# Braided notions of dialogue categories

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**Abstract**

A dialogue category is a monoidal category equipped with an exponentiating object $\bot$ called its tensorial pole. In a dialogue category, every object $x$ is thus equipped with a left negation $x \multimap \bot$ and a right negation $\bot \multimapinv x$. An important point of the definition is that the object $x$ is not required to coincide with its double negation. Our main purpose in the present article is to formulate two non commutative notions of dialogue categories – called helical and balanced dialogue categories. In particular, we show that the category of left $H$-modules of arbitrary dimension on a ribbon Hopf algebra $H$ defines a balanced dialogue category $Mod(H)$ whose tensorial pole $\bot$ is the underlying field $k$. We explain how to recover from this basic observation the well-known fact that the full subcategory of finite dimensional left $H$-modules defines a ribbon category.

## 1 Introduction

Much work has been devoted in the past thirty years in order to understand the interaction between the algebraic properties of quantum groups and the topological invariants of low-dimensional manifolds. These investigations have been generally performed in categories of finite dimensional representations where the duality is involutive, in the sense that the canonical map $A \to A^{**}$ transporting a space $A$ to its bidual space $A^{**}$ is invertible. These inquiries have lead to the discovery of several important notions of non commutative monoidal categories, most notably among them:

- the notion of cyclic category (also called pivotal category) formulated by Freyd and Yetter in [14]
- the notion of ribbon category (also called tortile category) formulated and studied by Turaev [29], Shum [27], Joyal and Street [17].

One primary ambition of the present work is to adapt these two well-established notions (cyclic and ribbon categories) to situations where the duality $A \mapsto A^*$ is not necessarily involutive. To that purpose, we start from the primitive kind of duality supported by the notion of dialogue category. Recall that a dialogue category is a monoidal category $\mathscr{C}$ equipped with a tensorial pole $(\bot, \varphi, \psi)$ defined as an object $\bot$ together with a representation

$$\varphi_{x,y} \quad : \quad \mathscr{C}(x \otimes y, \bot) \quad \cong \quad \mathscr{C}(y, x \multimap \bot)$$

of the functor

$$y \mapsto \mathscr{C}(x \otimes y, \bot) \quad : \quad \mathscr{C}^{op} \longrightarrow Set$$

for each object $x$, and with a representation

$$\psi_{x,y} \quad : \quad \mathscr{C}(x \otimes y, \bot) \quad \cong \quad \mathscr{C}(x, \bot \multimapinv y)$$

of the functor

$$x \mapsto \mathscr{C}(x \otimes y, \bot) \quad : \quad \mathscr{C}^{op} \longrightarrow Set$$

for each object $y$. The object $\bot$ is often called the tensorial pole of the dialogue category $\mathscr{C}$, without further mention of the natural bijections $\varphi$ and $\psi$. This slight abuse of terminology is justified by the fact that once the object $\bot$ of the tensorial pole $(\bot, \varphi, \psi)$ is known, the objects $x \multimap \bot$ and $\bot \multimapinv x$ are uniquely defined modulo isomorphism, and the natural bijections $\varphi$ and $\psi$ are uniquely defined modulo an automorphism of the object $\bot$. A typical example of dialogue category is provided by the category $Mod(H)$ of representations (that is, of left $H$-modules) of arbitrary dimension associated to a Hopf algebra $H$ whose antipode is invertible. The pole $\bot$ of the dialogue category $Mod(H)$ is typically defined as the base field $k$, but other choices are possible. As a matter of fact, any choice of a distinguished object $\bot$ in a monoidal biclosed category $\mathscr{C}$ (closed on the left and on the right) like the category $Mod(H)$ defines a dialogue category with tensorial pole $\bot$. This makes the notion of dialogue category quite ubiquitous... which justifies to look for cyclic as well as ribbon notions of dialogue categories. This question leads to the definitions of helical, cyclic, balanced and ribbon dialogue categories formulated in the course of the article.

Besides the connections to representation theory, this work is part of a research program in logic and computer science, whose general purpose is to recast the dialogical interpretation of proofs and programs in the language of contemporary algebra. This algebraic account of game semantics is supported by the analysis of a primitive logic of tensor and negation — called tensorial logic — whose formulas may be seen as dialogue games, and whose proofs describe total innocent strategies, see [23] for details. This tensorial logic is obtained by relaxing the hypothesis that negation $A \mapsto A^{\bot}$ is involutive in Girard's linear logic [15]. From the categorical point of view, shifting from linear logic to tensorial logic means shifting from $*$-autonomous categories to dialogue categories. Recall that

**Definition 1 ($*$-autonomous categories)** *A $*$-autonomous category is a dialogue category where the tensorial pole $\bot$ is dualizing — this meaning that the two canonical morphisms*

$$\eta : x \longrightarrow (\bot \multimapinv x) \multimap \bot \qquad\qquad \eta' : x \longrightarrow \bot \multimapinv (x \multimap \bot)$$

*are isomorphisms for all objects $x$.*

This leads to the question of extending to dialogue categories the rich body of tools and concepts already existing for linear logic and $*$-autonomous categories. A typical illustration is provided by the formulation of non commutative variants of linear logic – either cyclic [30] braided [11] ribbon [12] or non-commutative [1] where "non-commutative" means that two tensor products (cyclic, commutative) organize the logic. On the algebraic side, one should mention Blute's construction [5] of a braided $*$-autonomous category of representations of a quasitriangular Hopf algebra $H$, together with the subsequent paper with on models of Ruet's noncommutative linear logic [6]. Unfortunately, the limitation to $*$-autonomous categories makes the construction somewhat artificial, and severely restricts their scope of application. One motivation of the present paper is thus to demonstrate that these constructions extend in the most natural way to dialogue categories. The shift to dialogue categories enables to enscope the important example of the category $Mod(H)$ of modules over a braided Hopf algebra $H$. This category is not $*$-autonomous, but it contains a full subcategory: the category of finite dimensional representations, which is a ribbon category.

This article mainly introduces the notions of helical and balanced dialogue category, and illustrates them with elementary examples. A coherence theorem for these non commutative notions of dialogue categories is then established in a companion paper [24]. It should be mentioned that the coherence theorem is achieved by adapting traditional proof-theoretic ideas (cut-elimination, etc.) to this topological situation. Typically, the first step of the construction of the free balanced dialogue category is to introduce a braided and twisted variant of tensorial logic — called ribbon logic. One establishes then that the free balanced dialogue category has formulas of ribbon logic as objects, and proofs of ribbon logic (modulo proof equality) as morphisms. The coherence theorem is called the proof-as-tangle theorem because it reduces the equality of two proofs $\pi_1$ and $\pi_2$ of ribbon logic — modulo the syntactic equalities provided by the logic — to the equality of their interpretation $[\pi_1]$ and $[\pi_2]$ as ribbon tangles — modulo deformation. It appears in the end that the interpretation $\pi \mapsto [\pi]$ of a proof as a ribbon tangle living in the free dialogue category provides a topological refinement to the game-theoretic interpretation of proofs as interactive strategies, see [24] for details.

**Plan of the paper.** After introducing a fractional notation for tensor and negation in § 2, we formulate a notion of *helical* dialogue category in § 3. Somewhat surprisingly, we explain in § 4 that a notion of topological twist arises on the tensorial pole $\perp$ in every helical dialogue category. This observation leads to the definition of a *cyclic* dialogue category is defined as a helical dialogue category where the twist coincides with the identity. After recalling the notion of braided, balanced and ribbon category in § 5, we formulate our notion of *balanced* dialogue category in § 6. We conclude the article by illustrating in § 7 the notion of balanced dialogue category with a series of examples coming from quantum group theory and categorical semantics.

# 2 A fractional notation

In order to establish some of our coherence diagrams, we find useful to introduce a fractional notation for implication and for negation. We explain the notation for monoidal closed categories in § 2.1 and then specialize it to dialogue categories in § 2.2.

## 2.1 Monoidal closed categories

Recall that a monoidal closed category is defined as a monoidal category equipped with a family of functors

$$(x \multimap -) \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C} \qquad\qquad (- \multimapinv x) \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

such that $(x \multimap -)$ is right adjoint to the functor $(x \otimes -)$ and $(- \multimapinv x)$ is right adjoint to the functor $(- \otimes x)$ for every object $x$ of the category $\mathscr{C}$. In the fractional notation, we write

$$x\,y \;:=\; x \otimes y \qquad\qquad \frac{y}{x}\!\circ \;:=\; x \multimap y \qquad\qquad \circ\!\frac{y}{x} \;:=\; y \multimapinv x$$

We find this fractional notation convenient to manipulate towers of implications. A typical illustration is the fact that the evaluation and coevaluation maps

$$x \otimes (x \multimap y) \xrightarrow{lev} y \qquad\qquad (y \multimapinv x) \otimes x \xrightarrow{rev} y$$

$$y \xrightarrow{colev} x \multimap (x \otimes y) \qquad\qquad y \xrightarrow{corev} (y \otimes x) \multimapinv x$$

become oriented versions of the familiar fractional equations:

$$x \, \frac{y}{x}\!\circ \xrightarrow{lev} y \qquad\qquad \circ\!\frac{y}{x} \, x \xrightarrow{rev} y$$

$$y \xrightarrow{colev} \frac{xy}{x}\!\circ \qquad\qquad y \xrightarrow{corev} \circ\!\frac{yx}{x}$$

## 2.2 Dialogue categories

In a dialogue category, every implication $x \multimap y$ or $y \multimapinv x$ is restricted to the particular case when $y = \bot$. This enables us to remove the numerator $y = \bot$ from the fractional notation, while keeping the denominator $x$. The benefits of the fractional notation is best illustrated by applying it to the bestiary of canonical morphisms of a dialogue category.

