\mathscr{A}$ equipped with an exponential ideal $\mathscr{B}$ is recovered by relaxing the definition of monoidal closed chirality just given, in the following way:

- one removes the monoidal structure $(\mathscr{B}, \ovee, \mathbf{false})$ of the category $\mathscr{B}$,
- one removes the pseudo-action $\obslash$ and the functor $(-)^* : \mathscr{A} \to \mathscr{B}^{op(0,1)}$,
- one keeps the pseudo-action $\oslash$ and the functor ${}^*(-) : \mathscr{B}^{op} \to \mathscr{A}$.

One shifts in that way from a pure chirality where $\mathscr{B}^{op(0,1)}$ is equivalent to $\mathscr{A}$ to a mixed chirality $(\mathscr{A}, \mathscr{B})$ corresponding to the familiar notion of exponential ideal:

Definition 9 (exponential ideal) *An exponential ideal in a monoidal category $\mathscr{A}$ is a category $\mathscr{B}$ equipped with a functor ${}^*(-) : \mathscr{B}^{op} \to \mathscr{A}$ together with a pseudo-action (29) and a natural bijection (30) making the coherence diagrams (31) and (32) commute.*

The notion of exponential ideal is important, in particular for its role in the categorical semantics of linear logic. Recall that a model of intuitionistic linear logic is defined as a symmetric monoidal adjunction

$$\begin{tikzcd}
\mathscr{M} \arrow[r, bend left, "Lin"] \arrow[r, phantom, "\perp"] & \mathscr{L} \arrow[l, bend left, "Mult"]
\end{tikzcd}$$

where the category $(\mathscr{M}, \times, 1)$ is cartesian and where the category $(\mathscr{L}, \otimes, e)$ is symmetric monoidal closed. In that case, it is easy to show that the functor

$$Mult \quad : \quad \mathscr{L} \quad \longrightarrow \quad \mathscr{M}$$

together with the pseudo-action

$$(P, A) \mapsto Lin(P) \multimap A \quad : \quad \mathscr{M}^{op} \times \mathscr{L} \quad \longrightarrow \quad \mathscr{L}$$

defines an exponential ideal $\mathscr{L}^{op}$ on the cartesian category $\mathscr{M}$ equipped with the natural bijection

$$\mathscr{M}(P \times Q, Mult(A)) \quad \cong \quad \mathscr{M}(Q, Mult(Lin(P) \multimap A)).$$

In the case of tensorial logic, the symmetric monoidal closed category $\mathscr{L}$ is replaced by a dialogue category $\mathscr{D}$. It appears that the category $\mathscr{D}$ induces an exponential ideal for the category $\mathscr{M}$, this time defined by the composite functor

$$Mult \circ \neg \quad : \quad \mathscr{D}^{op} \quad \longrightarrow \quad \mathscr{D} \quad \longrightarrow \quad \mathscr{M}$$

together with the pseudo-action

$$(P, A) \mapsto Lin(P) \otimes A \quad : \quad \mathscr{M} \times \mathscr{D} \quad \longrightarrow \quad \mathscr{D}$$

and the natural bijection:

$$\mathscr{M}(P \times Q, Mult \neg A) \quad \cong \quad \mathscr{M}(Q, Mult \neg (Lin(P) \otimes A)).$$

Besides this connection to linear logic and to tensorial logic, the two-sided formulation of intuitionistic implication $\Rightarrow$ as a pseudo-action $\otimes$ of the cartesian category $\mathscr{A}$ of proofs on the category $\mathscr{B}$ of refutations sheds light on the notion of *stack* which plays a fundamental role in the compilation of programming languages as well as in the description of abstract machines, see [9] for instance. Indeed, suppose that the formula a living in $\mathscr{A}$ is the "negation" of the formula b living in the category $\mathscr{B}$ in the sense that $a = {}^*b$. In that case, the formula

$$F \quad = \quad a_1 \Rightarrow (a_2 \Rightarrow \cdots (a_n \Rightarrow a) \cdots)$$

which lives in the category $\mathscr{A}$ of proofs and programs may be reformulated as the negation $F = {}^*G$ of the formula

$$G \quad = \quad a_1 \otimes (a_2 \otimes \cdots (a_n \otimes b) \cdots)$$

which lives in the category $\mathscr{B}$ of counter-proofs and counter-programs. The fact that the connective $\otimes$ describes a pseudo-action of $\mathscr{A}$ on $\mathscr{B}$ implies that the formula G in the category $\mathscr{B}$ is provided by a stack of objects $a_1, \ldots, a_n$ of the category $\mathscr{A}$ "pushed" on the single object b of the category $\mathscr{B}$. This brings to light an interesting and somewhat unexpected connection between the notion of *stack* in computer science and the familiar notion of *action* in algebra.

