A micrological study of helix negation

Paul-André Melliès[*]

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Abstract

This paper is the second part of the micrological study of negation initiated in our companion paper. We have encountered there a discrepancy between left and right dialogue chiralities which we resolve here by introducing the notion of ambidextrous dialogue chirality. One main purpose of the paper is to disclose the topological nature of this logical notion. More specifically, we establish that an ambidextrous chirality is the same thing as a left chirality equipped with an helical structure on its tensorial negation. This topological insight enables us to conclude the project initiated in our companion paper, and to present ambidextrous chiralities in a purely combinatorial way.

1 Introduction

Among the several equivalent definitions of *dialogue chirality* formulated in the companion paper [6], one finds a definition based on the following combinators:

$$
\begin{array}{rclcl}
\mathbf{axiom}[m] & : & L(a) & \longrightarrow & m^* \oslash L(m \owedge a) \\
\mathbf{cut}[m] & : & m \owedge R(m^* \oslash b) & \longrightarrow & R(b)
\end{array}
$$

In this formulation inspired by proof theory, one requires that each pair of combinators defines what we call a *transjunction accross* the adjunction $L \dashv R$ for every object m of the category $\mathscr{A}$. In addition, one requires that the family

[*]CNRS, Laboratoire PPS, UMR 7126, Université Paris Diderot, Sorbonne Paris Cité, F-75205 Paris, France.

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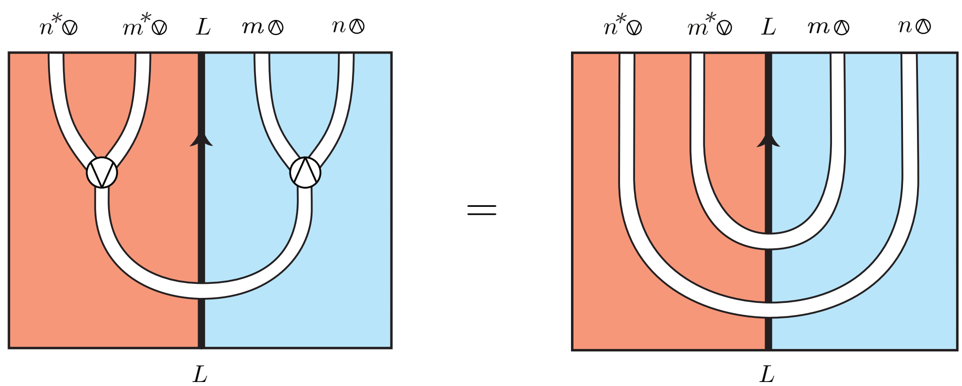
axiom$[-]$ is monoidal in the sense that the coherence diagram

$$\begin{array}{ccc}
n^* \ovee L(n \owedge a) & \xrightarrow{\mathbf{axiom}[m]} & n^* \ovee (m^* \ovee L(m \owedge (n \owedge a))) \\
\uparrow {\scriptstyle \mathbf{axiom}[n]} & & \downarrow {\scriptstyle \textit{associativity}} \\
 & & (n^* \ovee m^*) \ovee L((m \owedge n) \owedge a) \\
 & & \downarrow {\scriptstyle \textit{monoidality}} \\
La & \xrightarrow{\mathbf{axiom}[m \owedge n]} & (m \owedge n)^* \ovee L((m \owedge n) \owedge a)
\end{array} \qquad (1)$$

commutes for all objects a, m, n of the category $\mathscr{A}$. As explained in [6], one should think of this coherence diagram as an η-expansion law for the axiom link of a notion of proof-net for tensorial logic. The connection becomes even clearer when one depicts (1) in the graphical language of string diagrams:



(2)

As should be clear from the picture, the purpose of the η-expansion law is to decompose the axiom$[m \owedge n]$ link into the pair of more elementary axiom$[m]$ and axiom$[n]$ links. A natural question at this point is whether there exists a similar η-expansion law for the axiom link

$$La \xrightarrow{\mathbf{left.axiom}[Rm]} (Rm)^* \ovee L((Rm) \owedge a) \qquad (3)$$

associated to the negation Rm of an object m of the category $\mathscr{B}$. The idea would be to deduce it from the axiom link of the object m itself. Recall from [5] that a dialogue chirality consists (among other data) of two monoidal categories $(\mathscr{A}, \owedge, \mathrm{true})$ and $(\mathscr{B}, \ovee, \mathrm{false})$ related by a monoidal equivalence

$$\mathscr{A} \underset{{}^*(-)}{\overset{(-)^*}{\rightleftarrows}} \mathscr{B}^{op(0,1)} \quad (\textit{monoidal equivalence})$$

Note that we are indicated in (3) that we consider the *left* axiom link. The reason is that, in order to define an η-expansion for Rm, one needs to be careful about orientations. It appears indeed that the good decision is to start from the *right* axiom link

$$L(\mathbf{true}) \xrightarrow{\quad \mathbf{right.axiom}[m] \quad} L({}^{*}m) \ovee m$$

associated to the object m in the category $\mathscr{B}$. This morphism induces in turn the morphism

$$La \xrightarrow{\ \eta\ } L(RL(\mathbf{true}) \owedge a) \xrightarrow{\quad map \quad} L(R(L({}^{*}m) \ovee m) \owedge a)$$

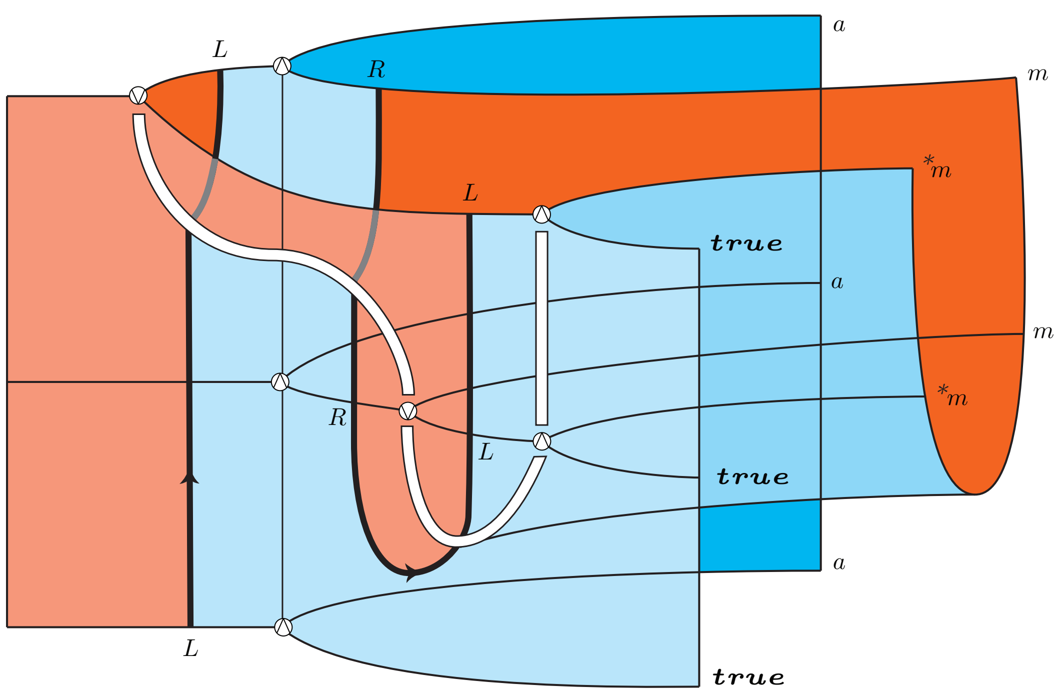
which may be composed with the distributivity law

$$L(R(L({}^{*}m) \ovee m) \owedge a) \xrightarrow{\quad distributivity \quad} L({}^{*}m) \ovee L((Rm) \owedge a)$$

The resulting morphism

$$La \xrightarrow{\qquad\qquad} L({}^{*}m) \ovee L((Rm) \owedge a) \tag{4}$$

may be depicted in the following way:



The different shades of blue and red are here to separate the different instances of categories $\mathscr{A}$ and $\mathscr{B}$ appearing in the 2-categorical diagram. Note that for graphical convenience, we depict a variant

$$L(\mathbf{true} \owedge a) \longrightarrow L(R(L(\mathbf{true} \owedge {}^*m) \ovee m) \owedge a) \longrightarrow L(\mathbf{true} \owedge {}^*m) \ovee L((Rm) \owedge a)$$

of the morphism (4). The last step in order to complete the construction is to require the existence of a family of isomorphisms

$$L({}^*m) \xrightarrow{\quad isomorphism \quad} (Rm)^* \tag{5}$$

natural in m. In this way, by composing (4) with (5), one obtains a morphism

$$La \xrightarrow{\qquad\qquad} (Rm)^* \ovee L((Rm) \owedge a) \tag{6}$$

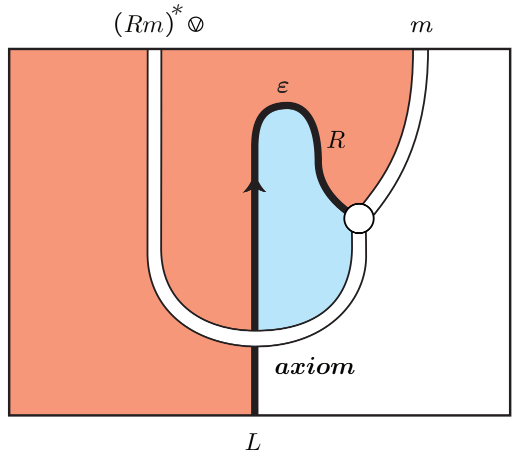
with the expected source and target. There remains to understand what coercion map (5) to pick in order to ensure that the two morphisms (3) and (6) coincide. We will see at the end of the paper that the equality (3) = (6) holds when the coercion isomorphism (5) makes the following diagram commute:

$$\begin{array}{ccc} L(\mathbf{true}) & \xrightarrow{\mathbf{right.axiom}[m]} & L({}^*m) \ovee m \\ {\scriptstyle \mathbf{left.axiom}[Rm]}\downarrow & & \downarrow{\scriptstyle coercion} \\ (Rm)^* \ovee LRm & \xrightarrow{\quad \varepsilon \quad} & (Rm)^* \ovee m \end{array} \tag{7}$$

This observation is important because it reveals the topological nature of the problem. The point is that the commutative diagram (7) characterizes the coercion isomorphism (5) which resolves our problem as the right decurrification of the composite morphism

$$L(\mathbf{true}) \xrightarrow{\mathbf{left.axiom}[Rm]} (Rm)^* \ovee LRm \xrightarrow{\ \varepsilon\ } (Rm)^* \ovee m \tag{8}$$

It turns out that this morphism secretly implements a *topoligical helix* in pure logic. This unexpected phenomenon becomes apparent when one depicts the morphism in string diagrams as:



(9)

Here, the white area represents the trivial category $\mathbf{1}$ with a unique object and one morphism. The object m of the category $\mathscr{B}$ is itself understood as a functor $\mathbf{1} \to \mathscr{B}$. This reveals that negation may be *twisted* during a logical dispute in exactly the same way as a topological ribbon embedded in a 3-dimensional manifold. As a matter of fact, we will consider a notion of helical or ambidextrous dialogue chirality where the group $\mathbb{Z}$ acts on the negation strand by twisting it with an angle $2n\pi$ for $n \in \mathbb{Z}$.

One primary purpose of the present work is to shed light on these topological aspects of negation by transferring ideas from knot theory and representation theory of quantum groups. Recall that the notion of *ribbon category* axiomatizes the properties of the category $Mod(H)$ of finite dimensional representations of a quantum group H, defined here as a quasi-triangular Hopf algebra equipped with a twist $\theta \in H$ whose antipode S is moreover invertible. Every object $\perp$ in a ribbon category $\mathscr{C}$ defines a pair of negation functors

$$x \multimap \perp := x^{\vee} \otimes \perp \qquad\qquad \perp \multimapinv x := \perp \otimes x^{\wedge}$$

where $x^{\vee}$ denotes the right dual and $x^{\wedge}$ denotes the left dual of the object x in the ribbon category. It is not difficult to see that these negation functors equip the ribbon category $\mathscr{C}$ with the structure of a dialogue category. The associated dialogue chirality $(\mathscr{A}, \mathscr{B})$ has its two sides defined as

$$(\mathscr{A}, \owedge, \mathbf{true}) = (\mathscr{C}, \otimes, \mathbf{1}) \qquad\qquad (\mathscr{B}, \ovee, \mathbf{false}) = (\mathscr{C}, \otimes, \mathbf{1})^{op(0,1)}$$

with monoidal equivalence $(-)^*$ and ${}^*(-)$ defined as the identity on $\mathscr{C}$. We wish to give a little precedence to the left axiom combinator here, and thus define L and R as the negation functors

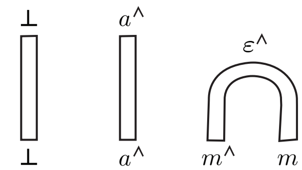
$$Lx = \perp \multimapinv x = \perp \otimes x^{\wedge} \qquad\qquad Rx = x \multimap \perp = x^{\vee} \otimes \perp.$$

The left axiom associated to the object m in the category $\mathscr{A} = \mathscr{C}$

$$La = \bot \otimes a^{\wedge} \xrightarrow{\textbf{left.axiom}[m]} m \varovee L({}^*m \oslash a) \quad \cong \quad \bot \otimes a^{\wedge} \otimes m^{\wedge} \otimes m$$

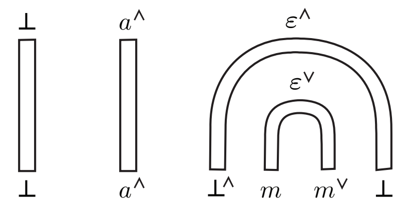
lives in the category $\mathscr{B} = \mathscr{C}^{\,op}$ and is thus defined as the counit of the dual pair $x \dashv x^{\vee}$ and depicted as follows in the category $\mathscr{C}$:

left.axiom$[m]$ $=$



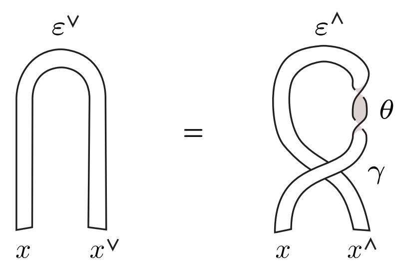
In particular,

left.axiom$[Rm]$ $=$



(10)

where for simplicity we identify the left dual $x^{\wedge}$ and the right dual $x^{\vee}$ of the object x. Note that the identification is always possible at the condition of adding the equality:

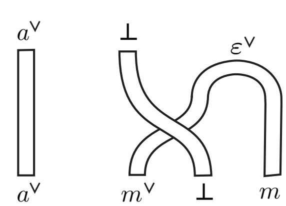


where γ and θ denote the braiding and the twist of the underlying ribbon category $\mathscr{C}$. The right axiom associated an object m in the category $\mathscr{A} = \mathscr{C}$

$$La = \bot \otimes a^{\wedge} \xrightarrow{\textbf{right.axiom}[m]} L(a \obslash m) \varovee m^{*} \quad \cong \quad m \otimes \bot \otimes m^{\wedge} \otimes a^{\wedge}$$

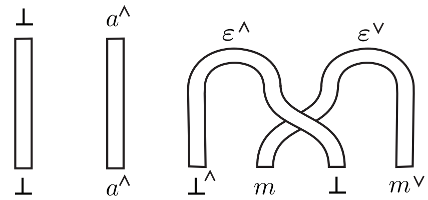
requires a braiding:

right.axiom$[m]$ $=$



The morphism (4) thus looks like:

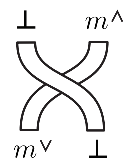
composite morphism (4) $=$



(11)

In that case, the appropriate coercion map (5) is simply given by the braiding

coercion map (5) $=$



(12)

of $m^\vee$ under $\bot$ which indeed transforms the framed ribbon tangle (11) into (10).

Now, we may take a completely different view on the ribbon category $\mathscr{C}$ and give precedence to the right axiom by defining the negation functors as

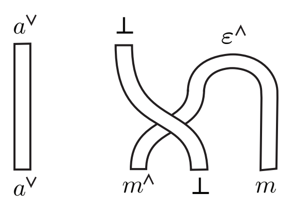
$$Lx = x \multimap \bot = x^\vee \otimes \bot \qquad\qquad Rx = \bot \multimapinv x = \bot \otimes x^\wedge.$$

In this situation, the left axiom requires a braiding:

$$La = a^\vee \otimes \bot \xrightarrow{\textbf{left.axiom}[m]} m^* \ovee L(m \owedge a) \cong a^\vee \otimes m^\vee \otimes \bot \otimes m$$

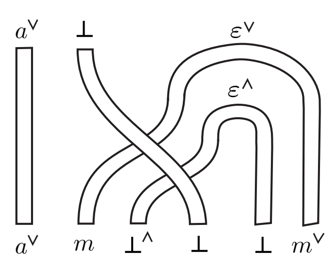
requires a braiding:

left.axiom$[m]$ $=$



In particular,

left.axiom$[Rm]$ $=$

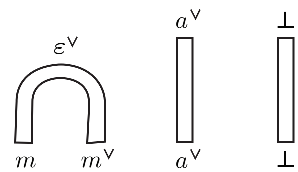


(13)

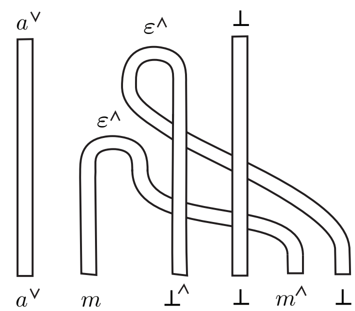
On the other hand, the right axiom associated to the object m in the category $\mathscr{C}$

$$La = a^\vee \otimes \bot \xrightarrow{\textbf{left.axiom}[m]} L(a \owedge {}^*m) \ovee m \quad \cong \quad m \otimes m^\vee \otimes a^\vee \otimes \bot$$

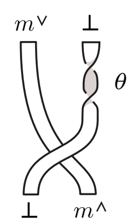
does not need the braiding and is thus defined as

right.axiom$[m]$ $=$ 

In that situation, the morphism (4) looks like:

composite morphism (4) $=$  (14)

and the coercion map (5) thus looks as follows: as in the previous case (12), one needs to permute the strand $m^\vee$ under the strand $\bot$ but also to twist the strand $\bot$ with an angle 2π as follows:

coercion map (5) $=$  (15)

in order to transform the framed ribbon tangle (14) into (13). One may check that in each situation, the coercion map (12) and (15) may be deduced as the right decurrification of the morphism (8), or alternatively defined as the unique coercion morphism making the diagram (7) commute.