**Basic evaluation maps.** Typically, the basic evaluation maps of a dialogue category

$$leval_x \; : \; x \otimes (x \multimap \bot) \longrightarrow \bot \qquad\qquad reval_x \; : \; (\bot \multimapinv x) \otimes x \longrightarrow \bot$$

are now written as

$$leval_x \quad : \quad x \, \frac{}{x}\!\circ \longrightarrow \bot \qquad\qquad reval_x \quad : \quad \circ\!\frac{}{x} \, x \longrightarrow \bot$$

**Evaluation maps.** We will make a great use in the paper of the two evaluation maps:

$$lev_{y,x} \;:\; y \otimes ((x \otimes y) \multimap \bot) \;\to\; x \multimap \bot \qquad rev_{y,x} \;:\; (\bot \multimapinv (x \otimes y)) \otimes x \;\to\; \bot \multimapinv y$$

which become in the fractional notation:

$$lev_{y,x} \;:\; y \;\frac{\quad\;\;\circ}{xy} \;\longrightarrow\; \frac{\quad\;\;\circ}{x} \qquad\qquad rev_{x,y} \;:\; \frac{\circ\;\;\quad}{xy}\; x \;\longrightarrow\; \frac{\circ\;\;\quad}{y}$$

**Double negation monads.** Every dialogue category $\mathscr{C}$ comes with two canonical adjunctions. The first adjunction describes the functor

$$L \;:\; \mathscr{C} \to \mathscr{C}^{op} \qquad L(x) \;=\; x \multimap \bot$$

as left adjoint to the functor

$$R \;:\; \mathscr{C}^{op} \to \mathscr{C} \qquad R(x) \;=\; \bot \multimapinv x$$

The second adjunction is obtained from the first one by applying the 2-functor $op : \mathscr{C} \mapsto \mathscr{C}^{op}$ which transports a category to its opposite category. It describes the functor

$$R^{op} \;:\; \mathscr{C} \to \mathscr{C}^{op} \qquad R^{op}(x) \;=\; \bot \multimapinv x$$

as left adjoint to the functor

$$L^{op} \;:\; \mathscr{C}^{op} \to \mathscr{C} \qquad L^{op}(x) \;=\; x \multimap \bot$$

The two adjunctions $L \dashv R$ and $R^{op} \dashv L^{op}$ induce two double negation monads $T = R \circ L$ and $T' = L^{op} \circ R^{op}$ on the category $\mathscr{C}$, thus defined as:

$$T \;:\; x \mapsto \bot \multimapinv (x \multimap \bot) \qquad\qquad T' \;:\; x \mapsto (\bot \multimapinv x) \multimap \bot.$$

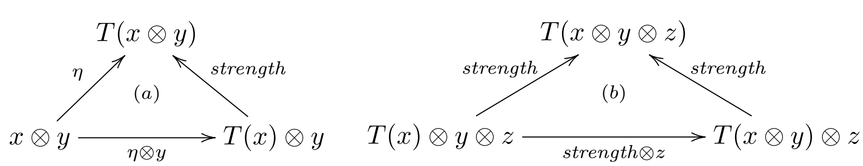
The fractional notation enables to write the two monads $T$ and $T'$ as:

$$T(x) \;=\; \frac{\circ\;\;\quad}{\frac{\quad\;\;\circ}{x}} \qquad\qquad T'(x) \;=\; \frac{\quad\;\;\circ}{\frac{\circ\;\;\quad}{x}}.$$

**Monadic strengths.** Recall that a right strength for a monad $T$ on a category $\mathscr{C}$ is defined as family of morphisms

$$strength_{x,y} \;:\; T(x) \otimes y \;\longrightarrow\; T(x \otimes y)$$

natural in $x$ and $y$, and making the diagrams below commute:



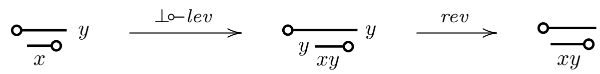
(1)

A left strength on a monad $T$ is defined as a family of morphisms

$$x \otimes T(y) \;\longrightarrow\; T(x \otimes y)$$

natural in $x$ and $y$, and making the symmetric diagrams commute. It is well-known that in any dialogue category, the double negation monad $T$ has a right strength and its companion $T'$ has a left strength. The right strength of $T$ is defined as the family of morphisms:

$strength_{x,y}$ :




(2)

It is easy to check that the family is natural in $x$ and in $y$ and that it makes the two diagrams (1) commute. The left strength of the monad $T'$ is defined in a similar way.

# 3 Helical dialogue categories

The notion of *helical dialogue category* is introduced in this section. The notion is defined in two different but equivalent ways.

- a helical dialogue category is described in § 3.1 and § 3.2 as a dialogue category equipped with a natural bijection between $\mathscr{C}(x \otimes y, \bot)$ and $\mathscr{C}(y \otimes x, \bot)$, satisfying an extra coherence property.
- a helical dialogue category is described in § 3.3 and § 3.4 as a dialogue category equipped with a natural isomorphism between the two negation functors $L : x \mapsto (x \multimap \bot)$ and $R : x \mapsto (\bot \circ\!\!- x)$, satisfying an extra coherence property.

We prove in § 3.5 that the second definition of helical dialogue category implies that the double negation monad $T$ is strong on the left as well as on the right. Once this important fact established, we show in § 3.6 that the two alternative definitions of helical dialogue categories are equivalent.

## 3.1 Helical presheaf

We introduce here the notion of *helical presheaf* on a monoidal category, which we then specialize in the next section to the case of dialogue categories.

**Definition 2 (helical presheaf)** *A helical presheaf on a monoidal category $\mathscr{C}$ is a presheaf*

$$\perp\!\!\!\perp \quad : \quad \mathscr{C}^{\,op} \quad \longrightarrow \quad \mathbf{Set}$$

*equipped with a family of bijections*

$$wheel_{x,y} \quad : \quad \perp\!\!\!\perp(y \otimes x) \quad \longrightarrow \quad \perp\!\!\!\perp(x \otimes y)$$

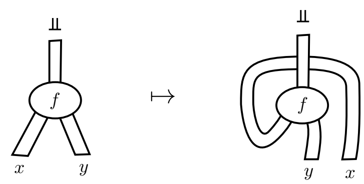
*natural in $x$ and $y$ and required to make the diagram*

$$\begin{array}{ccc}
\perp\!\!\!\perp((y \otimes z) \otimes x) & \xrightarrow{\ associativity\ } & \perp\!\!\!\perp(y \otimes (z \otimes x)) \\
\uparrow{\scriptstyle wheel_{x,y\otimes z}} & & \downarrow{\scriptstyle wheel_{y,z\otimes x}} \\
\perp\!\!\!\perp(x \otimes (y \otimes z)) & & \perp\!\!\!\perp((z \otimes x) \otimes y) \\
\downarrow{\scriptstyle associativity} & & \uparrow{\scriptstyle associativity} \\
\perp\!\!\!\perp((x \otimes y) \otimes z) & \xrightarrow{\ wheel_{x\otimes y,z}\ } & \perp\!\!\!\perp(z \otimes (x \otimes y))
\end{array} \qquad (3)$$

*commute for all objects $x, y, z$ of the category $\mathscr{C}$.*

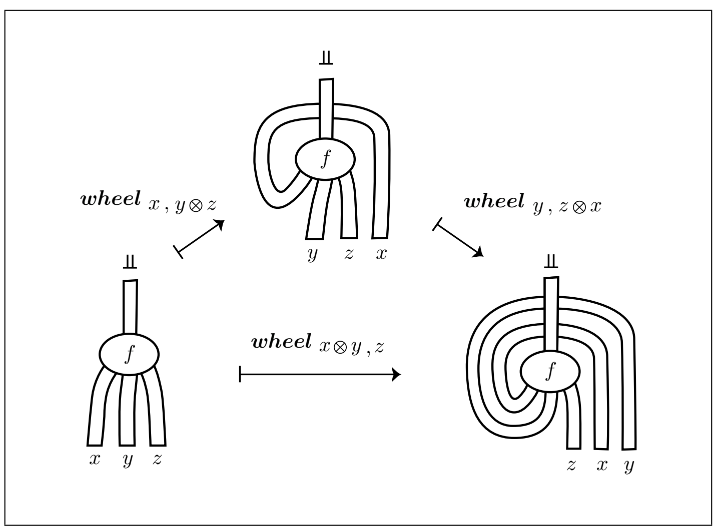
The action of $wheel_{x,y}$ on an element $f \in \perp\!\!\!\perp(x \otimes y)$ is depicted as follows:

$wheel_{x,y}$ :




(4)

In that graphical formulation, the coherence diagram expresses that the diagram below commutes:




It is not very difficult to deduce from the coherence diagram (3) that the diagram

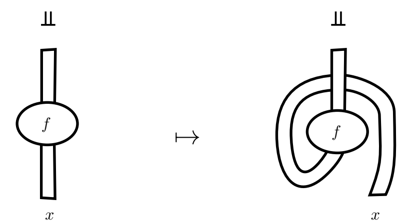
$$\begin{array}{ccc}
\perp\!\!\!\perp(x \otimes e) & \xrightarrow{\ wheel_{x,e}\ } & \perp\!\!\!\perp(e \otimes x) \\
\uparrow{\scriptstyle associativity} & & \downarrow{\scriptstyle associativity} \\
\perp\!\!\!\perp(x) & \xrightarrow{\ id\ } & \perp\!\!\!\perp(x)
\end{array}$$

commutes in any helical refutation category. On the other hand, the other expected coherence diagram

$$\begin{array}{ccc} ⫫(e \otimes x) & \xrightarrow{\;wheel_{e,x}\;} & ⫫(x \otimes e) \\ {\scriptstyle associativity}\uparrow & & \downarrow{\scriptstyle associativity} \\ ⫫(x) & \xrightarrow{\;id\;} & ⫫(x) \end{array} \tag{5}$$

does not commute in general. As a matter of fact, this is an important aspect of our definition: we will see in § 4 that this lack of coherence reflects the existence of a topological "twist" on the refutation presheaf ⫫ arising as follows:

$wheel_{x,I}$ :




## 3.2 Helical dialogue categories [ $wheel_{x,y}$ ]

Every dialogue category $\mathscr{C}$ comes equipped with the presheaf defined as

$$⫫ \quad : \quad x \quad \mapsto \quad \mathscr{C}(x, \bot).$$

Consequently, a helical structure on a dialogue category $(\mathscr{C}, \bot)$ is defined as a family of bijections

$$wheel_{x,y} \quad : \quad \mathscr{C}(x \otimes y, \bot) \quad \longrightarrow \quad \mathscr{C}(y \otimes x, \bot)$$

natural in $x$ and $y$, required to make the diagram below

$$\begin{array}{ccc} \mathscr{C}((y \otimes z) \otimes x, \bot) & \xrightarrow{\;associativity\;} & \mathscr{C}(y \otimes (z \otimes x), \bot) \\ {\scriptstyle wheel_{x,y\otimes z}}\uparrow & & \downarrow{\scriptstyle wheel_{y,z\otimes x}} \\ \mathscr{C}(x \otimes (y \otimes z), \bot) & & \mathscr{C}((z \otimes x) \otimes y, \bot) \\ {\scriptstyle associativity}\downarrow & & \uparrow{\scriptstyle associativity} \\ \mathscr{C}((x \otimes y) \otimes z, \bot) & \xrightarrow{\;wheel_{x\otimes y,z}\;} & \mathscr{C}(z \otimes (x \otimes y), \bot) \end{array} \tag{6}$$

commute for all objects $x, y, z$. Note that this diagram is the same as Diagram (3) where $\mathscr{C}(-, \bot)$ replaces the presheaf ⫫. This leads to the central definition of the article:

**Definition 3 (Helical dialogue category)** *A helical dialogue category is a dialogue category equipped with a helical structure.*

## 3.3 Helical dialogue categories [ $turn_x$ externally ]

In every dialogue category $\mathscr{C}$, there is a one-to-one relationship between the natural isomorphisms

$$turn_x \quad : \quad x \multimap \bot \quad \longrightarrow \quad \bot \multimapinv x \tag{7}$$

and the natural bijections

$$wheel_{x,y} \quad : \quad \mathscr{C}(x \otimes y, \bot) \longrightarrow \mathscr{C}(y \otimes x, \bot) \tag{8}$$

The relationship works as follows: given a natural transformation (7), one defines the function $wheel_{x,y}$ as

$$\begin{array}{ccc} \mathscr{C}(x \otimes y, \bot) & & \mathscr{C}(y \otimes x, \bot) \\ {\scriptstyle \varphi_{x,y}}\big\downarrow & & \big\uparrow{\scriptstyle \psi^{-1}_{y,x}} \\ \mathscr{C}(y, x \multimap \bot) & \xrightarrow{\mathscr{C}(y, turn_x)} & \mathscr{C}(y, \bot \multimapinv x) \end{array} \tag{9}$$

for all objects $x, y$ of the dialogue category $\mathscr{C}$. Conversely, given a natural transformation (8), one defines the morphism $turn_x$ as the image of the identity of $x \multimap \bot$ along the composite function

$$\begin{array}{ccc} \mathscr{C}(x \multimap \bot, x \multimap \bot) & & \mathscr{C}(x \multimap \bot, \bot \multimapinv x) \\ {\scriptstyle \varphi^{-1}_{x, x \multimap \bot}}\big\downarrow & & \big\uparrow{\scriptstyle \psi_{x \multimap \bot, x}} \\ \mathscr{C}(x \otimes (x \multimap \bot), \bot) & \xrightarrow{wheel_{x, x \multimap \bot}} & \mathscr{C}((x \multimap \bot) \otimes x, \bot) \end{array}$$

It is immediate that these define converse translations, and moreover that the transformation (7) is reversible if and only if the associated transformation (8) is reversible. In particular, the inverse of (7) is provided in that case by the image of the identity of $\bot \multimapinv x$ along the function

$$\begin{array}{ccc} \mathscr{C}(\bot \multimapinv x, \bot \multimapinv x) & & \mathscr{C}(\bot \multimapinv x, x \multimap \bot) \\ {\scriptstyle \psi^{-1}_{x, \bot \multimapinv x}}\big\downarrow & & \big\uparrow{\scriptstyle \varphi_{x, \bot \multimapinv x}} \\ \mathscr{C}((\bot \multimapinv x) \otimes x, \bot) & \xrightarrow{wheel^{-1}_{\bot \multimapinv x, x}} & \mathscr{C}(x \otimes (\bot \multimapinv x), \bot) \end{array}$$

Consequently,

**Proposition 1** *A helical structure on a dialogue category may be equivalently defined as a natural isomorphism (7) whose associated natural bijection (8) makes the coherence diagram (6) commute.*

## 3.4 Helical dialogue categories [ $turn_x$ internally ]

The equivalent definition of helical structure formulated in Proposition 1 at the end of the previous § 3.3 is not entirely satisfactory, because it expresses the coherence condition on $turn$ as a commutative diagram living inside the category ***Set*** rather than *inside* the category $\mathscr{C}$. The remaining task of the section will be thus to establish the following proposition.

**Proposition 2** *A helical structure on a dialogue category $\mathscr{C}$ is equivalently defined as a natural isomorphism*

$$turn_x \quad : \quad x \multimap \bot \quad \longrightarrow \quad \bot \multimapinv x$$

*making the coherence diagram*

$$\begin{array}{ccccc}
 & & \bot & & \\
 & \overset{reval}{\nearrow} & & \overset{leval}{\nwarrow} & \\
(\bot \multimapinv x) \otimes x & & & & y \otimes (y \multimap \bot) \\
\uparrow{\scriptstyle turn_x} & & & & \uparrow{\scriptstyle turn_y^{-1}} \\
(x \multimap \bot) \otimes x & & & & y \otimes (\bot \multimapinv y) \\
\uparrow{\scriptstyle lev} & & & & \uparrow{\scriptstyle rev} \\
y \otimes ((x \otimes y) \multimap \bot) \otimes x & & \xrightarrow{turn_{x \otimes y}} & & y \otimes (\bot \multimapinv (x \otimes y)) \otimes x
\end{array} \qquad (10)$$

*commute for all objects $x, y$ of the category $\mathscr{C}$.*

**Remark.** The careful reader may wonder whether the hypothesis of Proposition 2 should also include the coherence diagram below:

$$\begin{array}{ccccc}
 & & \bot & & \\
 & \overset{leval}{\nearrow} & & \overset{reval}{\nwarrow} & \\
I \otimes (I \multimap \bot) & & & & (\bot \multimapinv I) \otimes I \\
 & \underset{associativity}{\searrow} & & \underset{associativity}{\nearrow} & \\
 & I \multimap \bot & \xrightarrow{turn_I} & \bot \multimapinv I &
\end{array} \qquad (11)$$

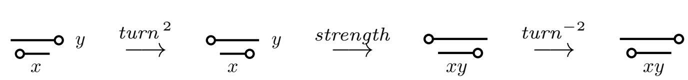
This additional property is not mentioned because it follows from the coherence diagram (10). Hint: take $x = y = I$ and check that the left and right evaluation maps

$$lev \,: I \otimes ((I \otimes I) \multimap I) \longrightarrow I \multimap I \qquad\qquad rev \,: (\bot \multimapinv (I \otimes I)) \otimes I \longrightarrow I \multimapinv I$$

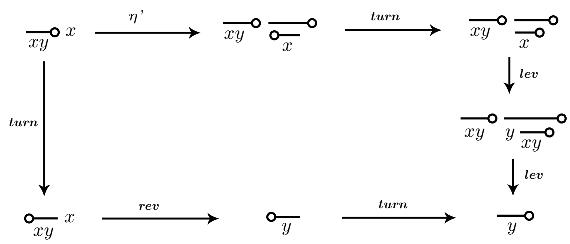
are provided by the expected canonical maps of monoidal categories.

## 3.5 A strong monad on the left and on the right

We have seen in § 2.2 that the double negation monad $T : x \mapsto \bot \multimapinv (x \multimap \bot)$ has a right strength defined in (2). One reason for requiring the coherence diagram (10) is that the companion monad $T' : x \mapsto (\bot \multimapinv x) \multimap \bot$ inherits the right strength of the monad $T$ in that case. The right strength of $T'$ is defined as

$strength'_{x,y}$ :  (12)

In order to establish that the natural transformation $strength'$ defines a right strength on the monad $T'$, one needs to show it makes the diagrams $(a)$ and $(b)$ of Equation (1) commute for the monad $T'$. As a matter of fact, it is nearly immediate from the definition of the map (12) that diagram $(b)$ commutes: in particular, one does not need the coherence diagram (10) in order to establish the property. On the other hand, we will make an extensive use of this coherence diagram in order to establish that Diagram $(a)$ commutes. The first step in the proof is to show that the diagram




(13)

commutes for all $x, y$. This is established by the diagram chase depicted in Figure 1. Every square in the diagram commutes by functoriality of $\otimes$ or by

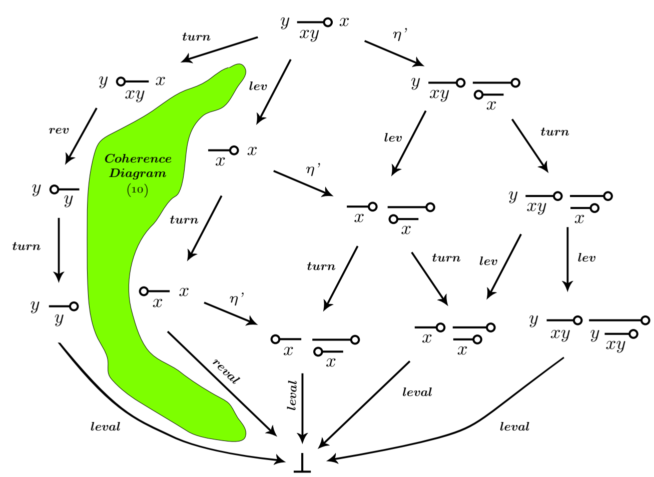



Figure 1: Diagram chase establishing that (13) commutes.