Interestingly enough, this specific pattern underlies the description of types in Krivine's classical realizability [26]. There, every formula A is interpreted as a set $\|A\|$ of stacks of "attackers", where a stack is defined as a sequence of λ-terms $t_1, \ldots, t_n$ "pushed" on top of a stack constant π in the following way

$$t_1 \centerdot (t_2 \centerdot \ldots (t_n \centerdot \pi) \ldots)$$

The main ingredient of classical realizability is provided by an orthogonality relation $\perp$ between λ-terms and stacks which describes when a λ-term t may be safely confronted with a stack π. The orthogonality relation is required to satisfy an appropriate closure condition with respect to reverse evaluation, see [26] for details. The interpretation $|A|$ of the formula A is then defined as the set of λ-terms orthogonal to its set of attackers:

$$|A| \quad = \quad \{ \quad t \quad | \quad \forall \pi \in \|A\|, \quad t \perp \pi \quad \}.$$

Although Krivine's classical realizability is not expressed at this stage in the language of category theory, orthogonality plays in his work the same role as negation in our setting. In particular, the type $A \Rightarrow B$ is interpreted as the orthogonal $|A \Rightarrow B|$ of the set of stacks $\|A \Rightarrow B\| := |A| \otimes \|B\|$ obtained by pushing a λ-term $t \in |A|$ realizing the formula A on top of a stack $\pi \in \|B\|$ attacking the formula B.

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References

[1] Jean-Marc Andreoli. *Logic programming with focusing proofs in linear logic.* J. Logic Comput. 2, no. 3, 297–347, 1992.

[2] John Baez and James Dolan. *Higher-Dimensional Algebra III: n-Categories and the Algebra of Opetopes.* Adv. Math. 135 (1998), 145-206.

[3] M. Barr. *Nonsymmetric $*$-autonomous categories*. Theoretical Computer Science 139, 115–130.

[4] Josh Berdine, Peter O'Hearn, Udday Reddy and Hayo Thielecke. *Linear continuation-passing.* Higher-Order and Symbolic Computation 15(2/3), 181?203. (2002)

[5] Josh Berdine. *Linear and Affine Typing of Continuation-Passing Style.* PhD thesis, Queen Mary, University of London, London, United Kingdom, 2004.

[6] Clemens Berger, Ieke Moerdijk. *The Boardman-Vogt resolution of operads in monoidal model categories*, Topology 45 (2006), 807-849

[7] Robin Cockett, Robert Seely. *Polarized category theory, modules, and game semantics*, Theory and Applications of Categories 18 (2) (2007) 4–101.

[8] Pierre-Louis Curien, Hugo Herbelin. *Duality of Computations.* Proc. International Conference on Functional Programming, September 2000, Montréal, IEEE (2000).

[9] Pierre-Louis Curien, Hugo Herbelin. *Abstract machines for dialogue games.* Published in *Interactive models of computation and program behaviour*. Pierre-Louis Curien, Hugo Herbelin, Jean-Louis Krivine, Paul-André Melliès. Panoramas et Synthèses 27, Société Mathématique de France, 2009.

[10] Vincent Danos, Jean-Baptiste Joinet and Harold Schellinx. *LKQ and LKT: sequent calculi for second order logic based upon dual linear decompositions of classical implication.* In Advances in Linear Logic, 211-224, Cambridge University Press (1995).

[11] Alberto Carraro, Thomas Ehrhard, Antonino Salibra, *The stack calculus.* Proc. 7th Workshop on Logical and Semantic Frameworks, with Application, EPTCS vol. 113, pp. 93-108 (2012).

[12] Andrzej Filinski. *Declarative continuations, an investigation of duality in programming language semantics.* Proceedings of Category Theory and Computer Science, LNCS 389, Springer Verlag 1989.

[13] Carsten Führmann. *Direct Models for the Computational Lambda Calculus*. Electronic Notes in Theoretical Computer Science 20, 245?292 (1999)

[14] Jean-Yves Girard. *A new constructive logic: classical logic.* Mathematical Structures in Computer Science 1 (3) (1991) 255–296.

[15] Jean-Yves Girard. *Locus Solum.* Mathematical Structures in Computer Science 11, pp. 301-506, 2001.

[16] Timothy Griffin. *A formulae-as-type notion of control*. Proceedings of Principles of Programming Languages 17: 47?58, 1990.

[17] Nick Gurski. *Biequivalences in tricategories.* arXiv:1102.0979. To appear in Theory and Applications of Categories.

[18] Masahito Hasegawa, *Classical linear logic of implications.* In Proc. 11th Annual Conference of the European Association for Computer Science Logic (CSL'02), Springer Lecture Notes in Comput. Sci. 2471, pp. 458–472.