Plan of the paper After introducing in §2 the two notions of helical dialogue categories and chiralities, we establish a 2-categorical equivalence between them in §3. Then, we introduce the notion of ambidextrous chirality in §4 and establish that it coincides with the notion of helical chirality. The notion of ambidextrous chirality is presented in the language of transjunctions in §5. This leads to the notion of linearly distributive chirality introduced in §6. We establish in §7 the main theorem which states that an ambidextrous chirality is the same thing as an ambidextrous linearly distributive category with a duality. In the very last section §8 of the paper, we establish the claim about η-expansion formulated in the introduction.

2 Helical dialogue categories and chiralities

The notion of helical dialogue category was introduced in the companion paper [7] among a series alternative notions of dialogue categories motivated by topology. The notion of helical dialogue category is important because it appears (at least in our current understanding) as the most primitive instance of these topological notions of dialogue categories. In particular, every cyclic or ribbon dialogue category is helical in a canonical way. Our main purpose in this section is to introduce and to justify the corresponding notion of *helical dialogue chirality*. In order to validate this 2-sided account of helical dialogue categories, we proceed in exactly the same way as for the notion of dialogue chirality, see [5] for details. We thus construct a 2-category ***HeliCat*** of helical dialogue categories in §2.2 and a 2-category ***HeliChir*** of helical dialogue chiralities in §2.3 and §2.4. Finally, we exhibit a 2-dimensional equivalence between the pair of 2-categories in §3.

2.1 Helical dialogue categories

A dialogue category is defined as a monoidal category $(\mathscr{C}, \otimes, e)$ equipped with an object $\perp$ coming together with a representation

$$\varphi_{x,y} \quad : \quad \mathscr{C}(x \otimes y, \perp) \quad \cong \quad \mathscr{C}(y, x \multimap \perp)$$

of the functor

$$y \mapsto \mathscr{C}(x \otimes y, \perp) \quad : \quad \mathscr{C}^{op} \longrightarrow Set$$

for each object x, and with a representation

$$\psi_{x,y} \quad : \quad \mathscr{C}(x \otimes y, \perp) \quad \cong \quad \mathscr{C}(x, \perp \multimapinv y)$$

of the functor

$$x \mapsto \mathscr{C}(x \otimes y, \perp) \quad : \quad \mathscr{C}^{op} \longrightarrow Set$$

for each object y. The following notion of dialogue category is introduced in [6].

Definition 1 (helical dialogue category) *A helical dialogue category is a dialogue category $\mathscr{C}$ equipped with a family of bijections*

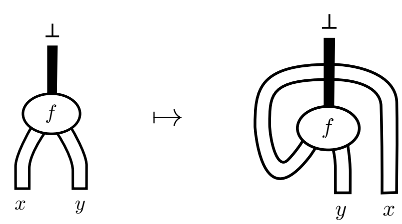
$$wheel_{x,y} \quad : \quad \mathscr{C}(x \otimes y, \bot) \quad \longrightarrow \quad \mathscr{C}(y \otimes x, \bot)$$

natural in x and y and required to make the diagram

$$\begin{array}{ccc}
\mathscr{C}((y \otimes z) \otimes x, \bot) & \xrightarrow{\ associativity\ } & \mathscr{C}(y \otimes (z \otimes x), \bot) \\
{\scriptstyle wheel_{x,y\otimes z}} \uparrow & & \downarrow {\scriptstyle wheel_{y,z\otimes x}} \\
\mathscr{C}(x \otimes (y \otimes z), \bot) & & \mathscr{C}((z \otimes x) \otimes y, \bot) \\
{\scriptstyle associativity} \downarrow & & \uparrow {\scriptstyle associativity} \\
\mathscr{C}((x \otimes y) \otimes z, \bot) & \xrightarrow{\ wheel_{x\otimes y,z}\ } & \mathscr{C}(z \otimes (x \otimes y), \bot)
\end{array} \qquad (16)$$

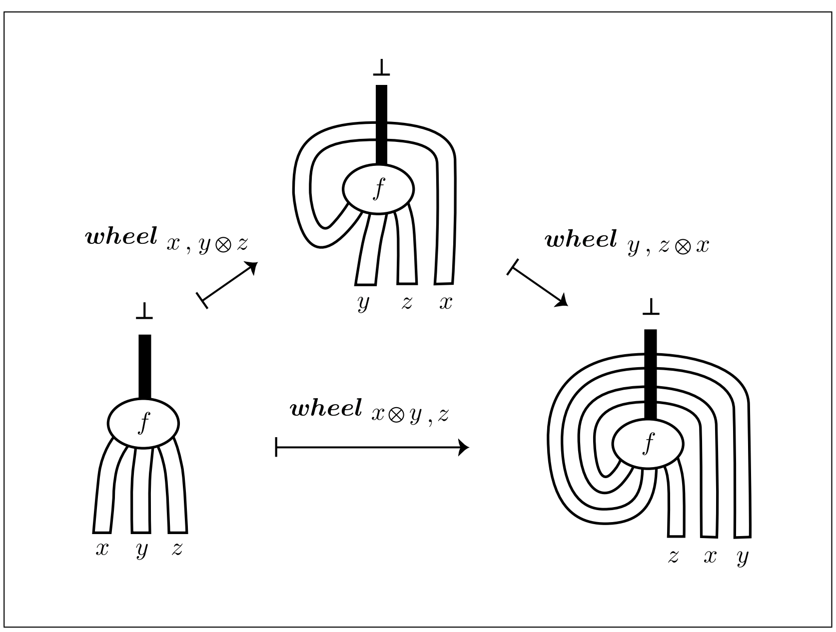
commute for all objects x, y, z of the category $\mathscr{C}$.

$wheel_{x,y} \quad :$



(17)

In that graphical formulation, the coherence diagram expresses that the diagram below commutes:



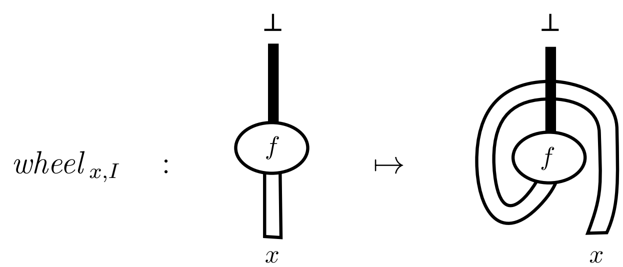
It is not very difficult to deduce from the coherence diagram (16) that the diagram

$$\begin{array}{ccc} \mathscr{C}(x \otimes e, \bot) & \xrightarrow{\quad wheel_{x,e} \quad} & \mathscr{C}(e \otimes x, \bot) \\ {\scriptstyle associativity}\uparrow & & \downarrow{\scriptstyle associativity} \\ \mathscr{C}(x, \bot) & \xrightarrow{\quad id \quad} & \mathscr{C}(x, \bot) \end{array}$$

commutes in any helical refutation category. On the other hand, the other expected coherence diagram

$$\begin{array}{ccc} \mathscr{C}(e \otimes x, \bot) & \xrightarrow{\quad wheel_{e,x} \quad} & \mathscr{C}(x \otimes e, \bot) \\ {\scriptstyle associativity}\uparrow & & \downarrow{\scriptstyle associativity} \\ \mathscr{C}(x, \bot) & \xrightarrow{\quad id \quad} & \mathscr{C}(x, \bot) \end{array} \tag{18}$$

does not commute in general. As a matter of fact, this is an important aspect of our definition: this lack of coherence reflects the existence of a topological "twist" of angle 2π on the negation functor R, arising as follows:



2.2 A 2-category of helical dialogue categories

We define a 2-category ***HeliCat*** with

- helical dialogue categories as 0-cells,
- helical functors as 1-cells,
- dialogue natural transformations as 2-cells.

The 1-dimensional cells. A helical functor between two helical dialogue categories is defined as a lax monoidal functor

$$F \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{D}$$

equipped with a morphism

$$\bot_F \quad : \quad F(\bot) \quad \longrightarrow \quad \bot$$

such that the diagram

$$
\begin{array}{ccccc}
\mathscr{C}(x \otimes y, \bot) & \xrightarrow{F} & \mathscr{D}(F(x \otimes y), F(\bot)) & \xrightarrow{coercion} & \mathscr{D}(F(x) \otimes F(y), \bot) \\
\downarrow{\scriptstyle wheel_{x,y}} & & & & \downarrow{\scriptstyle wheel_{F(x),F(y)}} \\
\mathscr{C}(y \otimes x, \bot) & \xrightarrow{F} & \mathscr{D}(F(y \otimes x), F(\bot)) & \xrightarrow{coercion} & \mathscr{D}(F(y) \otimes F(x), \bot)
\end{array}
\tag{19}
$$

commutes for all objects x, y of the category $\mathscr{C}$. In this diagram, the two coercion maps are deduced by precomposing with the lax monoidal structure of the functor F

$$m_{x,y} \quad : \quad F(x) \otimes F(y) \quad \longrightarrow \quad F(x \otimes y)$$

and by postcomposing with the map $\bot_F$.

The 2-dimensional cells. The 2-cells are defined in the same way as in the case of dialogue categories, see [5] for details. A dialogue natural transformation

$$\theta \quad : \quad (F, \bot_F) \quad \Rightarrow \quad (G, \bot_G)$$

is defined there as a natural transformation

$$\theta \quad : \quad F \quad \Rightarrow \quad G$$

making the diagram

$$
\begin{array}{ccc}
F(\bot) & & \\
 & \searrow^{\bot_F} & \\
\downarrow{\scriptstyle \theta_\bot} & & \bot \\
 & \nearrow_{\bot_G} & \\
G(\bot) & &
\end{array}
$$

commute. We leave the reader check that the expected notions of (horizontal and vertical) identity and composition define a 2-category ***HeliCat*** together with a forgetful 2-functor

$$U \quad : \quad \textbf{\textit{HeliCat}} \quad \longrightarrow \quad \textbf{\textit{DiaCat}}$$

to the 2-category ***DiaCat*** of dialogue categories constructed in [5]. Note that by construction, the 2-functor U is fully faithful on 2-dimensional cells.

2.3 Helical dialogue chiralities

Recall from [5, 6] that a dialogue chirality is defined as a pair of monoidal categories

$$(\mathscr{A}, \owedge, \mathbf{true}) \qquad\qquad (\mathscr{B}, \ovee, \mathbf{false})$$

equipped with an adjunction $L \dashv R$ and with a monoidal equivalence

$$\mathscr{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{B} \quad (\bot) \qquad\qquad \mathscr{A} \underset{{}^*(-)}{\overset{(-)^*}{\rightleftarrows}} \mathscr{B}^{op(0,1)} \quad (\textit{monoidal equivalence})$$

together with a family of bijections

$$\chi_{m,a,b} \quad : \quad \langle\, m \owedge a \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, a \,|\, m^* \ovee b \,\rangle \tag{20}$$

natural in a and b, where $\langle\, a \,|\, b \,\rangle$ is defined as

$$\langle\, a \,|\, b \,\rangle \quad = \quad \mathscr{A}(\, a \,,\, R\, b\,).$$

The family χ is moreover required to make the diagram

$$\begin{array}{ccccc}
\langle\, (m \owedge n) \owedge a \,|\, b \,\rangle & & \xrightarrow{\quad \chi_{m \owedge n} \quad} & & \langle\, a \,|\, (m \owedge n)^* \ovee b \,\rangle \\
\big\downarrow {\scriptstyle associativity} & & & & \big\uparrow {\scriptstyle associativity,\ monoidality\ of\ negation} \\
\langle\, m \owedge (n \owedge a) \,|\, b \,\rangle & \xrightarrow{\chi_m} & \langle\, n \owedge a \,|\, m^* \ovee b \,\rangle & \xrightarrow{\chi_n} & \langle\, a \,|\, n^* \ovee (m^* \ovee b) \,\rangle
\end{array} \tag{21}$$

commute. A helical dialogue chirality is then defined as a dialogue category equipped with a natural family of bijections *helix* permuting the two sides of the evaluation bracket $\langle - | - \rangle$.

Definition 2 (helical chirality) *A helical structure in a dialogue chirality is defined as a family of bijections*

$$helix_{a,b} \quad : \quad \langle\, a \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, {}^*b \,|\, a^* \,\rangle$$

natural in a and b, and making the diagram below commute:

$$\begin{array}{ccccc}
\langle\, a_1 \owedge a_2 \,|\, b \,\rangle & \xrightarrow{\ helix\ } & \langle\, {}^*b \,|\, (a_1 \owedge a_2)^* \,\rangle & = & \langle\, {}^*b \,|\, a_2^* \ovee a_1^* \,\rangle \\
\big\downarrow {\scriptstyle \chi_{a_1,a_2,b}} & & & & \big\uparrow {\scriptstyle \chi_{a_2,{}^*b,a_1^*}} \\
\langle\, a_2 \,|\, a_1^* \ovee b \,\rangle & & & & \langle\, a_2 \owedge {}^*b \,|\, a_1^* \,\rangle \\
\big\downarrow {\scriptstyle helix} & & & & \| \\
\langle\, {}^*(a_1^* \ovee b) \,|\, a_2^* \,\rangle & & & & \langle\, {}^*(b \ovee a_2^*) \,|\, a_1^* \,\rangle \\
\| & & & & \big\uparrow {\scriptstyle helix} \\
\langle\, {}^*b \owedge a_1 \,|\, a_2^* \,\rangle & \xrightarrow{\ \chi_{{}^*b,a_1,a_2^*}\ } & \langle\, a_1 \,|\, ({}^*b)^* \ovee a_2^* \,\rangle & = & \langle\, a_1 \,|\, b \ovee a_2^* \,\rangle
\end{array} \tag{22}$$

where every double edge is meant to describe a canonical coercion isomorphism induced by the monoidal equivalence between $\mathscr{A}$ and $\mathscr{B}^{op(0,1)}$. A helical dialogue chirality is then defined as a dialogue chirality equipped with a helical structure.

The family $helix$ induces a series of bijections

$$\mathscr{A}(a, Rb) \xrightarrow{helix} \mathscr{A}(^*b, Ra^*) \xrightarrow{adjunction} \mathscr{B}(L^*b, a^*) \xrightarrow{equivalence} \mathscr{A}(a, {}^*(L(^*b)))$$

each component of them, and thus their composite, natural in a and b. From this follows by the usual Yoneda argument that the natural family of bijections $helix$ may be alternatively formulated as or equivalently, as a family of isomorphisms

$$RL\text{-}helix_a \quad : \quad R(a^*) \quad \longrightarrow \quad {}^*(La) \tag{23}$$

natural in a. Similarly, the helix structure induces a natural bijection

$$\mathscr{B}(La, b) \xrightarrow{adjunction} \mathscr{A}(a, Rb) \xrightarrow{helix} \mathscr{A}(^*b, R(a^*)) \xrightarrow{equivalence} \mathscr{B}({}^*(R(a^*)), b)$$

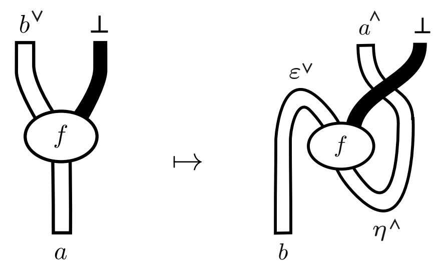
and it may be thus equivalently formulated as a natural family of isomorphisms

$$LR\text{-}helix_a \quad : \quad {}^*(La) \quad \longrightarrow \quad R(a^*) \tag{24}$$

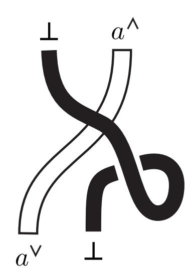
Interestingly, the identity

$$LR\text{-}helix_a \circ RL\text{-}helix_a \quad = \quad id_{R(a^*)} \quad : \quad R(a^*) \quad \longrightarrow \quad R(a^*)$$

is not necessarily satisfied. This has to do with the "helical" rather than simply "cyclic" structure of the dialogue chirality. Typically, consider the case of a ribbon category $\mathscr{C}$ equipped with an object $\bot$ considered in the introduction. This time, take $Rx = x \multimap \bot = x^\vee \otimes \bot$ and $Lx = \bot \multimapinv x = \bot \otimes x^\wedge$. Define the helix as follows

$helix_{a,b} \quad :$ 

where the left and right duals $x^\wedge$ and $x^\vee$ of an object x are identified.

$RL\text{-}helix_a \quad =$ 

$LR\text{-}helix_a \quad =$ 

2.4 A 2-category of helical dialogue chiralities

We define a 2-category ***HeliChir*** with helical dialogue chiralities as objects (or 0-dimensional cells). This construction will be compared in §3 with the 2-category ***HeliCat*** of helical categories just constructed in §2.4.