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dinaturality of left evaluation. The only non-trivial piece of the jigsaw is the triangle

$$\begin{array}{ccc} & \bot & \\ {\scriptstyle reval_x} \nearrow & & \nwarrow {\scriptstyle leval_{\bot \multimapinv x}} \\ (\bot \multimapinv x) \otimes x & \xrightarrow{(\bot \multimapinv x)\otimes\eta'} & (\bot \multimapinv x) \otimes ((\bot \multimapinv x) \multimap \bot) \end{array} \tag{14}$$

which happens to commute in every dialogue category, for somewhat obvious reasons. The second and last step of the proof is to establish that diagram $(a)$

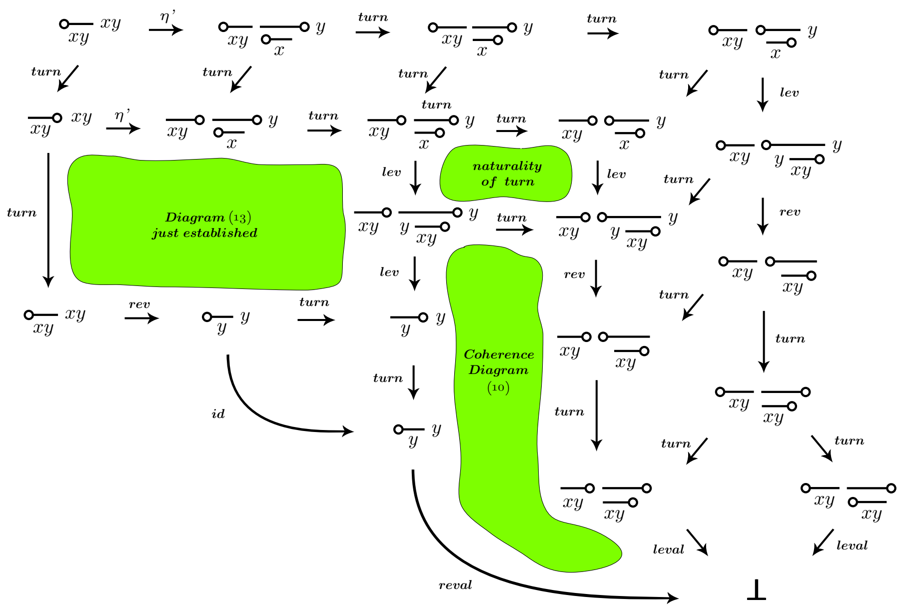



Figure 2: Diagram chase establishing that (12) is a right strength.

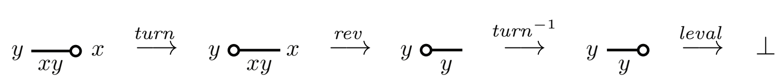
commutes, using the diagram chase in Figure (2). One shows symmetrically that the companion monad $T$ inherits a left strength from the double negation monad $T'$. This establishes that

**Proposition 3** *Suppose that dialogue category $\mathscr{C}$ is equipped with a natural isomorphism turn which satisfies the hypothesis of Proposition 2. Then, the two double negation monads $T$ and $T'$ are strong monads on the left as well as on the right.*

## 3.6 Proof of the main statement [Proposition 2]

At this stage, we are ready to show that the coherence axioms defining a helical structure *turn* in § 3.3 and § 3.4 are equivalent.

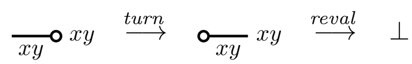
**Direction from §3.3 to §3.4.** This direction is the easy one to establish. The idea is to start from the righthand side of the coherence diagram (10) given by the composite morphism




and to apply to it the function

$$wheel_{\,y,((x\otimes y)\multimap\bot)\otimes x}$$

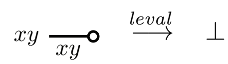
defined in Equation (9). It is not difficult to see that the resulting morphism is




One then applies the function

$$wheel^{-1}_{x\otimes y,(x\otimes y)\multimap\bot}$$

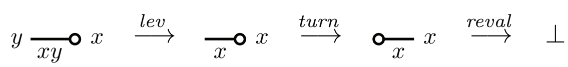
to this morphism and obtains the morphism




Finally, one applies the function

$$wheel_{x,y\otimes((x\otimes y)\multimap\bot)}$$

to this morphism, and obtains the morphism




which coincides with the lefthand side of the coherence diagram (10). Now, the external definition of a helical structure in §3.3 requires that the function

$$wheel_{x,y\otimes((x\otimes y)\multimap\bot)} \circ wheel^{-1}_{x\otimes y,(x\otimes y)\multimap\bot} \circ wheel_{y,((x\otimes y)\multimap\bot)\otimes x}$$

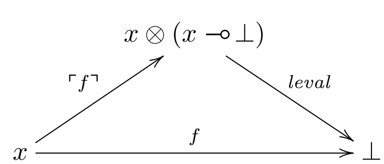
is the identity on the set of morphisms $\mathscr{C}(y \otimes ((x \otimes y) \multimap \bot) \otimes x, \bot)$. This establishes that the two morphisms defining the coherence diagram (10) are equal, and concludes the proof.

**Direction from §3.4 to §3.3.** In order to establish this direction, we take advantage of the notion of *name* of a morphism, introduced below.

**Definition 4 (left name)** *In any dialogue category $\mathscr{C}$, the left name $\ulcorner f \urcorner$ of a morphism $f : x \to \bot$ is defined as the morphism*

$$\ulcorner f \urcorner \quad = \quad \varphi_{x,I}(f) \quad : \quad I \quad \longrightarrow \quad x \multimap \bot.$$

Given the left name $\ulcorner f \urcorner$ of a morphism, one recovers the morphism $f$ itself by composing it with the evaluation map. The whole procedure is justified by the fact that the diagram




commutes for every morphism $f : x \to \bot$. Now, it is not difficult to see that the image of a morphism

$$f \quad : \quad x \otimes y \quad \longrightarrow \quad \bot$$

by the function $\mathit{wheel}_{x,y}(f)$ defined in Equation (9) coincides with the left name

$$y \otimes \ulcorner f \urcorner \otimes x \quad : \quad y \otimes x \quad \longrightarrow \quad y \underset{xy}{\multimap} x \tag{15}$$

composed with the morphism

$$y \underset{xy}{\multimap} x \quad \xrightarrow{lev} \quad \underset{x}{\multimap}\; x \quad \xrightarrow{turn} \quad \underset{x}{\multimapinv}\; x \quad \xrightarrow{reval} \quad \bot. \tag{16}$$

From now on, suppose given a morphism

$$f \quad : \quad x \otimes y \otimes z \quad \longrightarrow \quad \bot.$$

By the discussion above, the function $\mathit{wheel}_{x \otimes y, z}$ applied to the morphism $f$ is equal to the composite morphism

$$zxy \quad \xrightarrow{\ulcorner f \urcorner} \quad z \underset{xyz}{\multimap} xy \quad \xrightarrow{lev} \quad \underset{xy}{\multimap}\; xy \quad \xrightarrow{turn} \quad \underset{xy}{\multimapinv}\; xy \quad \xrightarrow{reval} \quad \bot.$$

In order to establish the coherence property of *wheel* stated in §3.3, we need to show that the composite function

$$\mathit{wheel}_{y, z \otimes x} \quad \circ \quad \mathit{wheel}_{x, y \otimes z}$$

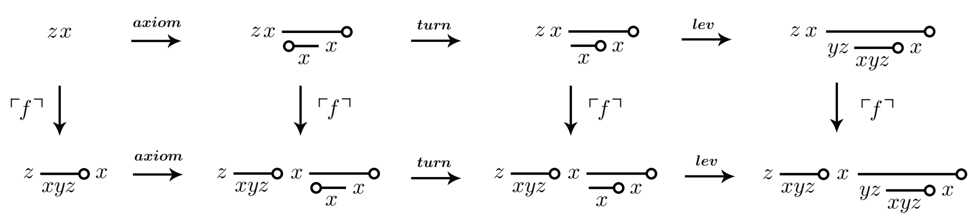
transports $f$ to the very same morphism. An easy computation shows that the name of $\mathit{wheel}_{x, y \otimes z}(f)$ is equal to the composite morphism

$$I \quad \xrightarrow{axiom} \quad \underset{\underset{x}{\multimapinv}\; x}{\multimap} \quad \xrightarrow{turn} \quad \underset{\underset{x}{\multimap}\; x}{\multimap} \quad \xrightarrow{lev} \quad \underset{yz \underset{xyz}{\multimap} x}{\multimap} \quad \xrightarrow{\ulcorner f \urcorner} \quad \underset{yzx}{\multimap}$$

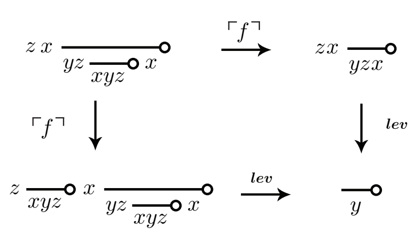
where the morphism *axiom* is defined as

$$\mathit{axiom} \quad = \quad \varphi_{(\bot \multimapinv x) \otimes x, I}(\mathit{reval}_x) \quad : \quad I \quad \longrightarrow \quad ((\bot \multimapinv x) \otimes x) \multimap \bot.$$

The series of commutative diagrams




completed with the dinaturality diagram of left evaluation



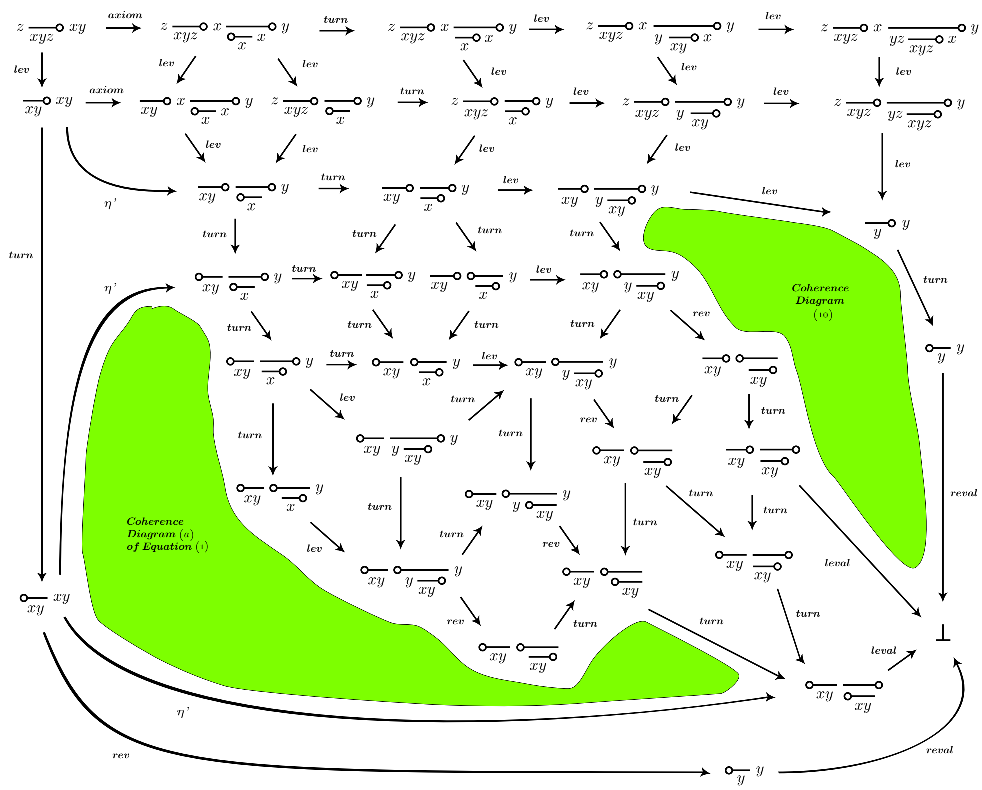
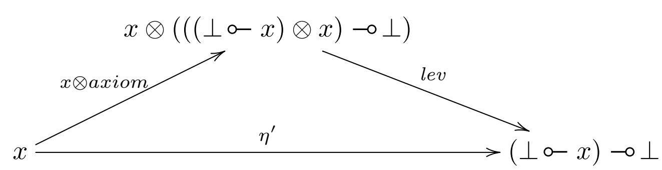



Figure 3: Diagram chase establishing Direction from §3.4 to §3.3.

shows that the equality

$$wheel_{y,z\otimes x}(f) \quad = \quad wheel_{y,z\otimes x} \quad \circ \quad wheel_{x,y\otimes z}(f)$$

reduces to the diagram chase in Figure 3. Every piece of the jigsaw is easily shown to commute, except possibly for the triangle diagram




which commutes by definition of the *axiom* map. This concludes the proof of Proposition 2.