[19] Masahito Hasegawa. *Semantics of linear continuation-passing in call-by-name.* In Proc. 7th International Symposium on Functional and Logic Programming (FLOPS2004), Nara.

[20] Martin Hofmann and Thomas Streicher. *Completeness of continuation models for lambda-mu-calculus.* Information and Computation, Volume 179, Issue 2, 15 December 2002, Pages 332-355.

[21] André Joyal and Ross Street. *An introduction to Tannaka duality and quantum groups.* Proceedings of Category Theory, Como 1990, Lecture Notes in Math. 1488, Springer Verlag 1991.

[22] Christian Kassel. *Quantum groups*. Graduate Textes in Mathematics 155. Springer Verlag, 1995.

[23] Max Kelly, Ross Street. *Review of the elements of 2-categories*, Category Seminar, LNM 420, 1974.

[24] Anders Kock. *On double dualization monads.* Mathematica Scandinavica 27, pp 151–165, 1970.

[25] Jean-Louis Krivine. *Opérateurs de mise en mémoire et traduction de Gödel.* Archive for Mathematical Logic, 1990, Volume 30, Issue 4, pp 241-267.

[26] Jean-Louis Krivine. *Realizability in classical logic.* Published in *Interactive models of computation and program behaviour*. Pierre-Louis Curien, Hugo Herbelin, Jean-Louis Krivine, Paul-André Melliès. Panoramas et Synthèses 27, Société Mathématique de France, 2009.

[27] Yves Lafont. *Negation versus implication.* Manuscript, 1991.

[28] Yves Lafont, Bernhard Reus and Thomas Streicher. *Continuations Semantics or Expressing Implication by Negation.* Technical Report 9321, Ludwig-Maximilians-Universitat, Munchen. Technical Report, 1993.

[29] Olivier Laurent. *Syntax vs. semantics: a polarized approach.* Theoretical Computer Science, 343(1-2):177-206. October 2005.

[30] Paul-André Melliès. *Categorical semantics of linear logic.* Published in *Interactive models of computation and program behaviour*. Pierre-Louis Curien, Hugo Herbelin, Jean-Louis Krivine, Paul-André Melliès. Panoramas et Synthèses 27, Société Mathématique de France, 2009.

[31] Paul-André Melliès. *Asynchronous games 3: An innocent model of linear logic.* Proceedings of the 10th Conference on Category Theory and Computer Science, Copenhagen, 2004.

[32] Paul-André Melliès and Nicolas Tabareau. *Resource modalities in tensor logic*. Annals of Pure and Applied Logic Volume 161, Issue 5, February 2010, Pages 632-653.

[33] Paul-André Melliès. *Game semantics in string diagrams.* Proceedings of *Logic In Computer Science*, LICS 2012.

[34] Paul-André Melliès. *Dialogue categories and Frobenius monoids.* Computation, Logic, Games, and Quantum Foundations: Essays Dedicated to Samson Abramsky. LNCS 7860, Springer Verlag, 2013.

[35] Paul-André Melliès. *Braided notions of dialogue categories.* Submitted. Manuscript available on the author's webpage.

[36] Guillaume Munch-Maccagnoni. *Models of a Non-Associative Composition.* Proceedings of the *International Conference on Foundations of Software Science and Computation Structures*, FoSSaCS 2014.

[37] Guillaume Munch-Maccagnoni. *Formulae-as-Types for an Involutive Negation.* Proceedings of CSL-LICS 2014.

[38] Chet Murthy. *A computational analysis of Girard's translation and LC.* Proceedings of *Logic in Computer Science*, LICS 1992.

[39] Michel Parigot. *Lambda-mu-calculus: An algorithmic interpretation of classical natural deduction.* Lecture Notes in Computer Science, Volume 624, pages 190-201, 1992.

[40] Peter Selinger. *Control categories and duality: on the categorical semantics of the lambda-mu calculus.* Mathematical Structures in Computer Science 11(2):207-260, 2001.

[41] Thomas Streicher and Bernard Reus. *Classical logic, continuation semantics and abstract machines.* Journal of Functional Programming, 8(6):543-572, 1998.

[42] Hayo Thielecke. *Continuation semantics and self-adjointness*. Proceedings of MFPS XIII, Electronic Notes in Theoretical Computer Science 6 (1997) 348–364.

[43] Vladimir Turaev. *Quantum Invariants of Knots and 3-Manifolds.* de Gruyter, 1994

[44] Philip Wadler. *Call-by-value is dual to call-by-name.* Proceedings of International Conference on Functional Programming, August 2003.

[45] Philip Wadler. *Call-by-value is dual to call-by-name, reloaded.* Proceedings of Rewriting Techniques and Applications (RTA), Nara, April 2005.

[46] Ittay Weiss. *Dendroidal sets*. PhD thesis available on the homepage of the author, September 2007.

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