The 1-dimensional cells. A 1-dimensional cell in ***HeliChir***

$$F \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is defined in essentially the same way as a 1-dimensional cell in the 2-category ***DiaChir*** of dialogue chiralities constructed in [5]. Hence, it is a quadruple $F = (F_\bullet, F_\circ, \widetilde{F}, \overline{F})$ consisting of a lax monoidal functor $F_\bullet : \mathscr{A}_1 \longrightarrow \mathscr{A}_2$, an oplax monoidal functor $F_\circ : \mathscr{B}_1 \longrightarrow \mathscr{B}_2$, a monoidal natural isomorphism

$$\begin{array}{ccc} \mathscr{A}_1 & \xrightarrow{\quad F_\bullet \quad} & \mathscr{A}_2 \\ {\scriptstyle (-)^*}\downarrow & \widetilde{F} \;\Swarrow & \downarrow{\scriptstyle (-)^*} \\ \mathscr{B}_1^{op(0,1)} & \xrightarrow[\quad F_\circ^{op(0,1)} \quad]{} & \mathscr{B}_2^{op(0,1)} \end{array} \tag{25}$$

together with a natural transformation:

$$\begin{array}{ccc} \mathscr{A}_1 & \xrightarrow{\quad F_\bullet \quad} & \mathscr{A}_2 \\ {\scriptstyle R}\uparrow & \overline{F} \;\Searrow & \uparrow{\scriptstyle R} \\ \mathscr{B}_1 & \xrightarrow[\quad F_\circ \quad]{} & \mathscr{B}_2 \end{array} \tag{26}$$

making the diagram

$$\begin{array}{ccc} \langle\, m \oslash a \,|\, b \,\rangle & \xrightarrow{\quad \chi_m \quad} & \langle\, a \,|\, m^* \obslash b \,\rangle \\ {\scriptstyle F_{m \oslash a, b}}\downarrow & & \downarrow{\scriptstyle F_{a, m^* \obslash b}} \\ \langle\, F_\bullet(m \oslash a) \,|\, F_\circ(b) \,\rangle & & \langle\, F_\bullet(a) \,|\, F_\circ(m^* \obslash b) \,\rangle \\ \downarrow & & \downarrow{\scriptstyle \textit{monoidality of } F_\circ} \\ {\scriptstyle \textit{monoidality of } F_\bullet}\downarrow & & \langle\, F_\bullet(a) \,|\, F_\circ(m^*) \obslash F_\circ(b) \,\rangle \\ \downarrow & & \downarrow{\scriptstyle \widetilde{F}} \\ \langle\, F_\bullet(m) \oslash F_\bullet(a) \,|\, F_\circ(b) \,\rangle & \xrightarrow{\quad \chi_{F_\bullet(m)} \quad} & \langle\, F_\bullet(a) \,|\, F_\bullet(m)^* \obslash F_\circ(b) \,\rangle \end{array} \tag{27}$$

commute for all objects a, m in $\mathscr{A}_1$ and b in $\mathscr{B}_1$. Here, the map

$$F_{a,b} \quad : \quad \langle\, a \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, F_\bullet(a) \,|\, F_\circ(b) \,\rangle$$

is defined as the composite

$$\begin{array}{ccccc} \langle\, a \,|\, b \,\rangle & & \xrightarrow{\quad F_{a,b} \quad} & & \langle\, F_\bullet(a) \,|\, F_\circ(b) \,\rangle \\ \| & & & & \| \\ \mathscr{A}_1(a, Rb) & \xrightarrow{F_\bullet} & \mathscr{A}_2(F_\bullet(a), F_\bullet(Rb)) & \xrightarrow{\overline{F}} & \mathscr{A}_2(F_\bullet(a), RF_\circ(b)) \end{array}$$

The only additional requirement compared to a 1-dimensional cell in ***DiaChir*** is that the diagram below commutes:

$$\begin{array}{ccccc} \langle\, a \,|\, b \,\rangle & & \xrightarrow{\quad F_{a,b} \quad} & & \langle\, F_\bullet(a) \,|\, F_\circ(b) \,\rangle \\ \Big\downarrow {\scriptstyle helix_{a,b}} & & & & \Big\downarrow {\scriptstyle helix_{F_\bullet(a), F_\circ(b)}} \\ \langle\, {}^*b \,|\, a^* \,\rangle & \xrightarrow{F_{{}^*b,a^*}} & \langle\, F_\bullet({}^*b) \,|\, F_\circ(a^*) \,\rangle & \xrightarrow{\widetilde{F}} & \langle\, {}^*(F_\circ(b)) \,|\, (F_\bullet(a))^* \,\rangle \end{array} \tag{28}$$

Here, the map $\widetilde{F}_{b,a}$ is defined by applying the natural isomorphism $\widetilde{F}$ on the object $F_\circ(a^*)$ in order to get the object $(F_\bullet(a))^*$ and *at the same time* its mate

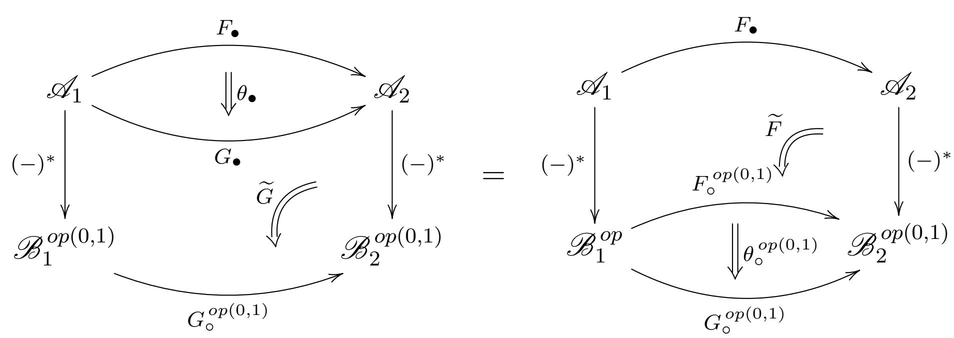
$$\widetilde{F}_{mate} \quad : \quad F_\bullet \circ {}^*(-) \quad \Rightarrow \quad {}^*(-) \circ F_\circ \quad : \quad \mathscr{B}_1^{op(0,1)} \quad \longrightarrow \quad \mathscr{A}_2$$

on the object $F_\bullet({}^*b)$ in order to get the object ${}^*(F_\circ(b))$.

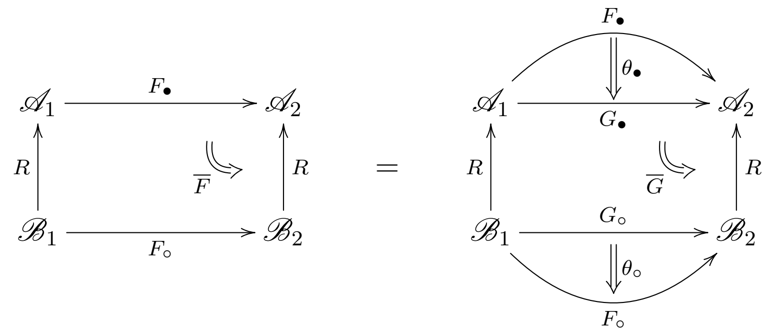
The 2-dimensional cells. A 2-dimensional cell in ***HeliChir***

$$\theta \quad : \quad F \Rightarrow G \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is defined in exactly the same way as a 2-dimensional cell in the 2-category ***DiaChir*** of dialogue chiralities constructed in [5]. It is a pair $(\theta_\bullet, \theta_\circ)$ of monoidal natural transformations $\theta_\bullet : F_\bullet \Rightarrow G_\bullet$ and $\theta_\circ : G_\circ \Rightarrow F_\circ$ satisfying the two equations below:



(29)



(30)

As in the case of the 2-category ***HeliCat*** in §2.2, we let the reader check that the expected notions of (horizontal and vertical) identity and composition define a 2-category ***HeliChir*** together with a forgetful 2-functor

$$U' \quad : \quad \textbf{\textit{HeliChir}} \quad \longrightarrow \quad \textbf{\textit{DiaChir}}$$

to the 2-category ***DiaChir*** of dialogue chiralities constructed in [5]. As in the case of the forgetful 2-functor from ***HeliCat*** in §2.2, the forgetful 2-functor U is fully faithful on 2-dimensional cells.

3 Construction of the 2-categorical equivalence

In this section, we show that the 2-categories ***HeliCat*** and ***HeliChir*** are equivalent in the appropriate 2-dimensional sense. The construction is a direct adaptation of the 2-dimensional equivalence between ***DiaCat*** and ***DiaChir*** exhibited in our companion paper [5]. So, it is not particularly difficult, but it should done with great care. In the same way as in the original case of dialogue categories, the equivalence may be understood as a coherence theorem for helical dialogue chiralities. In particular, it provides a general recipe to strictify a helical dialogue chirality into an equivalent helical dialogue category. Recall that a dialogue chirality $(\mathscr{A}, \mathscr{B})$ is called *strict* when $\mathscr{B} = \mathscr{A}^{op(0,1)}$ and moreover, the two functors ${}^*(-)$ and $(-)^*$ and their monoidal equivalence are trivial — that is, equal to the identity on $\mathscr{A}$.

3.1 From dialogue categories to dialogue chiralities

We start by constructing a 2-functor

$$\mathcal{F} \quad : \quad \textbf{\textit{HeliCat}} \quad \longrightarrow \quad \textbf{\textit{HeliChir}}$$

from the 2-category ***HeliCat*** of helical dialogue categories to the 2-category ***DiaChir*** of helical dialogue chiralities.

Definition of $\mathcal{F}$: the 0-dimensional cells. To every helical dialogue category $\mathscr{C}$, the 2-functor $\mathcal{F}$ associates the helical dialogue chirality defined in exactly the same way as in the case of basic dialogue categories:

$$(\mathscr{A}, \oslash, \mathbf{true}) \ := \ (\mathscr{C}, \otimes, e) \qquad\qquad (\mathscr{B}, \oslash, \mathbf{false}) \ := \ (\mathscr{C}, \otimes, e)^{op(0,1)}.$$

The monoidal equivalence between $\mathscr{A}$ and $\mathscr{B}^{op(0,1)}$ is defined as the identity functor on the monoidal category $\mathscr{C}$. The two adjoint functors L and R are defined as

$$L : x \mapsto \bot \multimapinv x \qquad\qquad R : x \mapsto x \multimap \bot$$

with the adjunction $L \dashv R$ witnessed by the series of bijections

$$\begin{array}{rcl} \mathscr{A}(x, R(y)) & = & \mathscr{C}(x, y \multimap \bot) \\ & \cong & \mathscr{C}(y \otimes x, \bot) \\ & \cong & \mathscr{C}(y, \bot \multimapinv x) \\ & = & \mathscr{B}(L(x), y) \end{array}$$

natural in x and y. The bijection $\chi_{m,x,y}$ is defined as the composite

$$\mathscr{C}(m \otimes x, y \multimap \bot) \xrightarrow{\varphi^{-1}_{x \otimes m, y}} \mathscr{C}(y \otimes m \otimes x, \bot) \xrightarrow{\varphi_{y \otimes m, x}} \mathscr{C}(x, (y \otimes m) \multimap \bot)$$

where for simplicity, we forget the associativity map between $y \otimes (m \otimes x)$ and $(y \otimes m) \otimes x$. Up to that stage, the 2-functor $\mathcal{F}$ is defined in exactly the same way in the original case of dialogue categories, see [5] for details. This already ensures that the data introduced for $(\mathscr{A}, \mathscr{B})$ define a dialogue chirality. The only novelty is the definition of the helical structure

$$\langle\, a \,|\, b \,\rangle = \mathscr{A}(a, Rb) = \mathscr{C}(a, b \multimap \bot) \quad \xrightarrow{helix_{a,b}} \quad \mathscr{C}(b, a \multimap \bot) = \mathscr{A}({}^*b, Ra^*) = \langle\, {}^*b \,|\, a^* \,\rangle$$

as the natural family of isomorphisms

$$\mathscr{C}(a, b \multimap \bot) \xrightarrow{\varphi^{-1}_{b,a}} \mathscr{C}(b \otimes a, \bot) \xrightarrow{wheel^{-1}_{a,b}} \mathscr{C}(a \otimes b, \bot) \xrightarrow{\varphi_{a,b}} \mathscr{C}(a, b \multimap \bot)$$

It is not difficult to check that this defines a helical dialogue chirality $(\mathscr{A}, \mathscr{B})$. The point is that the two coherence diagrams (16) for dialogue categories and (22) for dialogue chiralities essentially coincide.

Definition of $\mathcal{F}$: the 1-dimensional cells. To every dialogue functor

$$(F, \bot_F) \quad : \quad (\mathscr{C}, \bot_{\mathscr{C}}) \quad \longrightarrow \quad (\mathscr{D}, \bot_{\mathscr{D}})$$

the 2-functor $\mathcal{F}$ associates the 1-dimensional cell $\mathcal{F}(F)$ defined as the quadruple consisting of the lax monoidal functor

$$\mathcal{F}(F)_{\bullet} \ : \ \mathscr{C} \xrightarrow{F} \mathscr{D}$$

the oplax monoidal functor

$$\mathcal{F}(F)_{\circ} \ : \ \mathscr{C}^{op(0,1)} \xrightarrow{F^{op(0,1)}} \mathscr{D}^{op(0,1)}$$

the monoidal isomorphism $\widetilde{\mathcal{F}(F)}$ defined as the identity on the functor F, and the natural transformation

$$\overline{\mathcal{F}(F)} \quad : \quad R \circ F \quad \longrightarrow \quad F \circ R$$

whose components

$$F(x \multimap \bot_{\mathscr{C}}) \quad \longrightarrow \quad F(x) \multimap \bot_{\mathscr{D}}$$

is associated by currification $\varphi_{F(x),F(x \multimap \bot_{\mathscr{C}})}$ to the morphism

$$F(x) \otimes F(x \multimap \bot_{\mathscr{C}}) \longrightarrow F(x \otimes (x \multimap \bot_{\mathscr{C}})) \longrightarrow F(\bot_{\mathscr{C}}) \longrightarrow \bot_{\mathscr{D}}.$$

We know from [5] that this defines a 1-dimensional cell between dialogue chiralities. There remains to show that $\mathcal{F}(F)$ is compatible with the helical structure, in the technical sense that diagram (28) commutes. This fact is essentially immediate to deduce from the fact that F is compatible with $wheel$ in the technical sense that diagram (19) commutes.

Definition of $\mathcal{F}$: the 2-dimensional cells. The 2-functor $\mathcal{F}$ acts on 2-cells in exactly the same way as in the original case of dialogue categories considered in [5], see that paper for details.

3.2 From dialogue chiralities to dialogue categories

Now that the 2-functor $\mathcal{F}$ has been constructed, we complement it with a 2-functor in the reverse direction:

$$\mathcal{G} \quad : \quad \textbf{\textit{HeliChir}} \quad \longrightarrow \quad \textbf{\textit{HeliCat}}$$

from the 2-category of helical dialogue chiralities to the 2-category of helical dialogue categories.

Definition of $\mathcal{G}$: the 0-dimensional cells. The 2-functor transports every helical dialogue chirality $(\mathscr{A}, \mathscr{B})$ to the helical dialogue category defined as

$$(\mathscr{C}, \otimes, I) \quad := \quad (\mathscr{A}, \owedge, \mathbf{true})$$

equipped with the tensorial pole

$$\bot \quad := \quad R(\mathbf{false}).$$

together with the functors:

$$\bot \multimapinv x \;=\; {}^*(L(x)) \qquad\qquad x \multimap \bot \;=\; R(x^*).$$

The natural bijections φ and ψ are then defined by composing the series of natural bijections

$$\begin{array}{rcll}
\mathscr{C}(x \otimes y, \bot) & = & \mathscr{A}(x \owedge y, R(\mathbf{false})) & \text{by definition of } \mathscr{C} \text{ and of } \bot, \\
& \cong & \mathscr{A}(y, R(x^* \ovee \mathbf{false})) & \text{by applying } \chi_{x,y,\mathbf{false}}, \\
& \cong & \mathscr{A}(y, R(x^*)) & \text{by applying the unit law in } \mathscr{B}, \\
& \cong & \mathscr{B}(L(y), x^*) & \text{by the adjunction } L \dashv R, \\
& \cong & \mathscr{A}(x, {}^*(L(y))) & \text{by the adjunction } (-)^* \dashv {}^*(-), \\
& = & \mathscr{C}(x, {}^*(L(y))) & \text{by definition of } \mathscr{C}.
\end{array}$$

$$\begin{array}{rcll}
\mathscr{C}(x \otimes y, \bot) & = & \mathscr{A}(x \owedge y, R(\mathbf{false})) & \text{by definition of } \mathscr{C} \text{ and of } \bot, \\
& \cong & \mathscr{A}(y, R(x^* \ovee \mathbf{false})) & \text{by applying } \chi_{x,y,\mathbf{false}}, \\
& \cong & \mathscr{A}(y, R(x^*)) & \text{by applying the unit law in } \mathscr{B}, \\
& = & \mathscr{C}(y, R(x^*)) & \text{by definition of } \mathscr{C}.
\end{array}$$

The helical structure *wheel* on the dialogue category $\mathscr{C}$ is defined as the composite of natural bijections:

$$\begin{array}{ccccccc}
\mathscr{C}(x \otimes y, \bot) & & & \xrightarrow{\quad wheel_{x,y} \quad} & & & \mathscr{C}(y \otimes x, \bot) \\
\downarrow{\scriptstyle \varphi_{x,y}} & & & & & & \uparrow{\scriptstyle \varphi^{-1}_{y,x}} \\
\mathscr{C}(y, x \multimap \bot) & = & \langle\, {}^*y \mid x^* \,\rangle & \xrightarrow{\; helix^{-1}_{x,y} \;} & \langle\, x \mid y \,\rangle & = & \mathscr{C}(x, y \multimap \bot)
\end{array}$$

The fact that this defines a helical structure is essentially immediate: the reason already mentioned is that the two coherence diagrams (16) for dialogue categories and (22) for dialogue chiralities essentially coincide modulo translation of one into the other.

Definition of $\mathcal{G}$: the 1-dimensional cells. Every 1-dimensional cell

$$F = (F_\bullet, F_\circ, \widetilde{F}, \overline{F}) \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is transported to the dialogue functor $(F_\bullet, \bot_F)$ consisting of the functor

$$F_\bullet \quad : \quad \mathscr{A}_1 \quad \longrightarrow \quad \mathscr{A}_2$$

and of the morphism

$$\bot_F \quad : \quad F_\bullet(\bot_{\mathscr{A}_1}) \quad \longrightarrow \quad \bot_{\mathscr{A}_2}$$

defined as the composite

$$F_\bullet \circ R\,(\mathbf{false}) \xrightarrow{\overline{F}_{\mathbf{false}}} R \circ F_\circ\,(\mathbf{false}) \xrightarrow{\mathit{monoidality}} R\,(\mathbf{false})$$

There remains to show that $\mathcal{G}(F) = F_\bullet$ is compatible with the wheel structure, in the technical sense that diagram (19) commutes. This fact is essentially immediate to deduce from the diagram chase below:

$$\begin{array}{ccccc}
\mathscr{C}(x \otimes y, \bot) & \xrightarrow{F} & \mathscr{D}(F(x \otimes y), \bot) & \xrightarrow{\mathit{coercion}} & \mathscr{D}(F(x) \otimes F(y), \bot) \\
{\scriptstyle \psi_{x,y}} \downarrow & & (a) & & \downarrow {\scriptstyle \psi_{Fx,Fy}} \\
\mathscr{C}(x, \bot \multimapinv y) & \xrightarrow{F} & \mathscr{D}(Fx, F(\bot \multimapinv y)) & \longrightarrow & \mathscr{D}(Fx, \bot \multimapinv Fy) \\
{\scriptstyle \mathit{helix}^{-1}_{x,y}} \downarrow & & (b) & & \downarrow {\scriptstyle \mathit{helix}^{-1}_{Fx,Fy}} \\
\mathscr{C}(y, \bot \multimapinv x) & \xrightarrow{F} & \mathscr{D}(Fy, F(\bot \multimapinv x)) & \longrightarrow & \mathscr{D}(Fy, \bot \multimapinv Fx) \\
{\scriptstyle \psi^{-1}_{y,x}} \downarrow & & (c) & & \downarrow {\scriptstyle \psi^{-1}_{Fy,Fx}} \\
\mathscr{C}(y \otimes x, \bot) & \xrightarrow{F} & \mathscr{D}(F(y \otimes x), \bot) & \xrightarrow{\mathit{coercion}} & \mathscr{D}(F(y) \otimes F(x), \bot)
\end{array}$$

whose hexagons (a) and (c) commute in every dialogue category, and whose inner hexagon (b) commutes because diagram (28) commutes.