## 3.7 Cyclic dialogue categories

This discussion leads to a tentative definition of cyclic dialogue category.

**Definition 5** *A helical dialogue category is called* cyclic *when the diagram (5) commutes for every object $x$ of the category.*

The definition of cyclic dialogue category may be restated as a degeneracy property of the dialogical twist:

**Proposition 4** *A helical dialogue category is cyclic precisely when the morphism* $twist : \bot \to \bot$ *coincides with the identity on the tensorial pole.*

We would like to compare the resulting notion of cyclic dialogue category with the definition of cyclic $*$-autonomous category formulated by Blute, Lamarche and Ruet [6], and more recently studied in great detail by Egger and McCurdy [9]. Recall from the introduction (Definition 1) that a $*$-autonomous category is a dialogue category where the tensorial pole is dualizing – this meaning that the units $\eta$ and $\eta'$ of the two monads $T$ and $T'$ are isomorphisms. This leads to the following definition of cyclic $*$-autonomous category, which recovers the original definition in [6].

**Definition 6** *A cyclic* $*$*-autonomous category is a cyclic dialogue category whose tensorial pole* $\bot$ *is dualizing.*

# 4 Intermezzo: the dialogical twist

Now that the equivalence of the external and of the internal definitions of helical dialogue category provided in §3.3 and §3.4 has been established, we would like to come back to the dialogical twist map mentioned at the end of §3.1. We start by defining it externally in § 4.1, using the presheaf $\mathscr{C}(-,\bot)$. We then reformulate it internally in § 4.2 as the composite of familiar maps of the underlying dialogue category. This dialogical twist enables us to resolve a subtle difficulty of helical dialogue categories. Indeed, the isomorphism $turn$ enables to construct a natural isomorphism

$$\bot \multimapinv (x \multimap \bot) \quad \overset{turn}{\longrightarrow} \quad (x \multimap \bot) \multimap \bot \quad \overset{turn \multimap \bot}{\longrightarrow} \quad (\bot \multimapinv x) \multimap \bot \qquad (17)$$

between the double negation functors $T$ and $T'$. Somewhat surprisingly, this isomorphism is not a monad morphism. After establishing in § 4.3 that the dialogical twist satisfies two important commutative diagrams, we show in § 4.4 how it enables us to "twist" the isomorphism (17) in order to obtain a monad isomorphism between the two double negation monads $T$ and $T'$. We conclude the section by introducing in § 4.5 the notion of *cyclic* dialogue category, defined as a helical dialogue category where the twist is trivial.

## 4.1 The dialogical twist

In every dialogue category $\mathscr{C}$ equipped with a natural isomorphism

$$turn_x \quad : \quad x \multimap \bot \quad \longrightarrow \quad \bot \multimapinv x$$

there exists a natural family of bijections

$$\mathscr{C}(x,\bot) \xrightarrow{\varphi_{I,x}} \mathscr{C}(I, x \multimap \bot) \xrightarrow{\qquad turn_x \qquad} \mathscr{C}(I, \bot \multimapinv x) \xrightarrow{\psi^{-1}_{I,x}} \mathscr{C}(x,\bot)$$

which may be seen as an automorphism of the representable presheaf

$$\mathbf{y}_{\perp} \quad : \quad x \quad \mapsto \quad \mathscr{C}(x, \perp).$$

It follows from the Yoneda lemma that

**Proposition 5** *There exists a unique morphism* $twist : \perp \rightarrow \perp$ *making the diagram*

$$\begin{array}{ccc} \mathscr{C}(x, \perp) & \xrightarrow{\mathscr{C}(x, twist)} & \mathscr{C}(x, \perp) \\ \downarrow{\scriptstyle \varphi_{I,x}} & & \downarrow{\scriptstyle \psi_{I,x}} \\ \mathscr{C}(I, x \multimap \perp) & \xrightarrow{turn_x} & \mathscr{C}(I, \perp \multimapinv x) \end{array}$$

*commute for every object* $x$ *of the category* $\mathscr{C}$. *Moreover, this morphism is an isomorphism.*

By applying the Yoneda lemma again, one shows that this induces a natural isomorphism

$$twist \multimapinv x \quad : \quad \perp \multimapinv x \quad \longrightarrow \quad \perp \multimapinv x \tag{18}$$

whose component $twist \multimapinv x$ is characterized by the fact that it makes the diagram

$$\begin{array}{ccc} \mathscr{C}(x \otimes y, \perp) & \xrightarrow{\psi_{x,y}} & \mathscr{C}(x, \perp \multimapinv y) \\ \uparrow{\scriptstyle \mathscr{C}(x \otimes y, twist)} & & \uparrow{\scriptstyle \mathscr{C}(x, twist \multimapinv y)} \\ \mathscr{C}(x \otimes y, \perp) & \xrightarrow{\psi_{x,y}} & \mathscr{C}(x, \perp \multimapinv y) \end{array}$$

commute for all objects $x, y$. The natural isomorphism

$$x \multimap twist \quad : \quad x \multimap \perp \quad \longrightarrow \quad x \multimap \perp$$

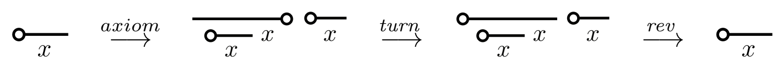
is defined in the same way. Note that the diagram below commutes:

$$\begin{array}{ccc} x \multimap \perp & \xrightarrow{turn_x} & \perp \multimapinv x \\ \downarrow{\scriptstyle x \multimap twist} & & \downarrow{\scriptstyle twist \multimapinv x} \\ x \multimap \perp & \xrightarrow{turn_x} & \perp \multimapinv x \end{array} \tag{19}$$

## 4.2 An internal definition

Now that the $twist \multimapinv x$ map has been defined in (18) by external means using the presheaf $\mathscr{C}(-, \perp)$, it makes sense to look for an internal definition, based on the familiar combinators of the underlying dialogue category $\mathscr{C}$. This is precisely the purpose of the next statement.

**Proposition 6** *In any dialogue category $\mathscr{C}$ equipped with a natural isomorphism $turn$, the induced map $twist \multimapinv x$ may be defined as the composite*

$twist \multimapinv x \quad :$ 


*for every object $x$ of the category.*

Proof: By the external definition of $twist \multimapinv x$ given in the previous section, the function

$$wheel_{(\bot \multimapinv x)\otimes x, I} \quad : \quad \mathscr{C}((\bot \multimapinv x) \otimes x) \quad \longrightarrow \quad \mathscr{C}((\bot \multimapinv x) \otimes x)$$

transports the morphism

$$reval_x \quad : \quad (\bot \multimapinv x) \otimes x \quad \longrightarrow \quad \bot.$$

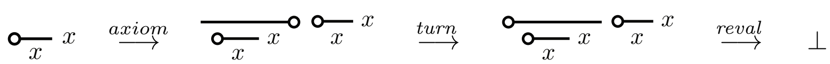
to the morphism

$$(\bot \multimapinv x) \otimes x \xrightarrow{twist \multimapinv x} (\bot \multimapinv x) \otimes x \xrightarrow{reval} \bot \; .$$

Now, recall that

$$axiom \quad : \quad I \quad \longrightarrow \quad ((\bot \multimapinv x) \otimes x) \multimap \bot$$

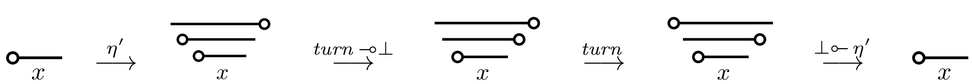
is defined as the left name of the morphism $reval_x$. This enables to apply the explicit definition of $wheel$ in § 3.6 based on Equations (15) and (16). This establishes that the function $wheel_{(\bot \multimapinv x)\otimes x, I}$ transports the morphism $reval_x$ to the composite morphism:




This leads to the statement of the proposition, and concludes the proof.

Now, suppose that the dialogue category $\mathscr{C}$ is helical in the sense of §3.4, this meaning that the coherence diagram (10) is satisfied. In that case,

**Proposition 7** *In a helical dialogue category $\mathscr{C}$, the dialogical twist $twist \multimapinv x$ may be defined as*




Proof: We start from the internal formulation of $twist \multimapinv x$ and post compose it with $turn^{-1}$. We obtain a map

$$f \quad : \quad \bot \multimapinv x \quad \longrightarrow \quad x \multimap \bot$$

Now, let us see it as

$$x \otimes (\bot \multimapinv x) \quad \xrightarrow{x \otimes f} x \otimes (x \multimap \bot) \quad \longrightarrow \quad \bot$$

The diagram chase in Figure 4 establishes then the property.