Definition of $\mathcal{G}$: the 2-dimensional cells. The 2-functor $\mathcal{G}$ acts on 2-dimensional cells in exactly the same way as in the original case of dialogue categories considered in [5]. We refer the reader to that paper for details. We simply recall here that every 2-dimensional cell $\theta = (\theta_\bullet, \theta_\circ)$ is transported to the dialogue transformation $\theta_\bullet$.

3.3 The pseudo-natural transformation Φ

The composite 2-functor

$$\textbf{\textit{HeliCat}} \xrightarrow{\ \mathcal{F}\ } \textbf{\textit{HeliChir}} \xrightarrow{\ \mathcal{G}\ } \textbf{\textit{HeliCat}}$$

coincides with the identity on the 2-category ***HeliCat*** of helical dialogue categories. In order to establish that ***HeliCat*** and ***HeliChir*** are biequivalent, we construct a pair of pseudo-natural transformations

$$\Phi \ : \ Id \longrightarrow \mathcal{F}\circ\mathcal{G} \qquad\qquad \Psi \ : \ \mathcal{F}\circ\mathcal{G} \longrightarrow Id$$

between the identity 2-functor on ***HeliChir*** and the 2-functor $\mathcal{F}\circ\mathcal{G}$. We then show that their components $\Phi_{(\mathscr{A},\mathscr{B})}$ and $\Psi_{(\mathscr{A},\mathscr{B})}$ define an equivalence in the 2-category ***DiaChir***, for every helical dialogue chirality $(\mathscr{A},\mathscr{B})$. Before proceeding further, we find convenient to give a detailed account of the helical dialogue chirality $(\mathscr{A},\mathscr{A}^{\,op(0,1)})$ obtained by applying the 2-functor $\mathcal{F}\circ\mathcal{G}$ to a given dialogue chirality $(\mathscr{A},\mathscr{B})$. The dialogue chirality $(\mathscr{A},\mathscr{A}^{\,op(0,1)})$ is equipped with the trivial monoidal equivalence:

$$\mathscr{A} \ \underset{id}{\overset{id}{\rightleftarrows}} \ \left(\mathscr{A}^{\,op(0,1)}\right)^{op(0,1)} \qquad \text{(monoidal equivalence)}$$

with the adjunction

$$\mathscr{A} \ \underset{R}{\overset{L}{\rightleftarrows}} \ \mathscr{B} \ \underset{((-)^{*})^{\,op(0,1)}}{\overset{({}^{*}(-))^{\,op(0,1)}}{\rightleftarrows}} \ \mathscr{A}^{\,op(0,1)} \qquad (\text{both } \perp)$$

From this follows that

$$\langle\, a_1 \,|\, a_2 \,\rangle_{(\mathscr{A},\mathscr{A}^{\,op(0,1)})} \quad = \quad \mathscr{A}(a_1, R\,(a_2^{*})) \quad = \quad \langle\, a_1 \,|\, a_2^{*} \,\rangle_{(\mathscr{A},\mathscr{B})}$$

The natural transformation $\chi_{(\mathscr{A},\mathscr{A}^{\,op(0,1)})}$ at instance (m,a,b) is defined as the composite function

$$\begin{array}{ccc}
\langle\, m \otimes a_1 \,|\, a_2^{*} \,\rangle & & \langle\, a_1 \,|\, (a_2 \otimes m)^{*} \,\rangle \\
\downarrow & & \uparrow \\
\langle\, m \otimes a_1 \,|\, a_2^{*} \parr \mathbf{false} \,\rangle & & \langle\, a_1 \,|\, (a_2 \otimes m)^{*} \parr \mathbf{false} \,\rangle \\
\big\downarrow{\scriptstyle (\chi_{(\mathscr{A},\mathscr{B})})^{-1}} & & \big\uparrow{\scriptstyle \chi_{(\mathscr{A},\mathscr{B})}} \\
\langle\, a_2 \otimes (m \otimes a_1) \,|\, \mathbf{false} \,\rangle & \longrightarrow & \langle\, (a_2 \otimes m) \otimes a_1 \,|\, \mathbf{false} \,\rangle
\end{array}$$

Similarly, the dialogue chirality $\mathcal{G} \circ \mathcal{F}(\mathscr{A}, \mathscr{B})$ has helical structure defined as

$$\langle\, a_1 \,|\, a_2^* \,\rangle_{(\mathscr{A},\mathscr{B})} \xrightarrow{\ helix_{a_1,a_2^*}\ } \langle\, {}^*(a_2^*) \,|\, a_1^* \,\rangle_{(\mathscr{A},\mathscr{B})} \xrightarrow{\ equivalence\ } \langle\, a_2 \,|\, a_1^* \,\rangle_{(\mathscr{A},\mathscr{B})}$$

where $helix$ is the helical structure of the original dialogue chirality $(\mathscr{A}, \mathscr{B})$.

After this detailed description of the "strictified" version $\mathcal{G} \circ \mathcal{F}(\mathscr{A}, \mathscr{B})$ obtained from the helical dialogue chirality $(\mathscr{A}, \mathscr{B})$, we are ready to introduce the pseudo-natural transformations Φ and Ψ. The construction is exactly the same as for dialogue chiralities in [5]. Our main concern is thus to check that the constructions are compatible with the helical structures of the original dialogue chirality $(\mathscr{A}, \mathscr{B})$ and of its strictified version.

The 1-dimensional cells $\Phi_{(\mathscr{A},\mathscr{B})}$. Recall from [5] that to every dialogue chirality $(\mathscr{A}, \mathscr{B})$ one associates the 1-cell

$$\Phi_{(\mathscr{A},\mathscr{B})} \quad : \quad (\mathscr{A}, \mathscr{B}) \quad \longrightarrow \quad (\mathscr{A}, \mathscr{A}^{\,op(0,1)})$$

defined as the pair of monoidal functors

$$(\Phi_{(\mathscr{A},\mathscr{B})})_{\bullet} \ : \ \mathscr{A} \xrightarrow{\ id\ } \mathscr{A} \qquad\qquad (\Phi_{(\mathscr{A},\mathscr{B})})_{\circ} \ : \ \mathscr{B} \xrightarrow{\ ({}^*(-))^{\,op(0,1)}\ } \mathscr{A}^{\,op(0,1)}$$

together with the monoidal natural isomorphism

$$\widetilde{\Phi_{(\mathscr{A},\mathscr{B})}} \quad = \quad \begin{array}{ccc} \mathscr{A} & \xrightarrow{\quad id \quad} & \mathscr{A} \\ {\scriptstyle (-)^*}\big\downarrow & \underline{\eta}\ \Swarrow & \big\downarrow{\scriptstyle id} \\ \mathscr{B}^{\,op(0,1)} & \xrightarrow[\quad {}^*(-) \quad]{} & (\mathscr{A}^{\,op(0,1)})^{\,op(0,1)} \end{array}$$

and the natural transformation

$$\overline{\Phi_{(\mathscr{A},\mathscr{B})}} \quad = \quad \begin{array}{ccc} \mathscr{A} & \xrightarrow{\quad id \quad} & \mathscr{A} \\ & & \big\uparrow{\scriptstyle R} \\ {\scriptstyle R}\big\uparrow & \underline{\varepsilon}^{\,op(0,1)}\ \Searrow & \mathscr{B} \\ & & \big\uparrow{\scriptstyle ((-)^*)^{\,op(0,1)}} \\ \mathscr{B} & \xrightarrow[\quad ({}^*(-))^{\,op(0,1)} \quad]{} & \mathscr{A}^{\,op(0,1)} \end{array}$$

where $\underline{\eta}$ and $\underline{\varepsilon}$ denote the unit and counit of the adjunction $(-)^* \dashv {}^*(-)$. We know from [5] that Φ defines a 1-dimensional cell between the dialogue chirality $(\mathscr{A}, \mathscr{B})$ and its strictified version. There remains to check that this 1-dimensional cell is compatible with the helical structures of the two dialogue chiralities. Technically speaking, this simply means that the coherence diagram (28) commutes when instantiated as follows:

$$
\begin{array}{ccccc}
\langle\, a \,|\, b \,\rangle & & \xrightarrow{\quad \Phi_{a,b} \quad} & & \langle\, a \,|\, ({}^*b)^* \,\rangle \\
 & & & & \big\downarrow {\scriptstyle helix_{a,({}^*b)^*}} \\
{\scriptstyle helix_{a,b}} \big\downarrow & & & & \langle\, {}^*(({}^*b)^*) \,|\, a^* \,\rangle \\
 & & & & \big\downarrow {\scriptstyle equivalence} \\
\langle\, {}^*b \,|\, a^* \,\rangle & \xrightarrow{\Phi_{{}^*b,a^*}} & \langle\, {}^*b \,|\, ({}^*(a^*))^* \,\rangle & \xrightarrow{\widetilde{\Phi}_{{}^*b,a^*}} & \langle\, {}^*b \,|\, a \,\rangle
\end{array}
$$

We leave the reader check that this diagram commutes by naturality of $helix$ and because all the bijections (except for the two instantiations of $helix$) involved in it are deduced from the equivalence $(-)^* \dashv {}^*(-)$.

The 2-dimensional cells Φ_F. Are constructed just as the 2-dimensional cells Φ_F in [5]. From this follows that each of them defines a 2-cell in the 2-category ***HeliChir*** and that the family Φ itself defines a pseudo-natural transformation.

3.4 The pseudo-natural transformation Ψ

The 1-dimensional cells $\Psi_{(\mathscr{A},\mathscr{B})}$. To every dialogue chirality $(\mathscr{A},\mathscr{B})$, one associates the 1-cell $\Psi_{(\mathscr{A},\mathscr{B})}$ defined as the pair of functors

$$
\big(\Psi_{(\mathscr{A},\mathscr{B})}\big)_\bullet \;:\; \mathscr{A} \xrightarrow{\;id\;} \mathscr{A} \qquad\qquad \big(\Psi_{(\mathscr{A},\mathscr{B})}\big)_\circ \;:\; \mathscr{A}^{\,op(0,1)} \xrightarrow{\;((-)^*)^{\,op(0,1)}\;} \mathscr{B}
$$

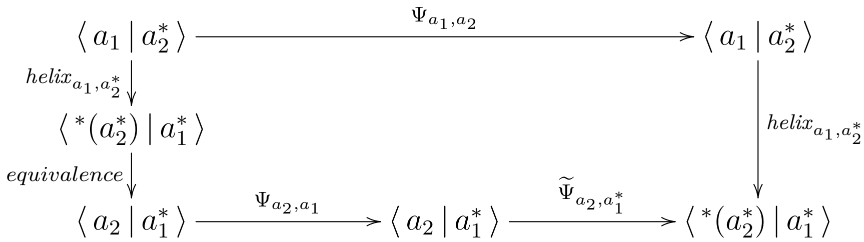
equipped with the trivial monoidal natural isomorphism

$$
\widetilde{\Psi_{(\mathscr{A},\mathscr{B})}} \quad = \quad
\begin{array}{ccc}
\mathscr{A} & \xrightarrow{\qquad id \qquad} & \mathscr{A} \\
{\scriptstyle id} \big\downarrow & \overset{id}{\swarrow\!\!\!\!\swarrow} & \big\downarrow {\scriptstyle (-)^*} \\
\big(\mathscr{A}^{\,op(0,1)}\big)^{\,op(0,1)} & \xrightarrow[\quad (-)^* \quad]{} & \mathscr{B}^{\,op(0,1)}
\end{array}
$$

and with the trivial natural transformation

$$
\overline{\Psi_{(\mathscr{A},\mathscr{B})}} \quad = \quad
\begin{array}{ccc}
\mathscr{A} & \xrightarrow{\qquad id \qquad} & \mathscr{A} \\
{\scriptstyle R} \big\uparrow & & \\
\mathscr{B} & \overset{id}{\searrow\!\!\!\!\searrow} & \big\uparrow {\scriptstyle R} \\
{\scriptstyle ((-)^*)^{\,op(0,1)}} \big\uparrow & & \\
\mathscr{A}^{\,op(0,1)} & \xrightarrow[\quad ((-)^*)^{\,op(0,1)} \quad]{} & \mathscr{B}
\end{array}
$$

We know from [5] that Ψ defines a 1-dimensional cell between the dialogue chirality $(\mathscr{A},\mathscr{B})$ and its strictified version. So, just as in the case of the 1-dimensional cell Φ in §3.4, we only need to check that this 1-dimensional cell Ψ is compatible with the helical structures of the two dialogue chiralities. The coherence diagram (28) is instantiated as follows in that case:



where we write $\langle a \mid b \rangle$ for the evaluation bracket $\langle a \mid b \rangle_{(\mathscr{A},\mathscr{B})}$ of the original dialogue chirality. We leave the reader check that this diagram commutes because Ψ_{a_1,a_2} and Ψ_{a_2,a_1} are equal to the identity, and because $\widetilde{\Psi}_{a_2,a_1^*}$ is obtained by applying the unit $\underline{\eta}$ of the equivalence $a_2 \to {}^*(a_2^*)$ inside the evaluation bracket.

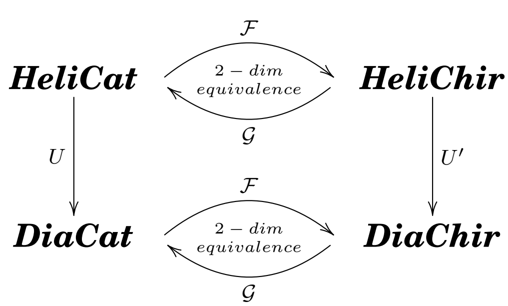
The 2-dimensional cells Ψ_F. Are constructed just as the 2-dimensional cells Ψ_F in [5]. From this follows that each of them defines a 2-cell in the 2-category ***HeliCat*** and that the family Ψ itself defines a pseudo-natural transformation.

3.5 Coherence theorem for helical dialogue chiralities

We have just established that

Theorem 1 (coherence theorem) *The pair of 2-functors $\mathcal{F}$ and $\mathcal{G}$ defines a biequivalence between the 2-categories* ***HeliCat*** *and* ***HeliChir***.

Note that the pair of forgetful 2-functors U and U' defines a homomorphism in the appropriate 2-dimensional sense between the biequivalences:



This homomorphism reflects the fact that strictification of a helical dialogue chirality $(\mathscr{A},\mathscr{B})$ is performed in the same way in ***HeliChir*** and in ***DiaChir***.

4 Ambidextrous chiralities

In this section, we introduce the notion of *ambidextrous chirality* which provides the most primitive notion of dialogue chirality equipped with a left *as well as* a righthand side currification. The notion is introduced in §4.1. Following our general policy, we construct in §4.2 a 2-category ***AmbiChir*** of ambidextrous chiralities. We establish in §4.3 our main result of the section, which states that the 2-category of ambidextrous chiralities is *isomorphic* to the 2-category ***HeliChir*** of helical dialogue chiralities. This very strong correspondance provides a purely logical justification for the topological notion of helical dialogue category.

4.1 Definition

Definition 3 (ambidextrous chirality) *An ambidextrous chirality is a pair of monoidal categories*

$$(\mathscr{A}, \owedge, \mathbf{true}) \qquad\qquad (\mathscr{B}, \ovee, \mathbf{false})$$

equipped with a monoidal equivalence

$$\mathscr{A} \underset{{}^{\circledast}(-)}{\overset{(-)^{\circledast}}{\rightleftarrows}} \mathscr{B}^{op(0,1)} \qquad \textit{monoidal equivalence}$$

and with two families of bijections

$$\begin{array}{rcccc} \chi^R_{m,a,b} & : & \langle\, a \owedge m \,|\, b \,\rangle & \longrightarrow & \langle\, a \,|\, b \ovee m^{\circledast} \,\rangle \\ \chi^L_{m,a,b} & : & \langle\, m \owedge a \,|\, b \,\rangle & \longrightarrow & \langle\, a \,|\, m^{\circledast} \ovee b \,\rangle \end{array}$$

natural in a, b and m, where the evaluation bracket is defined as

$$\langle - | - \rangle \quad := \quad \mathscr{A}(\,-\,,\, R(-)\,) \quad : \quad \mathscr{A}^{op} \times \mathscr{B} \quad \longrightarrow \quad Set$$

The families χ^L and χ^R are required to make the three diagrams commute:

$$\begin{array}{ccccc} \langle\, (m \owedge n) \owedge a \,|\, b \,\rangle & & \xrightarrow{\quad \chi^L_{m \owedge n} \quad} & & \langle\, a \,|\, (m \owedge n)^{\circledast} \ovee b \,\rangle \\ \downarrow{\scriptstyle associativity} & & & & \uparrow{\scriptstyle associativity \atop monoidality\ of\ negation} \\ \langle\, m \owedge (n \owedge a) \,|\, b \,\rangle & \xrightarrow{\chi^L_m} & \langle\, n \owedge a \,|\, m^{\circledast} \ovee b \,\rangle & \xrightarrow{\chi^L_n} & \langle\, a \,|\, n^{\circledast} \ovee (m^{\circledast} \ovee b) \,\rangle \end{array} \qquad (31)$$

$$\begin{array}{ccccc}
\langle\, a \owedge (m \owedge n) \,|\, b \,\rangle & & \xrightarrow{\qquad\qquad\chi^{R}_{m \owedge n}\qquad\qquad} & & \langle\, a \,|\, b \ovee (m \owedge n)^{\circledast} \,\rangle \\
\Big\downarrow {\scriptstyle associativity} & & & & \Big\uparrow {\scriptstyle associativity \; / \; monoidality\ of\ negation} \\
\langle\, (a \owedge m) \owedge n \,|\, b \,\rangle & \xrightarrow{\chi^{R}_{n}} & \langle\, a \owedge m \,|\, b \ovee n^{\circledast} \,\rangle & \xrightarrow{\chi^{R}_{m}} & \langle\, a \,|\, (b \ovee n^{\circledast}) \ovee m^{\circledast} \,\rangle
\end{array} \tag{32}$$

$$\begin{array}{ccccc}
\langle\, (m \owedge a) \owedge n \,|\, b \,\rangle & \xrightarrow{\chi^{R}_{n}} & \langle\, m \owedge a \,|\, b \ovee n^{\circledast} \,\rangle & \xrightarrow{\chi^{L}_{m}} & \langle\, a \,|\, m^{\circledast} \ovee (b \ovee n^{\circledast}) \,\rangle \\
\Big| {\scriptstyle associativity} & & & & \Big| {\scriptstyle associativity} \\
\langle\, m \owedge (a \owedge n) \,|\, b \,\rangle & \xrightarrow{\chi^{L}_{m}} & \langle\, a \owedge n \,|\, m^{\circledast} \ovee b \,\rangle & \xrightarrow{\chi^{R}_{n}} & \langle\, a \,|\, (m^{\circledast} \ovee b) \ovee n^{\circledast} \,\rangle
\end{array} \tag{33}$$

for all objects a, m, n *of the category* $\mathscr{A}$ *and all objects* b *of the category* $\mathscr{B}$.