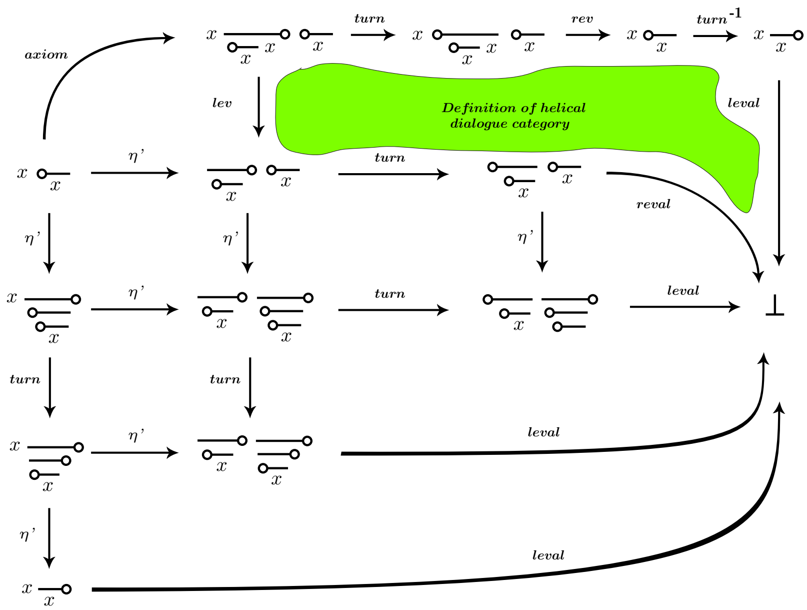
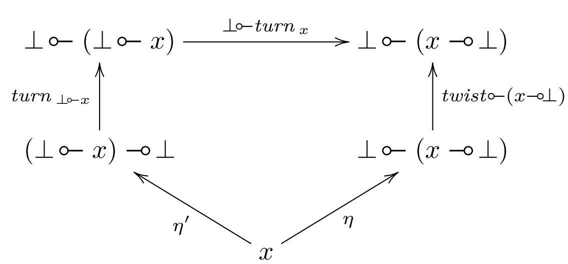



Figure 4: Diagram chase establishing Proposition 7.

### 4.3 Two commutative diagrams

One reason for taking the dialogical twist seriously is the following statement.

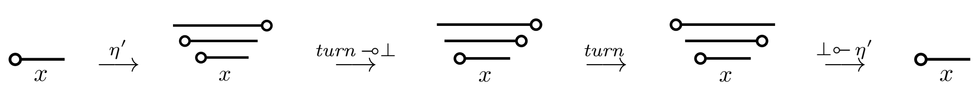
**Proposition 8** *In a helical dialogue category $\mathscr{C}$, the diagram*



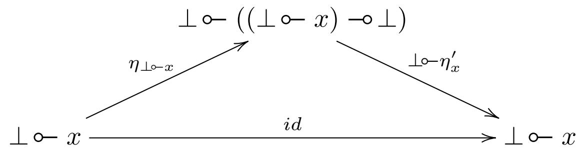

(20)

*commutes for every object $x$ of the category.*

Proof: we have already established that the isomorphism (18) may be alternatively defined as:



So, we may replace $twist \multimap (x \multimapinv \bot)$ in Diagram (20). It appears that commutativity of Diagram (20) reduces then to the fact that the diagram



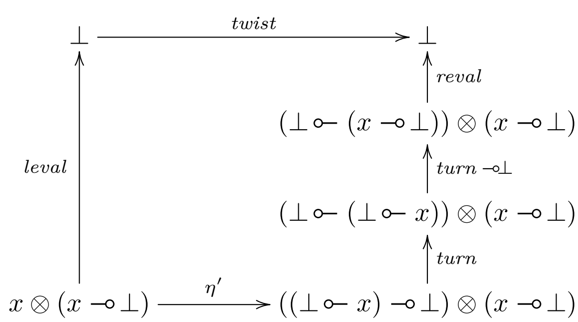

commutes. This is true in every dialogue category, since

$$\eta_{\bot \multimapinv x} \;=\; \eta \circ R(x) \qquad\qquad \bot \multimapinv \eta'_x \;=\; R \circ \varepsilon(x)$$

This is the triangular law of the adjunction $L \dashv R$.

This leads to the following statement which plays a fundamental role in proof theory because it expresses an $\eta$-expansion law for axioms (evaluation maps) on a negated formula.

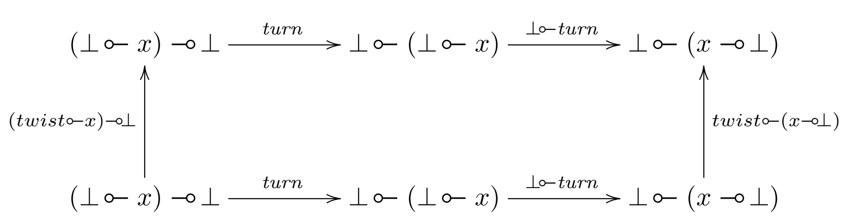
**Proposition 9** *In every helical dialogue category $\mathscr{C}$, the diagram*




*commutes for every object $x$ of the category.*

Proof: simply combine Diagram (20) in Proposition 8 with Diagram (14) and definition of $twist \multimapinv x$.

**Proposition 10** *In a helical dialogue category $\mathscr{C}$, the diagram*




*commutes.*

## 4.4 The monad isomorphism

We use the dialogical twist to construct an isomorphism between the two double negation monads $T$ and $T'$. The important point is that

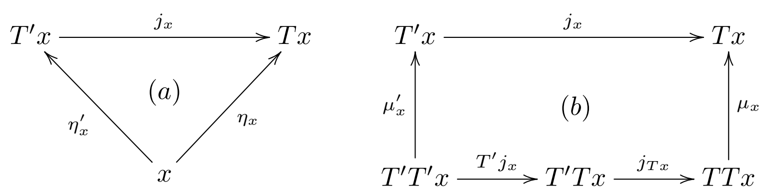
**Proposition 11** *In a helical dialogue category $\mathscr{C}$, the natural family*

$$(\bot \multimapinv x) \multimap \bot \xrightarrow{turn} \bot \multimapinv (\bot \multimapinv x) \xrightarrow{turn} \bot \multimapinv (x \multimap \bot) \xrightarrow{twist^{-1}} \bot \multimapinv (x \multimap \bot)$$

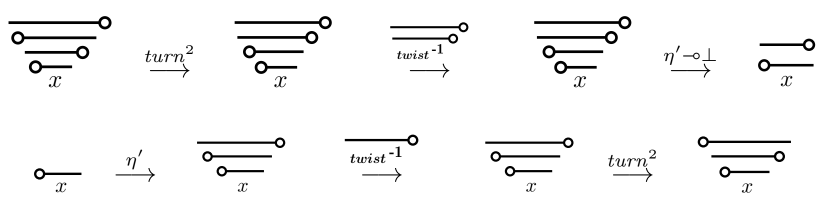
*defines a monad morphism*

$$j \quad : \quad (T', \mu', \eta') \quad \longrightarrow \quad (T, \mu, \eta).$$

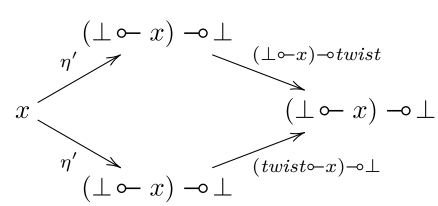
Proof. By monad morphism, one means that the two diagrams below




commute for every object $x$ of the category. We have established in Proposition 8 that the lefthand side Diagram $(a)$ commutes. Commutativity of Diagram $(b)$ reduces to the fact that the multiplication $\mu'_x$ coincides with the morphism




This boils down to the commutativity of the diagram




## 4.5 Cyclic dialogue categories

Recall that a cyclic dialogue category is a helical dialogue category where the diagram (5) commutes for every object $x$ of the category.

**Proposition 12** *A helical dialogue category is cyclic precisely when the morphism $twist : \bot \to \bot$ coincides with the identity on the tensorial pole.*

We would like to compare the resulting notion of cyclic dialogue category with the definition of cyclic $*$-autonomous category formulated by Blute, Lamarche and Ruet [6], and more recently studied in great detail by Egger and McCurdy [9]. Recall from the introduction (Definition 1) that a $*$-autonomous category is a dialogue category where the tensorial pole is dualizing – this meaning that the units $\eta$ and $\eta'$ of the two monads $T$ and $T'$ are isomorphisms. This leads to the following definition of cyclic $*$-autonomous category, which recovers the original definition in [6].

**Definition 7** *A cyclic $*$-autonomous category is a cyclic dialogue category whose tensorial pole $\perp$ is dualizing.*

# 5 Ribbon categories

To that purpose, we start by recalling the definition of monoidal categories in § 5.1, of braided monoidal categories in § 5.2, of balanced monoidal categories in § 5.3 and of ribbon categories in § 5.4.

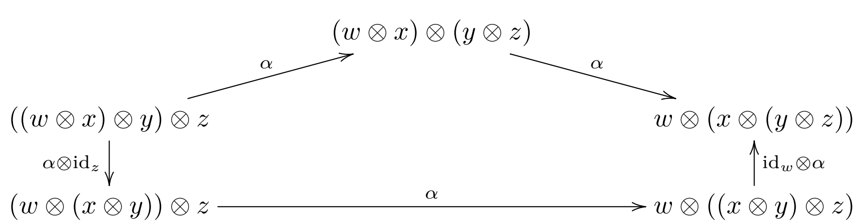
## 5.1 Monoidal categories

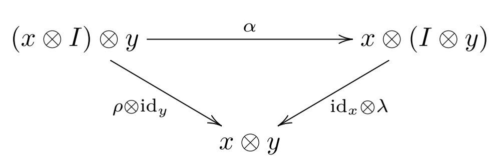
In order to fix notations, we recall that a monoidal category $\mathscr{C}$ is a category equipped with a functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ and an object $I$ and three natural isomorphisms

$$\alpha_{x,y,z} \quad : \quad (x \otimes y) \otimes z \quad \longrightarrow \quad x \otimes (y \otimes z)$$

$$\lambda_x \quad : \quad I \otimes x \quad \longrightarrow \quad x \qquad\qquad \rho_x \quad : \quad x \otimes I \quad \longrightarrow \quad x$$

making the two coherence diagrams below commute.