The two first coherence diagrams (31) and (32) may be seen as left and right instances of the familiar coherence diagram (21) for left currification in dialogue chiralities. The last coherence diagram (33) requires that the left and right currification are compatible in the expected sense. We will see that this last requirement has the somewhat unexpected consequence of equipping the ambidextrous chirality with a helical structure described in §4.3.

4.2 The 2-category of ambidextrous chiralities

Here, we define the 2-category ***AmbiChir*** whose 0-dimensional cells are the ambidextrous chiralities, and whose 1 and 2-dimensional cells are defined as follows.

The 1-dimensional cells. An ambidextrous homomorphism is defined as a quadruple

$$F = (F_{\bullet}, F_{\circ}, \widetilde{F}, \overline{F})$$

consisting of a lax monoidal functor $F_{\bullet} : \mathscr{A}_1 \longrightarrow \mathscr{A}_2$, an oplax monoidal functor $F_{\circ} : \mathscr{B}_1 \longrightarrow \mathscr{B}_2$, a monoidal natural isomorphism (25) and a natural isomorphism (26) making the diagram (27) commute for $\chi = \chi^L$ together with the corresponding diagram for the right currification χ^R, given below:

$$
\begin{array}{ccc}
\langle\, a \owedge m \,|\, b \,\rangle & \xrightarrow{\ \chi^R_m\ } & \langle\, a \,|\, b \ovee m^{\circledast} \,\rangle \\
\big\downarrow{\scriptstyle F_{a \owedge m, b}} & & \big\downarrow{\scriptstyle F_{a, b \ovee m^{\circledast}}} \\
\langle\, F_\bullet(a \owedge m) \,|\, F_\circ(b) \,\rangle & & \langle\, F_\bullet(a) \,|\, F_\circ(b \ovee m^{\circledast}) \,\rangle \\
 & & \big\downarrow{\scriptstyle monoidality\ of\ F_\circ} \\
\big\downarrow{\scriptstyle monoidality\ of\ F_\bullet} & & \langle\, F_\bullet(a) \,|\, F_\circ(b) \ovee F_\circ(m^{\circledast}) \,\rangle \\
 & & \big\downarrow{\scriptstyle \widetilde{F}} \\
\langle\, F_\bullet(a) \owedge F_\bullet(m) \,|\, F_\circ(b) \,\rangle & \xrightarrow{\ \chi^R_{F_\bullet(m)}\ } & \langle\, F_\bullet(a) \,|\, F_\circ(b) \ovee F_\bullet(m)^{\circledast} \,\rangle
\end{array}
\qquad (34)
$$

The 2-dimensional cells. The 2-dimensional cells are defined in exactly the same way as in the 2-categories ***DiaChir*** and ***HeliChir***.

4.3 An isomorphism between *AmbiChir* and *HeliChir*

We establish here that

Theorem 2 *The 2-category **AmbiChir** of ambidextrous chiralities is isomorphic to the 2-category **HeliChir** of helical chiralities.*

To that purpose, we construct a pair of 2-functors

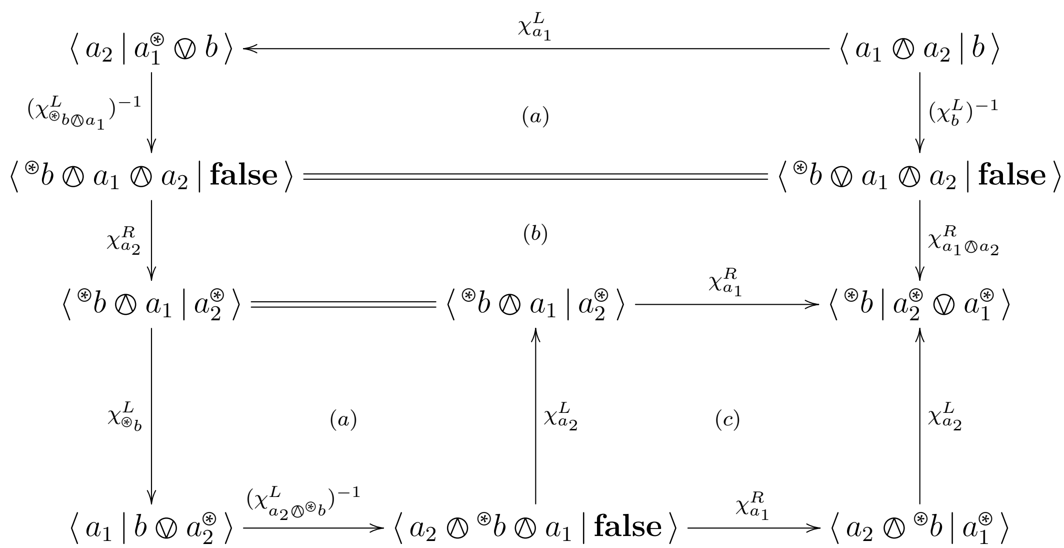
$$
\mathcal{F} \ : \ \textbf{\textit{AmbiChir}} \ \longrightarrow \ \textbf{\textit{HeliChir}} \qquad\qquad \mathcal{G} \ : \ \textbf{\textit{HeliChir}} \ \longrightarrow \ \textbf{\textit{AmbiChir}}
$$

and then show that they define an isomorphism between the 2-categories.

Ambidextrous $\Rightarrow$ Helical. The 2-functor $\mathcal{F}$ transports every ambidextrous chirality $(\mathscr{A}, \mathscr{B}, \chi^L, \chi^R)$ to the underlying dialogue chirality $(\mathscr{A}, \mathscr{B}, \chi)$ with left currification $\chi = \chi^L$. The dialogue chirality is moreover equipped with the following helical structure:

$$
\begin{array}{ccccc}
\langle\, a \,|\, b \,\rangle & & \xrightarrow{\qquad helix_{a,b} \qquad} & & \langle\, {}^{\circledast}b \,|\, a^{\circledast} \,\rangle \\
\big\| & & & & \big\| \\
\langle\, a \,|\, ({}^{\circledast}b)^{\circledast} \ovee \mathbf{false} \,\rangle & \xrightarrow{\ (\chi^L_{{}^{\circledast}b})^{-1}\ } & \langle\, {}^{\circledast}b \owedge a \,|\, \mathbf{false} \,\rangle & \xrightarrow{\ \chi^R_a\ } & \langle\, {}^{\circledast}b \,|\, \mathbf{false} \ovee a^{\circledast} \,\rangle
\end{array}
$$

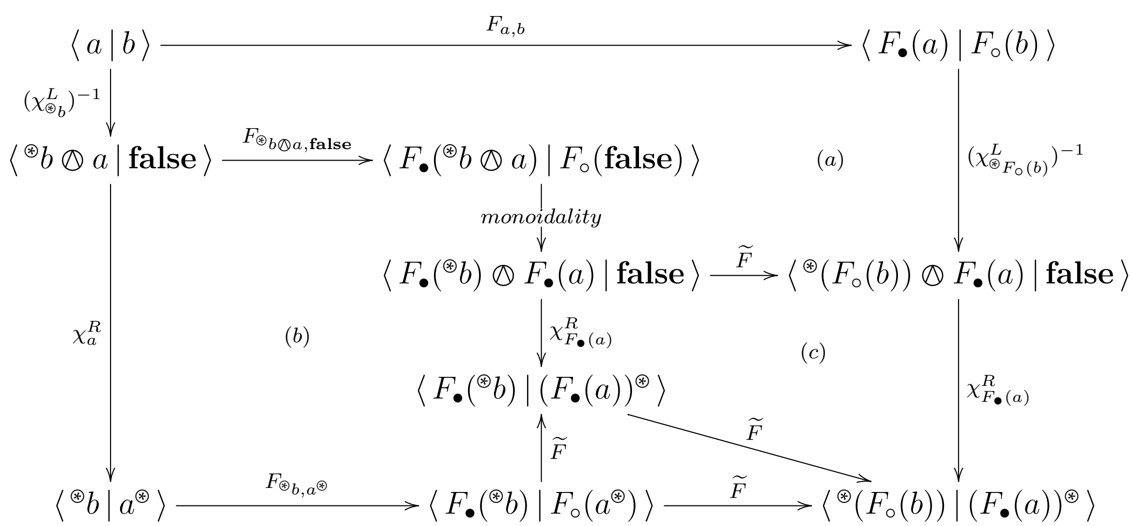
In order to show that the coherence diagram (22) commutes in every ambidextrous chirality, we decompose it into a jigsaw of four pieces:



where each component (a), (b) and (c) commutes as an instance of the coherence diagram (31), (32) and (33) respectively. This establishes that $\mathcal{F}$ transports every ambidextrous chirality to a helical chirality. Now, we would like to show that every ambidextrous homomorphism

$$F = (F_{\bullet}, F_{\circ}, \widetilde{F}, \overline{F}) \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

between ambidextrous chiralities defines a helical homomorphism between the corresponding helical chiralities. To that purpose, we need to check that the coherence diagram (28) commutes.

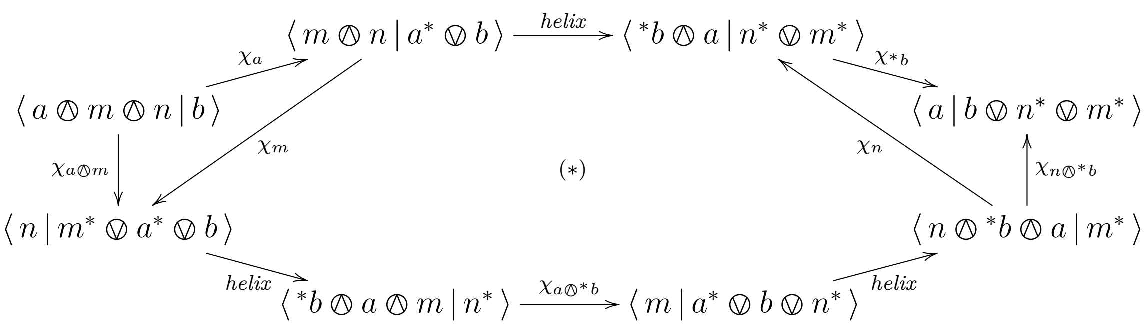


where (a) commutes because of (27) (b) commutes because of (34) and (c) because of naturality of χ^R.

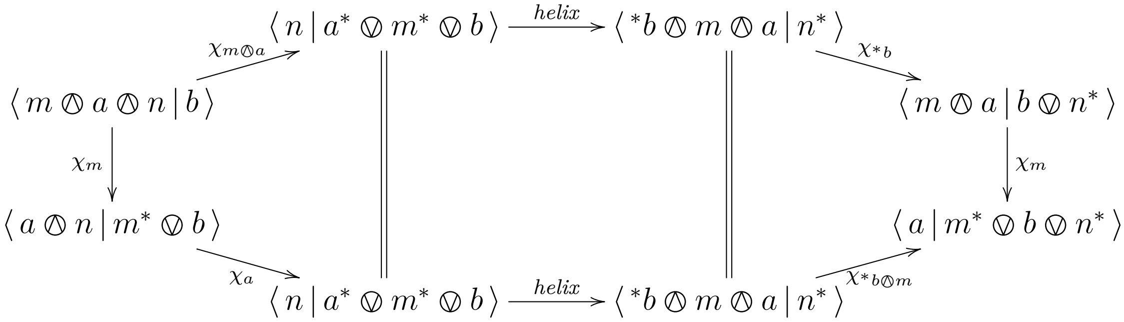
Helical ⇒ Ambidextrous. The 2-functor $\mathcal{G}$ transports every helical chirality $(\mathscr{A}, \mathscr{B}, \chi)$ into the ambidextrous chirality $(\mathscr{A}, \mathscr{B}, \chi^L, \chi^R)$ with left currification χ^L defined as the currification χ of the original chirality, and right currification χ^R defined as the composite morphism

$$\begin{array}{ccccc}
\langle\, a \oslash m \,|\, b \,\rangle & \xrightarrow{\chi^R_{m,a,b}} & \langle\, a \,|\, b \owedge m^* \,\rangle & = & \langle\, a \,|\, ({}^*b)^* \owedge m^* \,\rangle \\
\downarrow{\scriptstyle \chi_{a,m,b}} & & & & \uparrow{\scriptstyle \chi_{{}^*b,a,m^*}} \\
\langle\, m \,|\, a^* \owedge b \,\rangle & \xrightarrow{helix} & \langle\, {}^*(a^* \owedge b) \,|\, m^* \,\rangle & = & \langle\, {}^*b \oslash a \,|\, m^* \,\rangle
\end{array} \tag{35}$$

We need to establish that the two coherence diagrams (32) and (33) of ambidextrous categories commute. Once instantiated with the definitions just given, the coherence diagram (32) looks as follows:



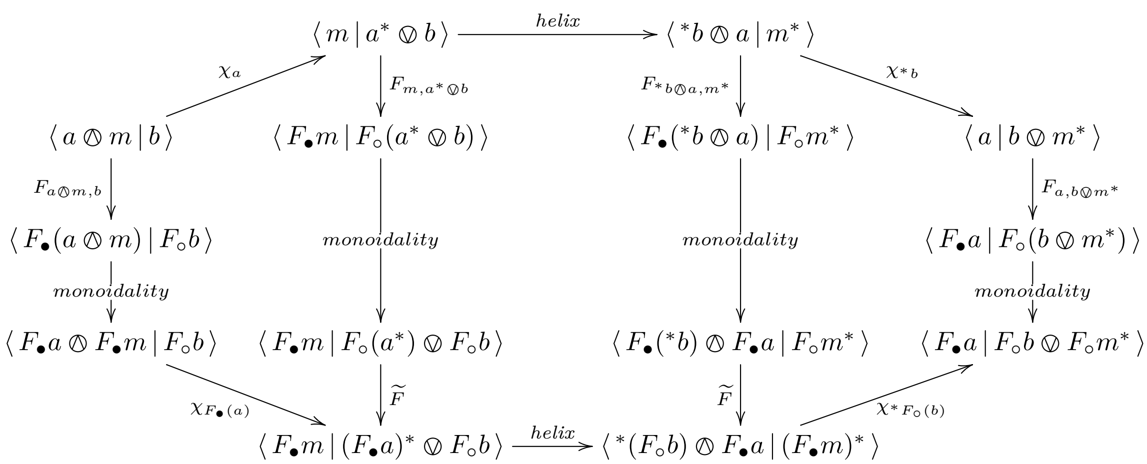
The point is that the inner diagram $(*)$ commutes because it coincides with (22). The two triangles commute by definition of dialogue chiralities and more specifically the coherence diagram (21). Once instantiated with the definition of χ^R above, the coherence diagram (33) is easily shown to commute:



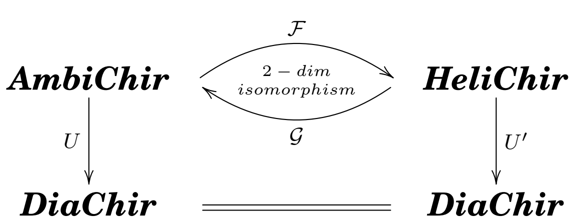
in every helical dialogue chirality. This establishes that the 2-functor $\mathcal{G}$ transports every helical chirality to an ambidextrous chirality. Now, we want to prove that every helical homomorphism

$$F = (F_\bullet, F_\circ, \widetilde{F}, \overline{F}) \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

defines an ambidextrous homomorphism between the underlying ambidextrous chiralities $\mathcal{G}(\mathscr{A}_1, \mathscr{B}_1)$ and $\mathcal{G}(\mathscr{A}_2, \mathscr{B}_2)$. To that purpose, we need to check that the coherence diagram (34) commutes for F with χ^R defined as in (35).



Isomorphism. An easy computation convinces that $\mathcal{F}$ and $\mathcal{G}$ are reverse transformations and thus establish a one-to-one relationship between the ambidextrous chiralities and the helical chiralities. Moreover, the 2-functors do not alter the dialogue chirality underlying a given ambidextrous or helical dialogue chirality. From this follows that the isomorphism lives above the 2-category ***DiaChir*** in the sense that the diagram below commutes.

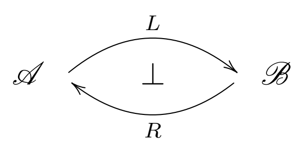


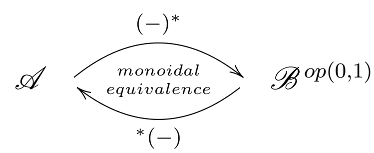
5 Ambidextrous chiralities by transjunctions

We proceed in the same way as we did for dialogue chiralities in our companion paper [6] and reformulate the notion of ambidextrous chirality in the language of transjunctions. A first presentation is given in §5.1 with emphasis given to the properties of the axiom combinators, then a second one with emphasis given to cut combinators in §5.2. One purpose of this presentation based on transjunctions is to provide an algebraic counterpart to the notions of axiom and cut links in proof-net of linear or tensorial logic. The connection becomes visually explicit in the graphical account of the coherence diagrams in string diagrams, see §5.3 for details.

5.1 Formulation based on the axiom combinator

It is not difficult to reformulate the notion of ambidextrous chirality using the language of transjunctions. An ambidextrous chirality is a pair of monoidal categories $(\mathscr{A}, \owedge, \mathbf{true})$ and $(\mathscr{B}, \ovee, \mathbf{false})$ equipped with an adjunction $L \dashv R$ and with a monoidal equivalence



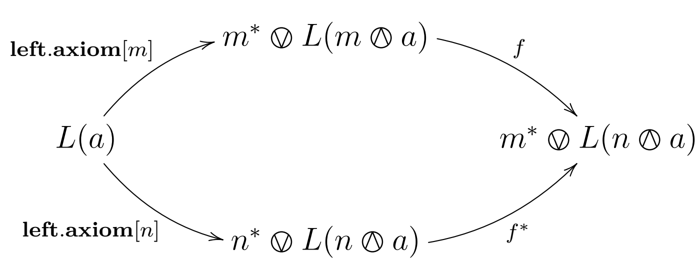


together with two families of transjunctions

$$\begin{array}{rcccl} \mathbf{left}.\mathbf{axiom}[m] & : & L(a) & \longrightarrow & m^* \ovee L(m \owedge a) \\ \mathbf{left}.\mathbf{cut}[m] & : & m \owedge R(m^* \ovee b) & \longrightarrow & R(b) \end{array}$$

$$\begin{array}{rcccl} \mathbf{right}.\mathbf{axiom}[m] & : & L(a) & \longrightarrow & (a \owedge m) \ovee m^* \\ \mathbf{right}.\mathbf{cut}[m] & : & (b \ovee m^*) \owedge m & \longrightarrow & R(b) \end{array}$$

One requires moreover the two coherence diagrams for the left axiom combinator



$$
\begin{array}{ccc}
n^* \ovee L(n \owedge a) & \xrightarrow{\ \mathbf{left}.\mathbf{axiom}[m]\ } & n^* \ovee (m^* \ovee L(m \owedge (n \owedge a))) \\
\Big\uparrow {\scriptstyle \mathbf{left}.\mathbf{axiom}[n]} & & \Big\downarrow {\scriptstyle associativity} \\
 & & (n^* \ovee m^*) \ovee L((m \owedge n) \owedge a) \\
 & & \Big\downarrow {\scriptstyle monoidality} \\
La & \xrightarrow{\ \mathbf{left}.\mathbf{axiom}[m \owedge n]\ } & (m \owedge n)^* \ovee L((m \owedge n) \owedge a)
\end{array}
\tag{36}
$$

for every morphism $f : m \longrightarrow n$ of the category $\mathscr{A}$, followed by the two coherence diagrams for the right axiom combinator

$$
\begin{array}{ccccc}
 & \nearrow^{\mathbf{right}.\mathbf{axiom}[m]} & L(a \owedge m) \ovee m^* & \searrow^{f} & \\
L(a) & & & & L(a \owedge n) \ovee m^* \\
 & \searrow_{\mathbf{right}.\mathbf{axiom}[n]} & L(a \owedge n) \ovee n^* & \nearrow_{f^*} &
\end{array}
$$

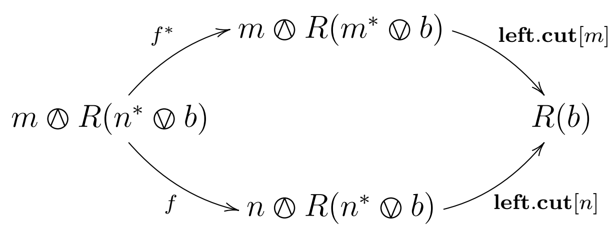
$$
\begin{array}{ccc}
L(a \owedge m) \ovee m^* & \xrightarrow{\ \mathbf{right}.\mathbf{axiom}[n]\ } & (L((a \owedge m) \owedge n) \ovee n^*) \ovee m^* \\
\Big\uparrow {\scriptstyle \mathbf{right}.\mathbf{axiom}[m]} & & \Big\downarrow {\scriptstyle associativity} \\
 & & L(a \owedge (m \owedge n)) \ovee (n^* \ovee m^*) \\
 & & \Big\downarrow {\scriptstyle monoidality} \\
La & \xrightarrow{\ \mathbf{right}.\mathbf{axiom}[m \owedge n]\ } & L(a \owedge (m \owedge n)) \ovee (m \owedge n)^*
\end{array}
\tag{37}
$$

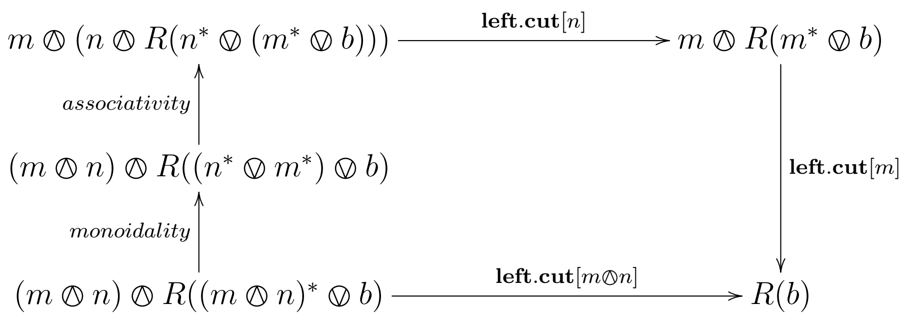
Finally, the important additional coherence diagram tells that the left and the right axiom combinators may be commuted in the following sense:

$$
\begin{array}{cccc}
 & L(a \owedge n) \ovee n^* & \xrightarrow{\ \mathbf{left}.\mathbf{axiom}[m]\ } & m^* \ovee (L(m \owedge (a \owedge n)) \ovee n^*) \\
\nearrow^{\mathbf{right}.\mathbf{axiom}[n]} & & & \Big\| {\scriptstyle associativity} \\
La & & & \\
\searrow_{\mathbf{left}.\mathbf{axiom}[m]} & & & \\
 & m^* \ovee L(m \owedge a) & \xrightarrow{\ \mathbf{right}.\mathbf{axiom}[n]\ } & (m^* \ovee L((m \owedge a) \owedge n)) \ovee n^*
\end{array}
\tag{38}
$$

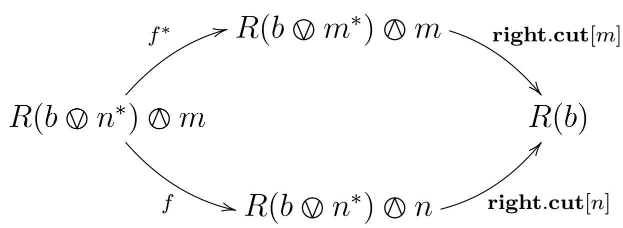
5.2 Alternative formulation based on cuts

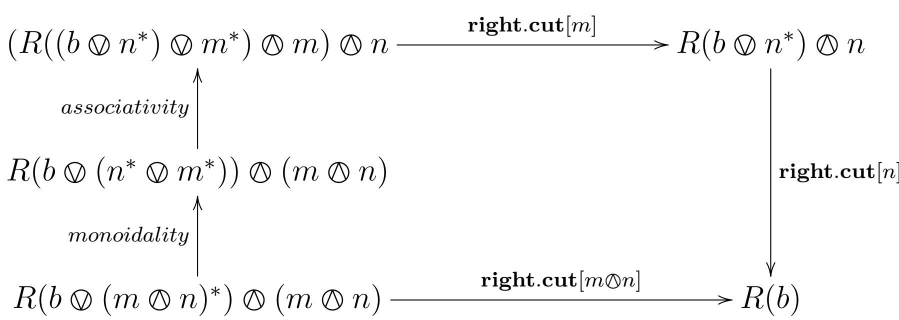
Here, for the sake of completeness, we review the alternative formulation of ambidextrous chiralities based this time on the cut combinator. Just as in the case of the formulation based on the axiom combinator, it combines a series of two coherence diagrams for left cut:





with a series of two coherence diagrams for the right cut:





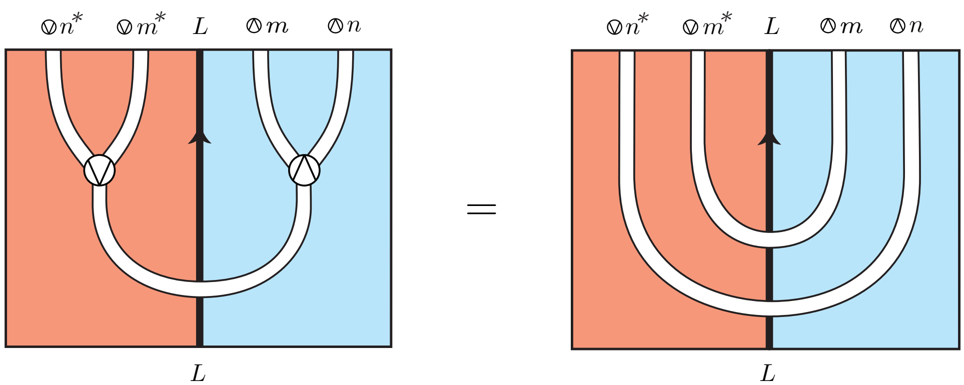
as well as an additional coherence diagram which ensures that the left and the right cut commute in the following sense:

$$
\begin{array}{ccc}
(m \owedge R(m^* \ovee (a \ovee n^*))) \owedge n & \xrightarrow{\ \mathbf{left.cut}[m]\ } & n^* \ovee R(n \owedge a) \\
\Big| & & \big\downarrow \mathbf{right.cut}[n] \\
\scriptstyle associativity & & R(b) \\
\Big| & & \big\uparrow \mathbf{left.cut}[m] \\
m \owedge (R((m^* \ovee a) \ovee n^*) \owedge n) & \xrightarrow{\ \mathbf{right.cut}[n]\ } & m^* \ovee R(m \owedge a)
\end{array}
\tag{39}
$$

These diagrams should commute for all objects a, m, n of the category $\mathscr{A}$, all objects b of the category $\mathscr{B}$ and all morphisms $f : m \longrightarrow n$ of the category $\mathscr{A}$.

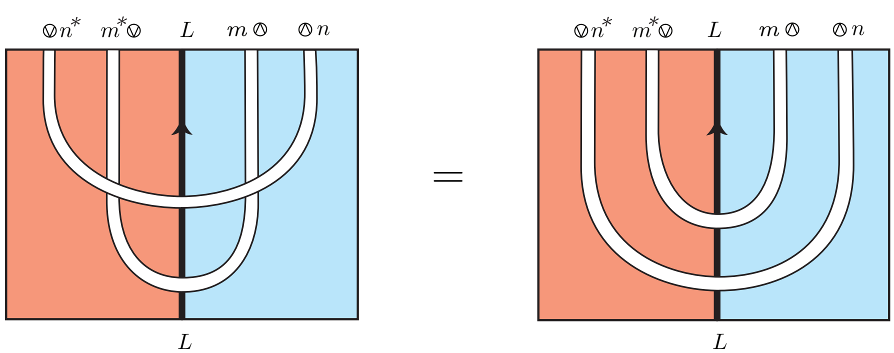
5.3 A pictorial account based on string diagrams

The coherence diagram (36) for the left axiom combinator has been already depicted as (2) in the introduction, while its righthand side variant (37) is depicted in exactly the same way:



(40)

The coherence diagram (38) is drastically different from (36) and (37) and is depicted as



where the apparent braiding is here to reflect the associativity rule which typically permutes the actions of $(m \otimes -)$ with the action of $(- \otimes n)$.

6 Linearly distributive chiralities

In this section, we adapt to an ambidextrous situation the notion of *linearly distributive chirality* introduced in [6]. The adaptation is done in the most straightforward way in §6.1 since it essentially consists in putting side by side a left and a right linearly distributive chirality in the sense of [6]. The notion of duality introduced in [6] is adapted accordingly. Once again, the resulting definition of duality is obtained by putting together a right and a left duality — except for the apparition of the coherence diagram (42) whose task is precisely to combine the left and the right axiom combinators.

6.1 Definition

A linearly distributive chirality is defined as a pair of monoidal categories

$$(\mathscr{A}, \owedge, \mathbf{true}) \qquad\qquad (\mathscr{B}, \ovee, \mathbf{false})$$

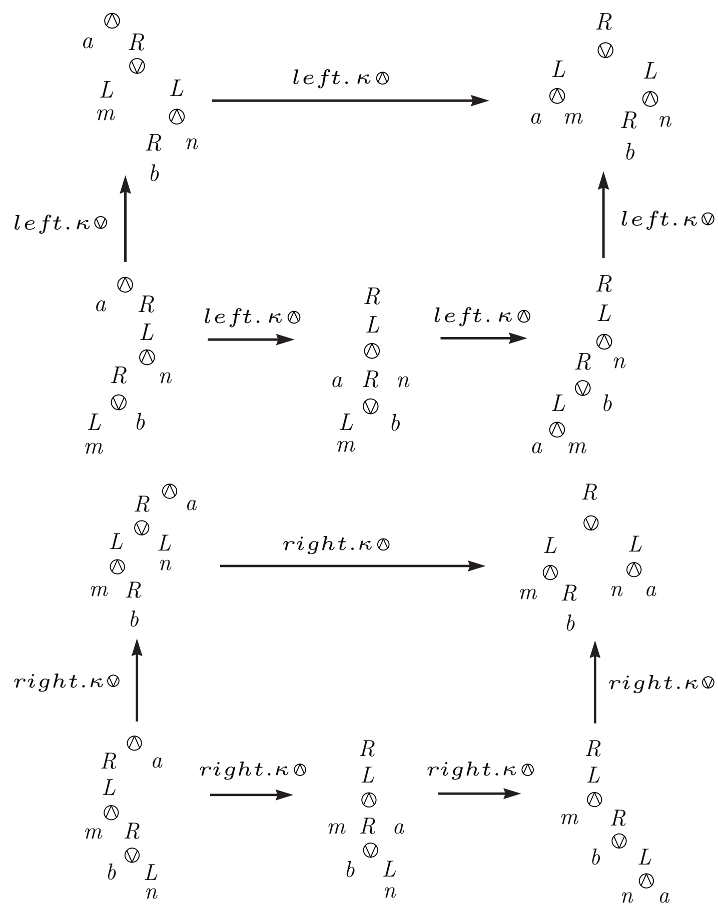
equipped with an adjunction

$$\mathscr{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{B} \qquad (L \dashv R)$$

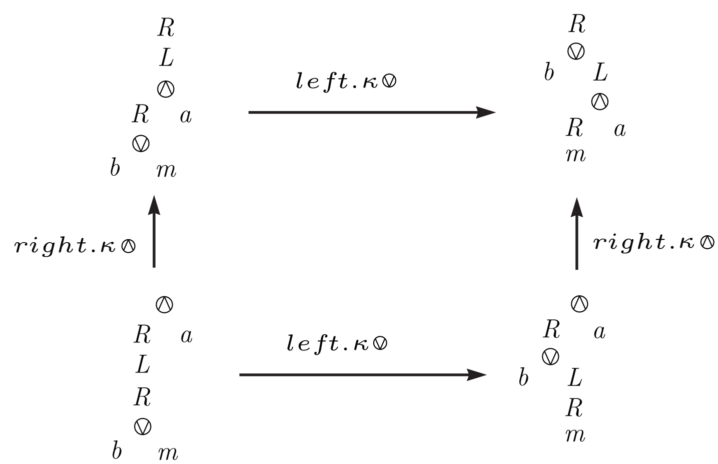
together with the four bimonads

$$\begin{array}{rccccc}
\mathbf{left}.\kappa^{\owedge} & : & m \owedge R(L(a) \ovee b) & \longrightarrow & R(L(m \owedge a) \ovee b) \\
\mathbf{left}.\kappa^{\ovee} & : & L(R(n \ovee b) \owedge a) & \longrightarrow & n \ovee L(R(b) \owedge a) \\
\mathbf{right}.\kappa^{\owedge} & : & R(b \ovee L(a)) \owedge m & \longrightarrow & R(b \ovee L(a \owedge m)) \\
\mathbf{right}.\kappa^{\ovee} & : & L(a \owedge R(b \ovee n)) & \longrightarrow & L(a \owedge R(b)) \ovee n
\end{array} \qquad (41)$$

between the $\owedge$-tensor product and the $\mathscr{B}$-monad of $\mathscr{A}$ on the one hand, and between the $\ovee$-tensor product and the $\mathscr{A}$-comonad of $\mathscr{B}$ on the other hand. Besides the resulting series of commutative diagrams, we ask that the two diagrams below commute



for all objects a, m, n of the category $\mathscr{A}$ and all object b of the category $\mathscr{B}$. Plus a series of other diagrams required in the proof of the following lemma.

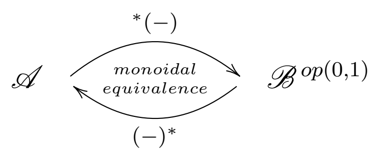


Note that these coherence diagrams are not justified by any of the previous discussions.

6.2 Dualities

We adapt the notion of duality introduced in [6] to an ambidextrous situation.

Definition 4 (duality) *A duality in a linearly distributive chirality* $(\mathscr{A}, \mathscr{B})$ *is defined as a monoidal equivalence*



together with four families of morphisms

$$\mathbf{right}.\mathbf{AX}[m] \quad : \quad \mathbf{true} \quad \longrightarrow \quad R(L(m) \ovee {}^*m)$$

$$\mathbf{right}.\mathbf{CUT}[m] \quad : \quad L(R({}^*m) \owedge m) \quad \longrightarrow \quad \mathbf{false}$$

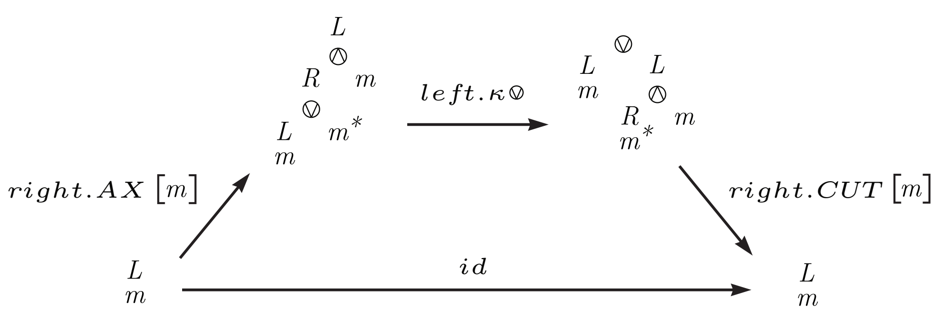
$$\mathbf{left}.\mathbf{AX}[m] \quad : \quad \mathbf{true} \quad \longrightarrow \quad R({}^*m \ovee L(m))$$

$$\mathbf{left}.\mathbf{CUT}[m] \quad : \quad L(m \owedge R({}^*m)) \quad \longrightarrow \quad \mathbf{false}$$

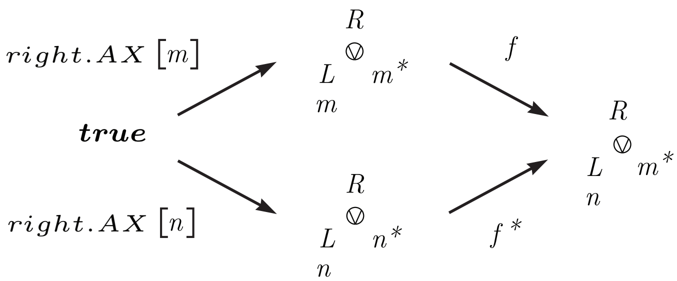
each of them parametrized by the objects m *of the category* $\mathscr{A}$.