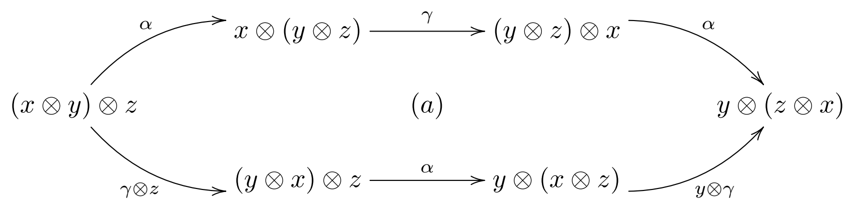

## 5.2 Braided categories

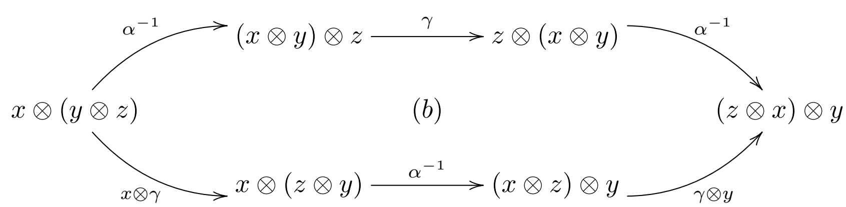
The notion of braided monoidal category $\mathscr{C}$ is introduced in ....

**Definition 8 (braiding)** *A braiding in a monoidal category $\mathscr{C}$ is a family of isomorphisms*

$$\gamma_{x,y} \quad : \quad x \otimes y \quad \longrightarrow \quad y \otimes x$$

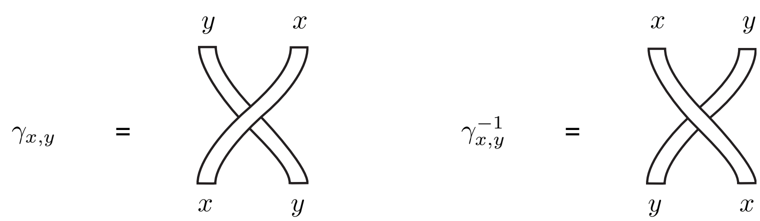
*natural in $x$ and $y$ such that the two diagrams*



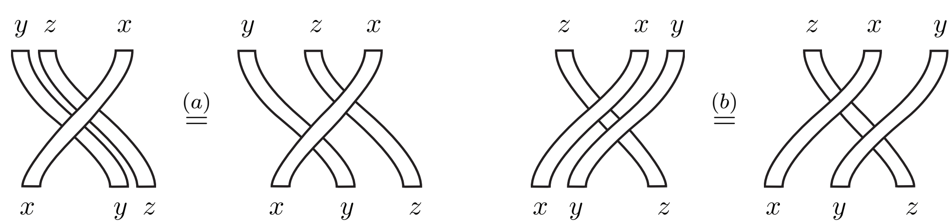




*commute.*

The braiding map $\gamma_{x,y}$ is depicted in string diagrams as a positive braiding of the ribbon strands $x$ and $y$ where its inverse is depicted as the negative braiding:




The two coherence diagrams $(a)$ and $(b)$ are then depicted as equalities between string diagrams:




## 5.3 Balanced categories

The notion of balanced category is introduced in...

**Definition 9 (balanced category)** *A balanced category $\mathscr{C}$ is a braided monoidal category equipped with a family of morphisms*

$$\theta_x \quad : \quad x \quad \longrightarrow \quad x$$

*natural in $x$, satisfying the equality*

$$\theta_I \quad = \quad \mathrm{id}_I$$

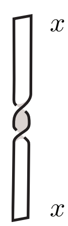
*and making the diagram*

$$\begin{array}{ccc} x \otimes y & \xrightarrow{\gamma_{x,y}} & y \otimes x \\ {\scriptstyle \theta_{x\otimes y}} \downarrow & & \downarrow {\scriptstyle \theta_y \otimes \theta_x} \\ x \otimes y & \xleftarrow[\gamma_{y,x}]{} & y \otimes x \end{array} \tag{21}$$

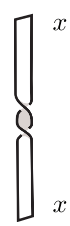
*commute for all objects $x$ and $y$ of the category $\mathscr{C}$.*

The twist $\theta_x$ is depicted as the ribbon $x$ twisted positively in the trigonometric direction with an angle $2\pi$ whereas its inverse $\theta_x^{-1}$ is depicted as the same ribbon $x$ twisted this time negatively with an angle $-2\pi$:

$\theta_x$ =

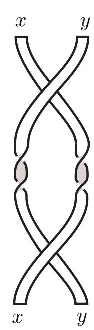



$\theta_x^{-1}$ =




This notation enables us to give a topological motivation to the axioms of a balanced category. The first requirement that $\theta_I$ is the identity means that the ribbon strand $I$ should be thought as ultra thin. The second requirement that the coherence Diagram 21 commutes says that topological equality between string diagrams:

$\theta_{x\otimes y}$ =




## 5.4 Ribbon categories

**Definition 10 (dual pairs)** *A dual pair in a monoidal category $\mathscr{C}$ is a quadruple $(x, y, \eta, \varepsilon)$ consisting of two objects $x$ and $y$ and two morphisms*

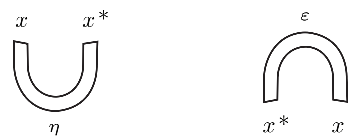
$$\eta : I \longrightarrow x \otimes y \qquad\qquad \varepsilon : y \otimes x \longrightarrow I$$

*making the two diagrams below commute:*

$$\begin{array}{ccc} (x \otimes y) \otimes x & \xrightarrow{\alpha^{-1}} & x \otimes (y \otimes x) \\ {\scriptstyle \eta \otimes x} \uparrow & & \downarrow {\scriptstyle x \otimes \varepsilon} \\ x & \xrightarrow{\mathrm{id}_x} & x \end{array} \qquad \begin{array}{ccc} y \otimes (x \otimes y) & \xrightarrow{\alpha^{-1}} & (y \otimes x) \otimes y \\ {\scriptstyle y \otimes \eta} \uparrow & & \downarrow {\scriptstyle \varepsilon \otimes y} \\ y & \xrightarrow{\mathrm{id}_y} & y \end{array}$$

*We will often write $x \dashv y$ in that case, and say that $x$ is a left dual of $y$, or equivalently that $y$ is a right dual of $x$.*

The unit $\eta$ and counit $\varepsilon$ are depicted as $U$-turns:



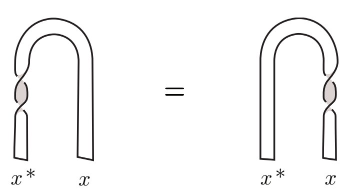

The two coherence diagrams express how a $U$-turn combines with a $U$-turn in the other direction:


**Definition 11 (ribbon category)** *A ribbon category $\mathscr{C}$ is a balanced category where every object $x$ is equipped with an object $x^*$ and two morphisms $\eta_x$ and $\varepsilon_x$ defining a dual pair $(x, x^*, \eta_x, \varepsilon_x)$ and making the diagram*

$$\begin{array}{ccc} x^* \otimes x & \xrightarrow{x^* \otimes \theta_x} & x^* \otimes xx \\ {\scriptstyle \theta_{x^*} \otimes x} \downarrow & & \downarrow {\scriptstyle \varepsilon_x} \\ x^* \otimes x & \xrightarrow{\varepsilon_x} & I \end{array}$$

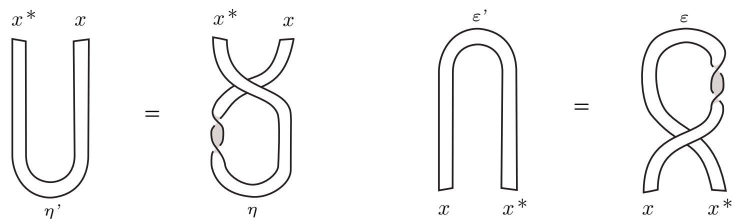
*commute for every object $x$ of the category $\mathscr{C}$.*

The coherence diagram of ribbon categories is depicted as

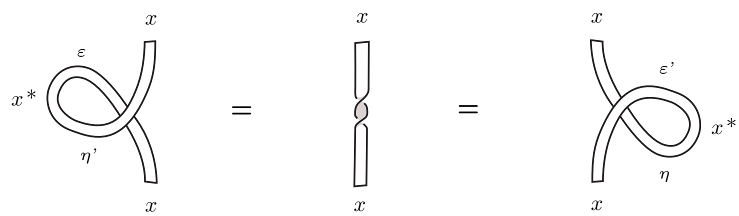



(22)

Note that in a ribbon category, every object $x^*$ is also right dual to the object $x$ thanks to the dual pair $x \dashv x^*$ with unit $\eta'$ and counit $\varepsilon'$ defined as



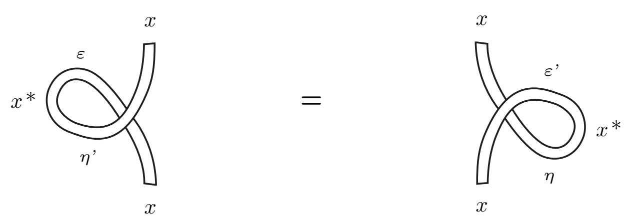

This implies in particular that the following equality holds in every ribbon category:




(23)

This leads to a concise definition of ribbon category, which does not mention the balanced structure:

**Proposition 13** *A ribbon category is the same thing as a braided category where every object $x$ is equipped with a dual pair $(x, x^*, \eta, \varepsilon)$ and a dual pair $(x^*, x, \eta', \varepsilon')$ satisfying the equality:*




*Note that the object $x^*$ is at the same time a left dual and a right dual of the object $x$.*

# 6 Balanced dialogue categories

At this stage, we are ready to introduce the notion of *balanced dialogue category* which provides a functorial bridge between proof theory and knot theory. The notion is defined in § 6.1. We show in § 6.2 that every balanced dialogue category induces two helical structures on the tensorial pole $\perp$. We conclude the section by illustrating in § 7.3 the notion of balanced dialogue category with an example coming from representation theory of quantum groups: the category of (finite and infinite dimensional) $H$-modules associated to a ribbon Hopf algebra $H$.