These morphisms are moreover required to make the three coherence diagrams below commute.

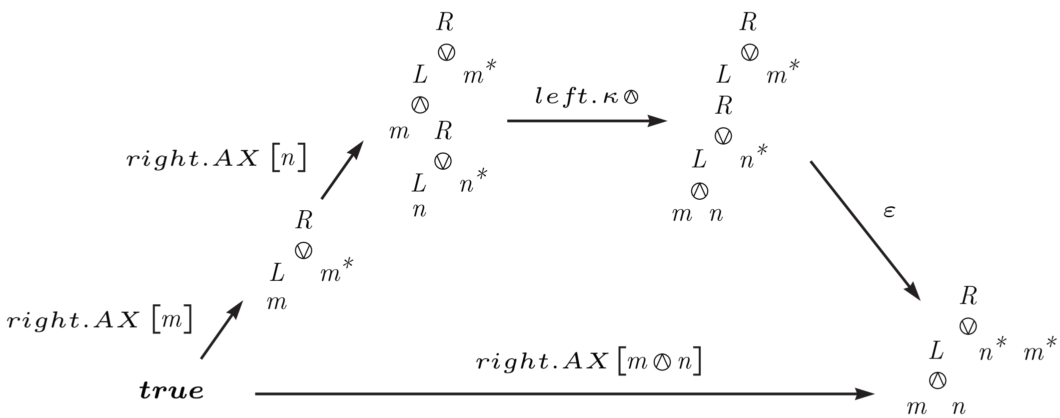
The right coherence diagrams. The first coherence diagram adapts the usual triangular axiom of adjunctions:

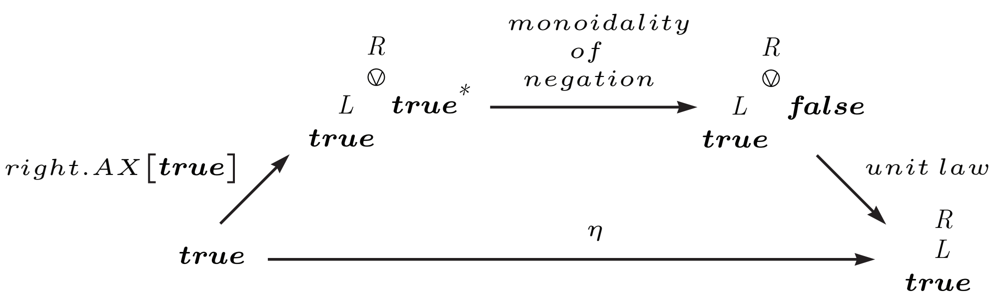


The second coherence diagram means that the family of combinators $\mathbf{AX}[-]$ is dinatural:



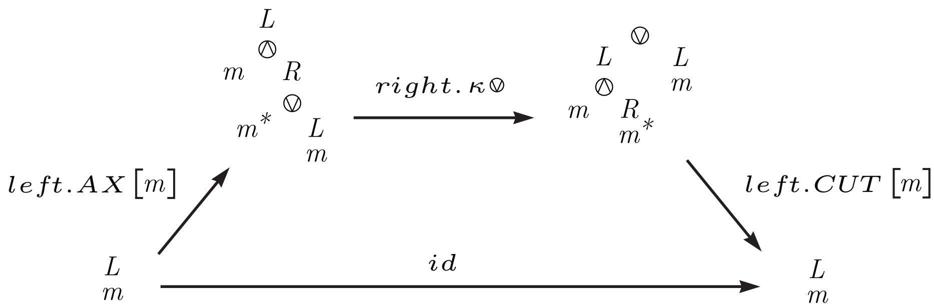
The third coherence diagram expresses a monoidality of the family $\mathbf{AX}[-]$:

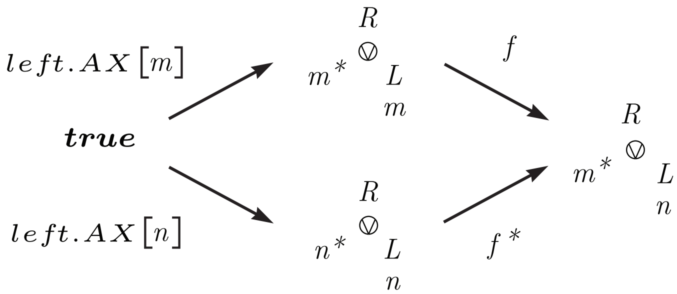


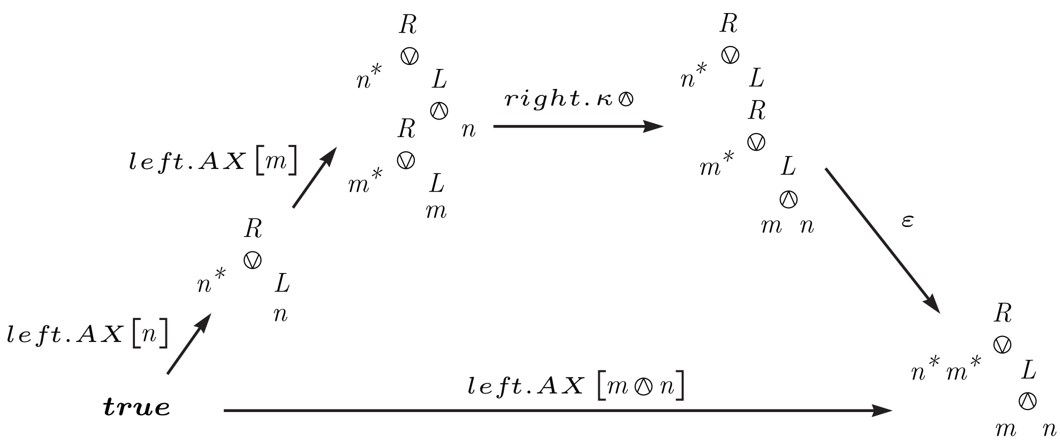


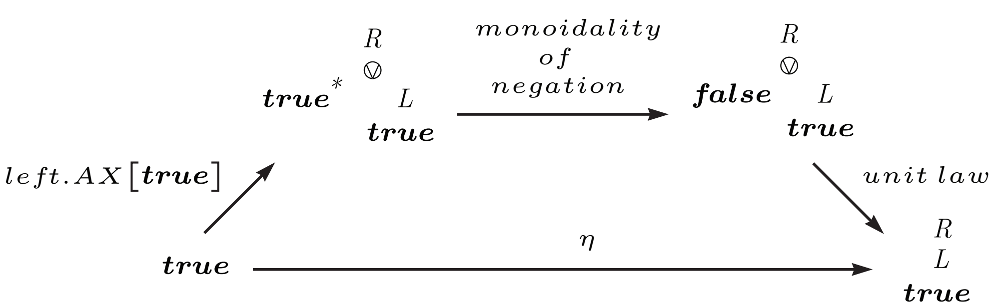
The four coherence diagrams hold for all objects m, n and all morphisms $f : m \to n$ of the category $\mathscr{A}$.

The left coherence diagrams. We need to give the same coherence diagrams on the left side.

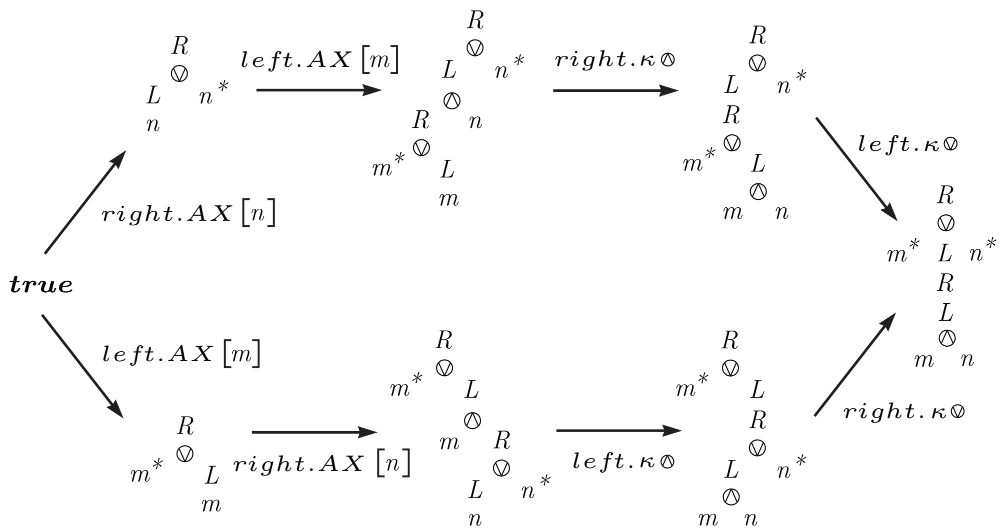

right.κ⊘
left.AX [m]
left.CUT [m]
id


left.AX [m]
left.AX [n]
true
f
f *


right.κ⊘
left.AX [m]
left.AX [n]
ε
true
left.AX [m ⊘ n]


monoidality
of
negation
left.AX [true]
unit law
true
η
false

The mixed coherence diagrams. We also ask this one, which ensures that the two left and right axioms commute:



(42)

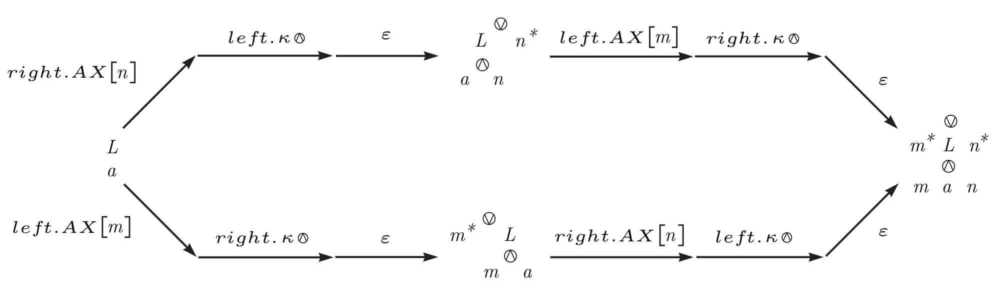
Important remark. Each of these coherence diagrams should be dualized and repeated for the combinator CUT. This is in sharp contrast with the notion of ambidextrous chirality introduced in §4. Indeed, there is apparently no way to recover the coherence diagram for the cut combinator from the corresponding diagrams for the combinator AX.

7 Main theorem

7.1 Preliminary result on helicality

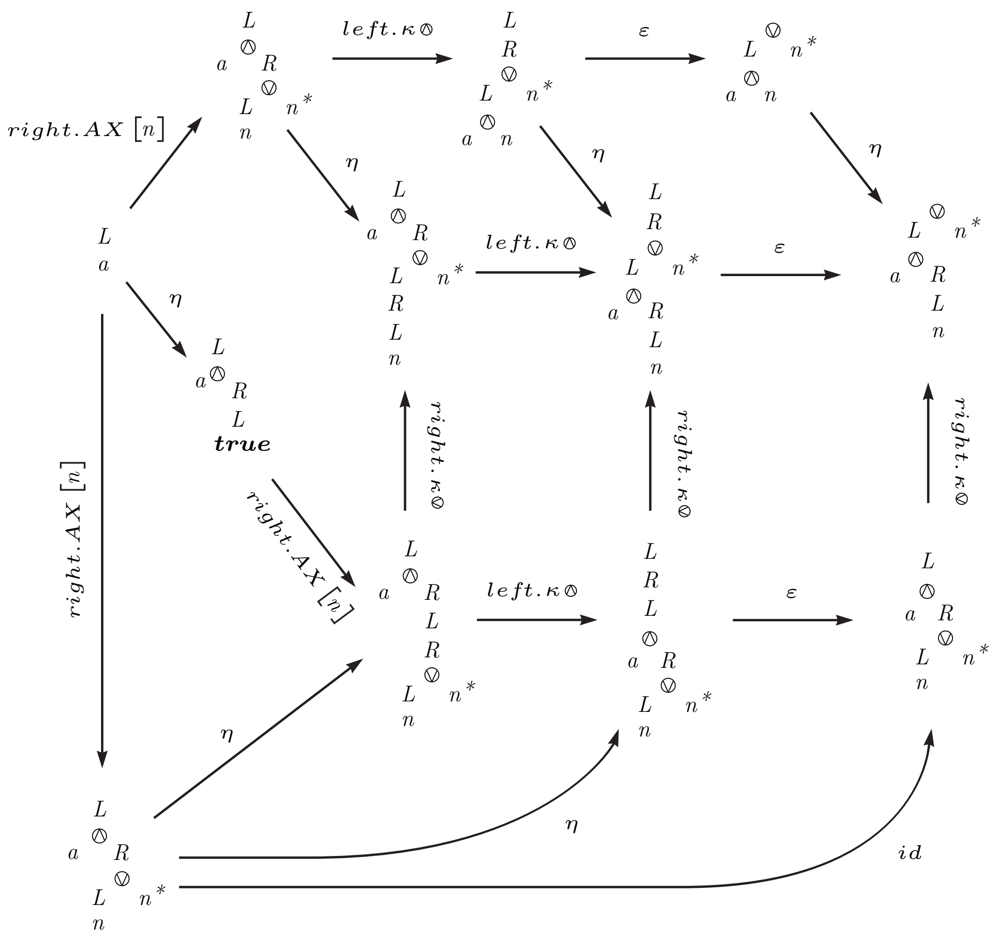
One main benefit of introducing the coherence diagram (42) is that we can establish the following property, which states that the dialogue category is helical.

Proposition 1 *Suppose given a linearly distributive chirality equipped with a duality. In that case, the following diagram*

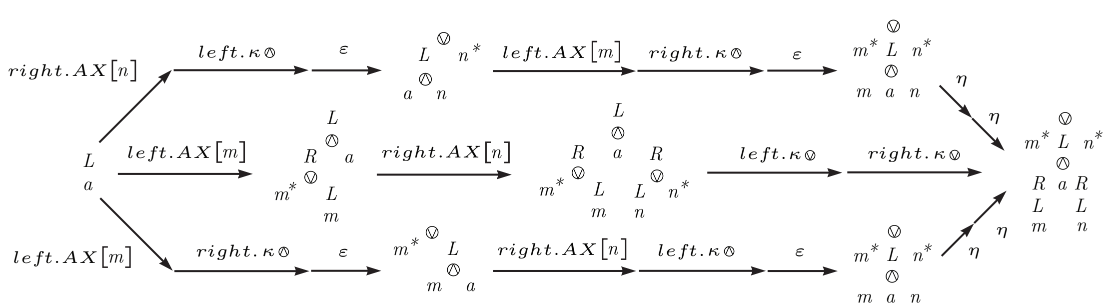


commutes.

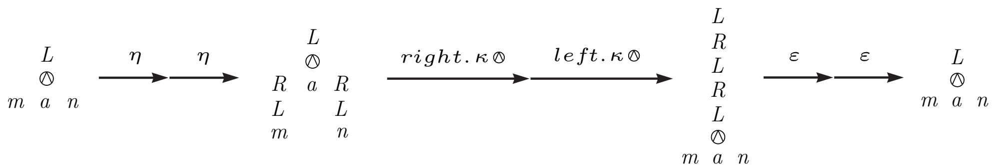
The proof that the left and the right axioms commute is based on the commutative diagram below:



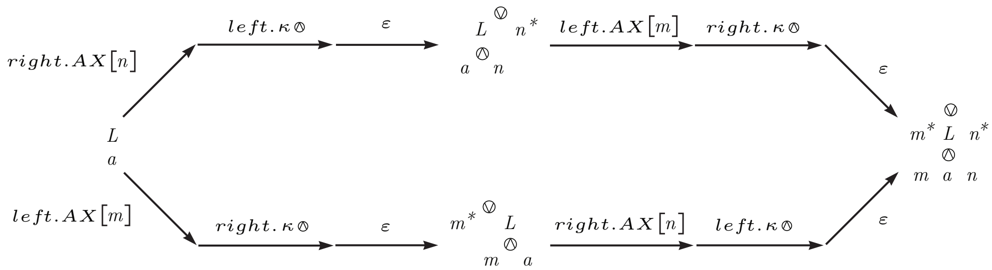
Note that we only need a special case of the mixed coherence diagram (42) with the combinator η : **true** $\to$ RL(**true**) identified with left.**AX**[**true**] up to isomorphism, commuting with the combinator right.**AX**[n]. This commutative diagram enables to establish in turn that the diagram below commutes:



In order to conclude, there simply remains to establish that the two morphisms η appearing in the previous diagram are monos. This immediately follows from the fact that



is equal to the identity., this esta that each η morphism involved in the previous diagram are monos. We conclude that the expected diagram



commutes.

Corollary 2 *Suppose given a linearly distributive chirality with a duality. In that case, the induced dialogue chirality is ambidextrous.*

7.2 From ambidextrous to linearly distributive and back

Given an ambidextrous chirality, one constructs a linearly distributive chirality equipped with a duality.

Proposition 3 *The ambidextrous chirality deduced from the linearly distributive chirality and its duality coincide with the original ambidextrous chirality.*

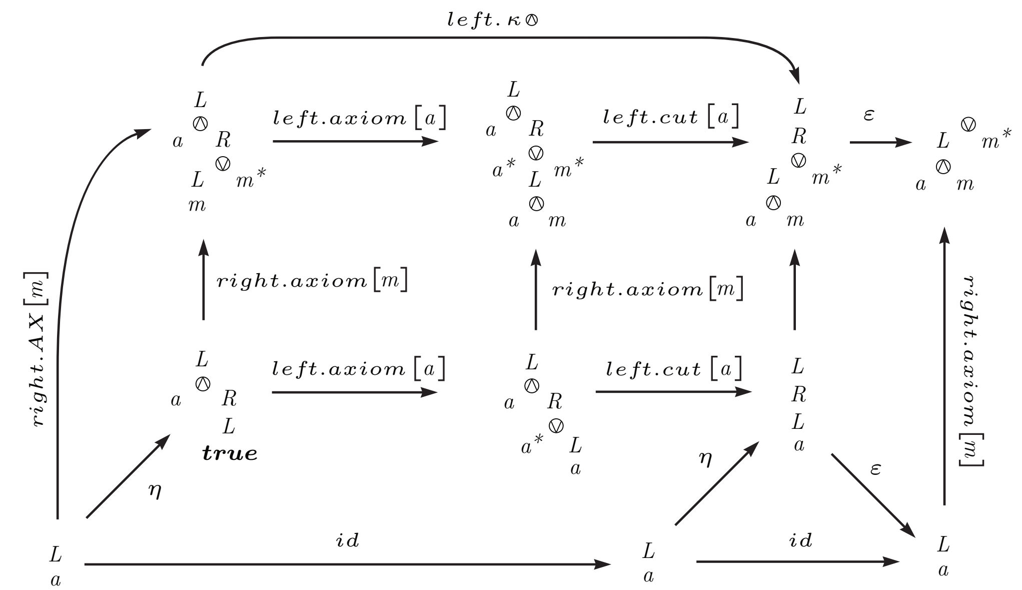
The property is easy to establish. This essentially reduces to establishing that the morphism

$$\mathbf{right.axiom}[m] \quad : \quad L(a) \quad \longrightarrow L(a \oslash m) \owedge {}^*m$$

in the original helical chirality $(\mathscr{A}, \mathscr{B})$ coincides with the morphism

$$L(a) \xrightarrow{\mathbf{right.AX}[m]} \xrightarrow{\mathbf{left.}\kappa^{\oslash}} \xrightarrow{\varepsilon} L(a \oslash m) \owedge {}^*m$$

recovered from the associated discursive pair. This is established by the simple diagram chase below



(43)

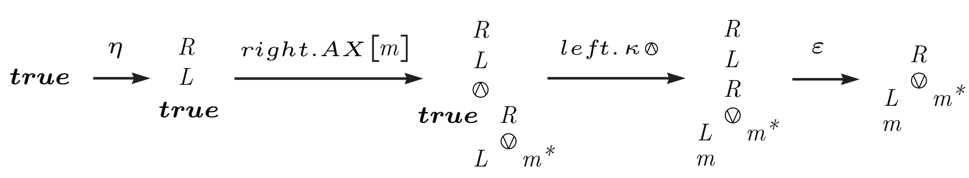
One then needs to check that the morphisms **left**.**axiom**$[m]$, **right**.**cut**$[m]$ and **left**.**cut**$[m]$ coincide with their reconstruction in the discursive pair. Each of the three facts is established by one of the three possible symmetric variants of the diagram (43).

7.3 From linearly distributive to ambidextrous and back

Suppose given a linearly distributive chirality equipped with a duality. We have seen how to construct an ambidextrous chirality from it. The question we would like to address here is whether the associated linearly distributive chirality coincides with the original one. The first step is to check that the morphism

$$\mathbf{right}.\mathbf{AX}[m] \quad : \quad \mathbf{true} \quad \longrightarrow \quad R(L(m) \oslash {}^{*}m)$$

of the original duality coincides with the morphism



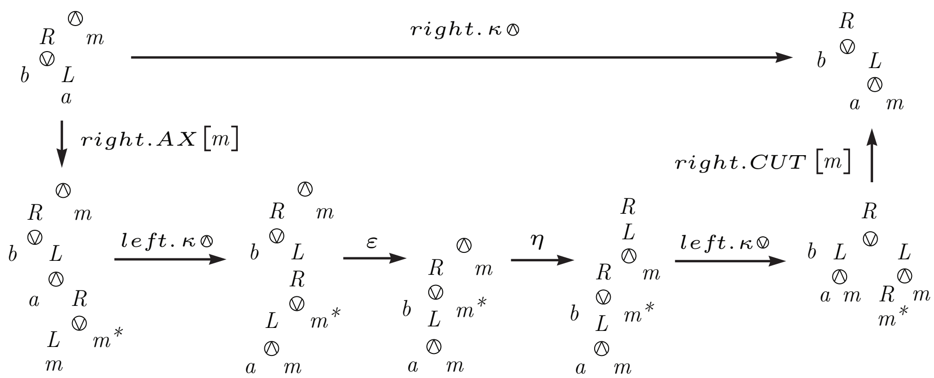
which corresponds to the morphism reconstructed from the chirality. This is essentially immediate. One needs to do the same for the three other components

left.**AX**$[m]$, **right**.**CUT**$[m]$ and **left**.**CUT**$[m]$ of the original duality. This is done in just the same way, by applying the appropriate symmetry to the case just treated.

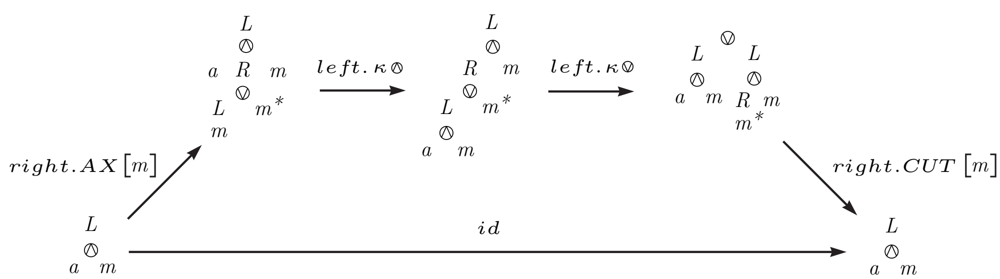
Now, the main difficulty lies in the second part of the proof, which consists in establishing that the distributivity law

$$\textbf{right}.\kappa^{\oslash} \quad : \quad R(b \oslash L(a)) \oslash m \quad \longrightarrow \quad R(b \oslash L(a \oslash m))$$

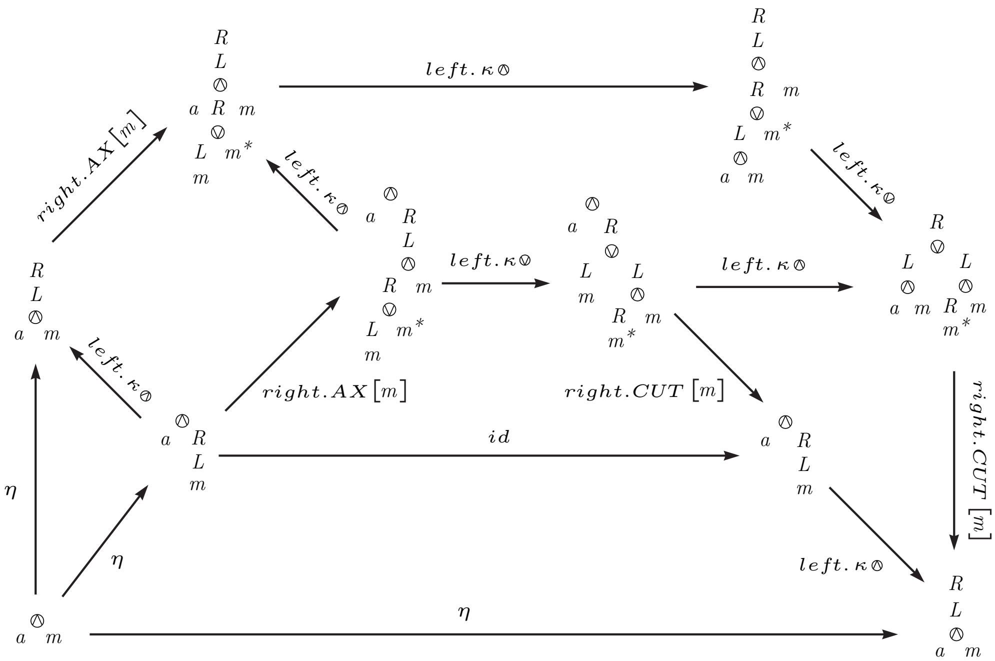
of the original discursive pair coincides with the morphism reconstructed from the associated chirality. This amounts to establishing that the diagram below



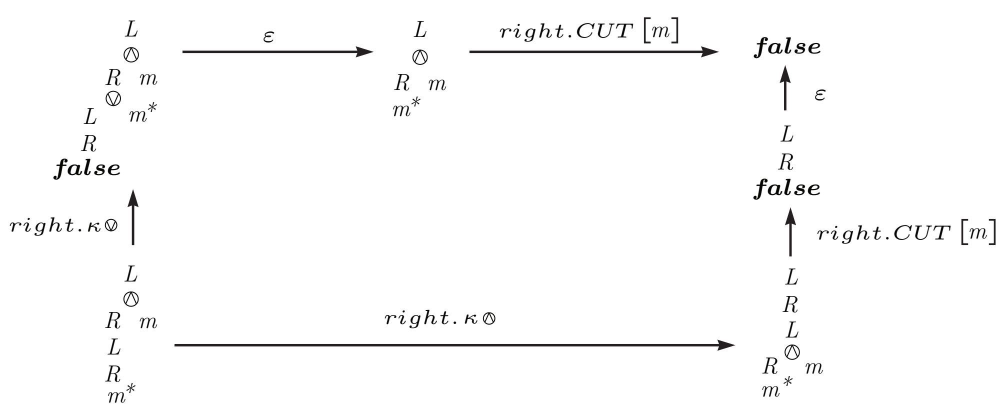
commutes. This is true, but not so easy to establish, although it boils down to producing the appropriate diagram chase. We start by establishing that



commutes as follows:



We then observe that the diagram below



commutes. Then, we get the final diagram chase:

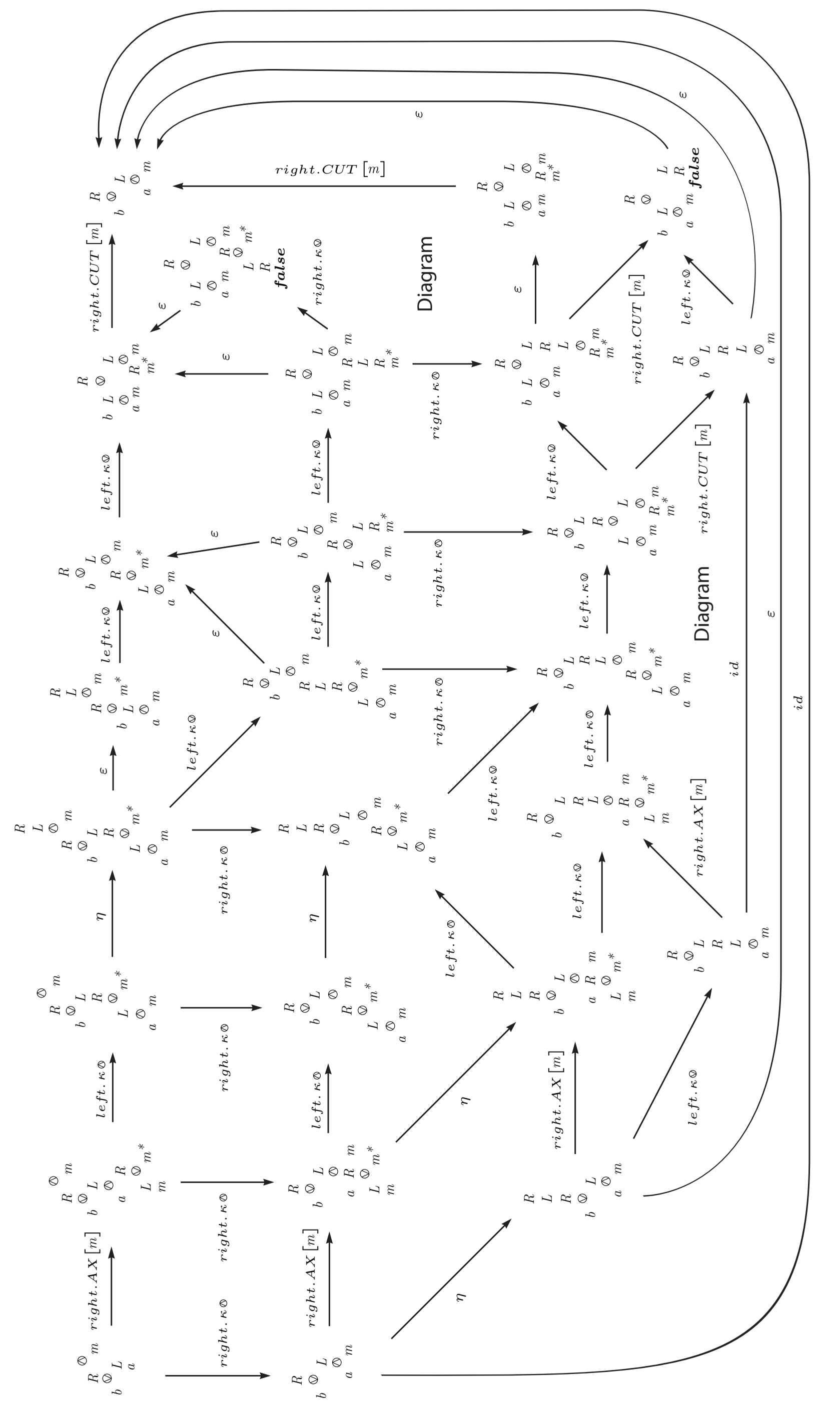


Figure 1: Commutative diagram

7.4 Main theorem

Proposition 4 *There is a one-to-one relationship between the following notions:*

- *an ambidextrous dialogue chirality,*
- *a linearly distributive chirality with a duality.*

8 Back to the eta-expansion law

At this point, we are ready to show that the recipe given in the introduction in order to η-expand the $\mathbf{left.axiom}[Rm]$ combinator works, at least when one picks the coercion isomorphism (5) uniquely characterized by the property of making the diagram (7) commute. Given a coercion isomorphism (whatsoever)

$$L({}^*m) \xrightarrow{\quad isomorphism \quad} (Rm)^*$$

we let

$$\mathbf{candidate}[m] \quad : \quad La \longrightarrow (Rm)^* \ovee L((Rm) \owedge a)$$

denote the composite morphism (42) defined in the introduction. We are looking for a definition of the coercion (5) to ensure that $\mathbf{candidate}[a]$ coincides with $\mathbf{left.axiom}[(Rm)]$. A first observation is that it is sufficient to establish the equality for the particular case where $a = \mathbf{true}$. The reason is that in an ambidextrous category, the $\mathbf{left.axiom}[Rm]$ combinator factors as

$$\begin{array}{ccc}
L(RL(\mathbf{true}) \owedge a) & \xrightarrow{\mathbf{left.axiom}[Rm]} & L(R((Rm)^* \ovee LRm) \owedge a) \\
\uparrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \mathbf{right.}\kappa^{\owedge}} \\
 & & RL((Rm)^* \ovee L((Rm) \owedge a)) \\
 & & \downarrow{\scriptstyle \varepsilon} \\
La & \xrightarrow{\mathbf{left.axiom}[Rm]} & (Rm)^* \ovee L((Rm) \owedge a)
\end{array}$$

Similarly, it is possible to apply the mixed coherence diagram in order to establish that the composite morphism $\mathbf{candidate}[m]$ factors in exactly the same way:

$$\begin{array}{ccc}
L(RL(\mathbf{true}) \owedge a) & \xrightarrow{\mathbf{candidate}[m]} & L(R((Rm)^* \ovee LRm) \owedge a) \\
\uparrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \mathbf{right.}\kappa^{\owedge}} \\
 & & RL((Rm)^* \ovee L((Rm) \owedge a)) \\
 & & \downarrow{\scriptstyle \varepsilon} \\
La & \xrightarrow{\mathbf{candidate}[m]} & (Rm)^* \ovee L((Rm) \owedge a)
\end{array}$$

From this follows that it is sufficient to check the identity in the specific case when $a = \mathbf{true}$. This amounts to showing that the diagram

$$\begin{array}{ccccc} LRL(\mathbf{true}) & \xrightarrow{\mathbf{right.axiom}[m]} & LR(L(^*m) \ovee m) & \xrightarrow{\mathbf{left}.\kappa^{\ovee}} & L(^*m) \ovee LRm \\ \uparrow{\scriptstyle \eta} & & & & \downarrow{\scriptstyle coercion} \\ L(\mathbf{true}) & & \xrightarrow{\qquad\mathbf{left.axiom}[Rm]\qquad} & & (Rm)^* \ovee LRm \end{array} \tag{44}$$

commutes for a well-chosen coercion map. Then, the left distributivity law

$$\mathbf{left}.\kappa^{\ovee} \quad : \quad LR(L(^*m) \ovee m) \quad \longrightarrow \quad L(^*m) \ovee LRm$$

used in the construction of $\mathbf{candidate}[m]$ factors as:

$$\begin{array}{ccccc} & \overset{\mathbf{left.axiom}[L(^*m)]}{\nearrow} & L(^*m) \ovee L(^*(L(^*m)) \owedge R(L(^*m) \ovee m) & \overset{\mathbf{left.cut}[L(^*m)]}{\searrow} & \\ LR(L(^*m) \ovee m) & & \xrightarrow{\qquad\mathbf{left}.\kappa^{\ovee}\qquad} & & L(^*m) \ovee LRm \end{array}$$

From this follows after an elementary computation that the diagram (44) commutes when the diagram below commutes:

$$\begin{array}{ccccc} & \overset{\mathbf{right.axiom}[m]}{\nearrow} & ^*(L(^*m)) \owedge R(L(^*m) \ovee m) & \overset{\mathbf{left.cut}[L(^*m)]}{\searrow} & \\ ^*(L(^*m)) \owedge RL(\mathbf{true}) & & & & Rm \\ \uparrow{\scriptstyle \eta} & & & & \downarrow{\scriptstyle coercion} \\ ^*(L(^*m)) & & ===================== & & ^*(L(^*m)) \end{array}$$

This last diagram commutes when one replaces $\mathbf{right.axiom}[m]$ by the following diagram:

$$\begin{array}{ccc} L(\mathbf{true}) & \xrightarrow{\mathbf{right.axiom}[m]} & L(^*m) \ovee m \\ {\scriptstyle \mathbf{left.axiom}[Rm]}\downarrow & & \downarrow{\scriptstyle coercion} \\ (Rm)^* \ovee LRm & \xrightarrow{\varepsilon} & (Rm)^* \ovee m \end{array}$$

This shows that if the coercion is an isomorphism which makes the diagram commutes, then it resolves the η-expansion problem.

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