## 6.1 Balanced dialogue categories

**Definition 12 (balanced dialogue category)** *A balanced dialogue category is a dialogue category $\mathscr{C}$ equipped with a braiding and a twist defining a bal-*

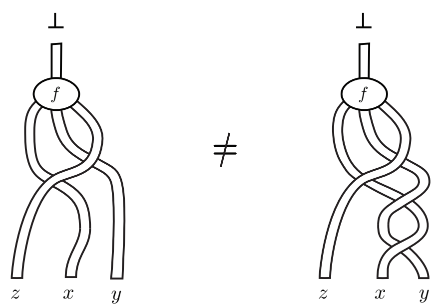
*anced category.*

## 6.2 Every balanced dialogue category is helical

Every dialogue category $\mathscr{C}$ whose underlying monoidal category is braided comes equipped with a natural bijection

$$\begin{array}{rccc} wheel_{x,y} & : & \mathscr{C}(x \otimes y, \bot) & \longrightarrow & \mathscr{C}(y \otimes x, \bot) \\ & & f & \longmapsto & f \circ \gamma_{y,x} \end{array}$$

Unfortunately, the bijection does not satisfy the coherence diagram (6) required of a helical structure in § 3.2. The trouble comes from the fact that the two diagrams below are not necessarily equal because the category $\mathscr{C}$ is braided, rather than symmetric:




So, in order to obtain the desired equality

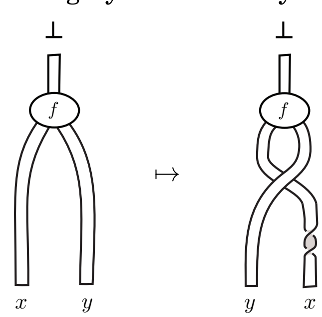
$$wheel_{y,z\otimes x} \quad = \quad wheel_{y,z\otimes x} \quad \circ \quad wheel_{x,y\otimes z}$$

one needs to define *wheel* in a slightly different way. However, braided categories are not sufficient to that purpose. This is precisely the reason for shifting to balanced categories, since they provide us with a satisfactory solution based on the ability to twist ribbon strands. Indeed, every balanced dialogue category $\mathscr{C}$ comes equipped with a natural bijection *wheel* defined this time as:

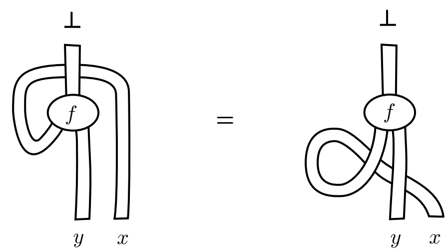
$$\begin{array}{rccc} wheel_{x,y} & : & \mathscr{C}(x \otimes y, \bot) & \longrightarrow & \mathscr{C}(y \otimes x, \bot) \\ & & f & \longmapsto & f \circ \gamma_{y,x} \circ (\mathrm{id}_y \otimes \theta_x) \end{array} \tag{24}$$

This bijection satisfies the coherence diagram (6) and thus defines a helical structure on the balanced dialogue category $\mathscr{C}$. Pictorially:

$wheel_{x,y}$ :



This definition is also supported (at least informally) by the topological equality which relates the pictorial notation for *wheel* on the one hand, and the topological reformulation in (23) of the twist map on the other hand.




(25)

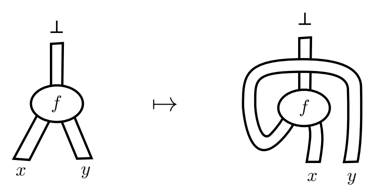
Although this diagram does not make sense in the general case of a balanced dialogue category, we will see in § 6.2 that it becomes meaningful in the particular case of a balanced dialogue category coming from a ribbon category. One shows moreover that

**Proposition 14** *In every balanced dialogue category $\mathscr{C}$, the dialogical twist associated to the helical structure (24) in Proposition 5 coincides with the twist map $\theta_\perp$ associated to the tensorial focus.*

## 6.3 Every balanced dialogue category is helical [twice]

In the same way as we defined in § 6.2 a wheel obtained by permuting the strand $x$ "under" the strand $\perp$ as depicted in Equation (4), we consider here another construction of the wheel, obtained this time by permuting the strand $x$ "above" the strand $\perp$, as depicted below.

$\overline{wheel}_{x,y}$ :

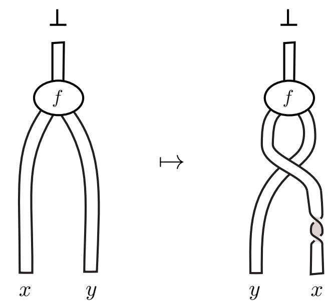



(26)

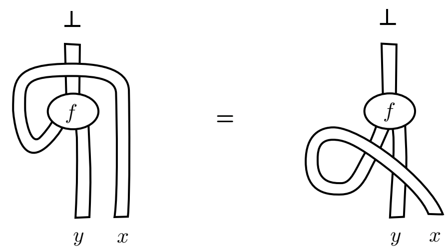
Although this picture makes only sense in ribbon categories, it leads to the observation that there exists another helical structure in every balanced dialogue category, defined this time as:

$$\begin{array}{rcccc} \overline{wheel}_{x,y} & : & \mathscr{C}(x \otimes y, \perp) & \longrightarrow & \mathscr{C}(y \otimes x, \perp) \\ & & f & \longmapsto & f \circ \gamma^{-1}_{x,y} \circ (\mathrm{id}_y \otimes \theta^{-1}_x) \end{array} \tag{27}$$

In picture:

$\overline{wheel}_{x,y}$ : 


Just as in § 6.2, the definition is supported by the topological equality below, which only makes sense in ribbon categories.




(28)

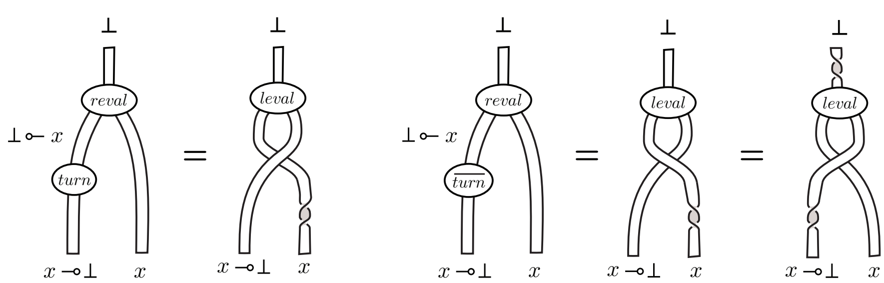
Note that the dialogical twist induced by the helical structure $\overline{wheel}$ is equal to the inverse $\theta_{\perp}^{-1}$ of the dialogical twist $\theta_{\perp}$ associated to the helical structure *wheel*.

## 6.4 Thin pole in a balanced dialogue category

By the one-to-one relationship described in § 3.3, each helical structure *wheel* and $\overline{wheel}$ induces a natural family of isomorphisms

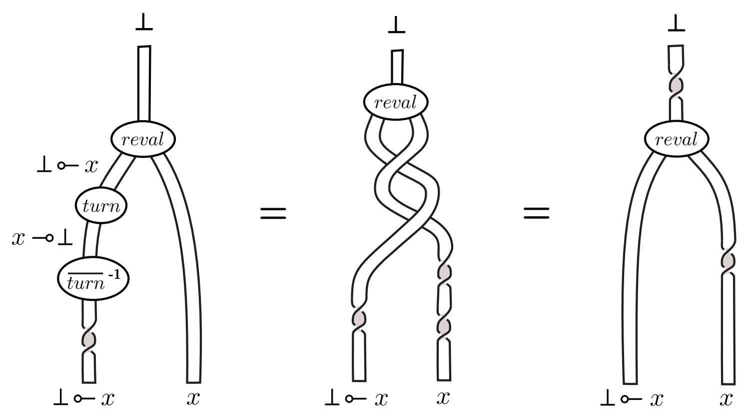
$$turn_x, \overline{turn}_x \quad : \quad \perp \multimapinv x \quad \longrightarrow \quad x \multimap \perp$$

which satisfies the coherence diagram (10). By construction, the isomorphisms are characterized by the graphical equalities:




(29)

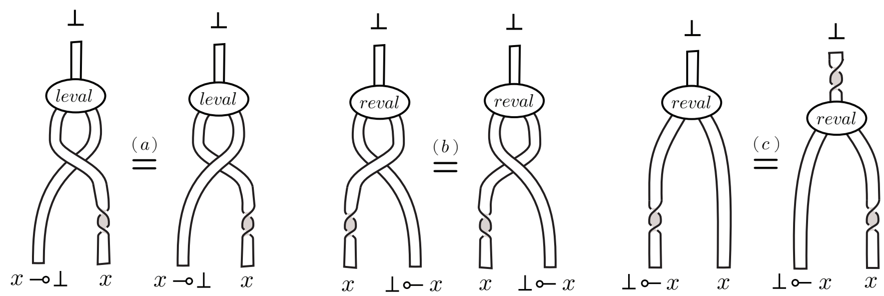
It is immediate to deduce from these equalities that




(30)

These equations hold in any balanced dialogue categories, and provide a dialogical variant of Equation (22) for ribbon categories. In particular,

**Proposition 15** *For every object $x$ of a balanced dialogue category $\mathscr{C}$, each of the three equalities $(a)$, $(b)$ and $(c)$ below*




(31)

*is equivalent to the equality $turn_x = \overline{turn}_x$.*

This leads to the definition of an ultra-thin pole, which adapts to dialogue categories the Equation (22) used in § 5.3 in order to define the notion of ribbon category.

**Definition 13 (ultra-thin pole)** *The tensorial pole $(\bot, \varphi, \psi)$ of a balanced dialogue category $\mathscr{C}$ is called utra-thin when one of the three equalities of Equation (31) (and thus all of them) is satisfied for every object $x$ of the category.*

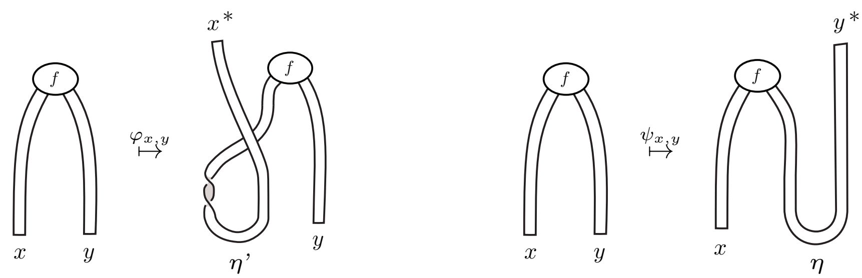
# 7 Four illustrations

## 7.1 First illustration: ribbon categories

We introduce the following notion in order to clarify the axiomatic position of balanced dialogue categories with respect to ribbon categories.

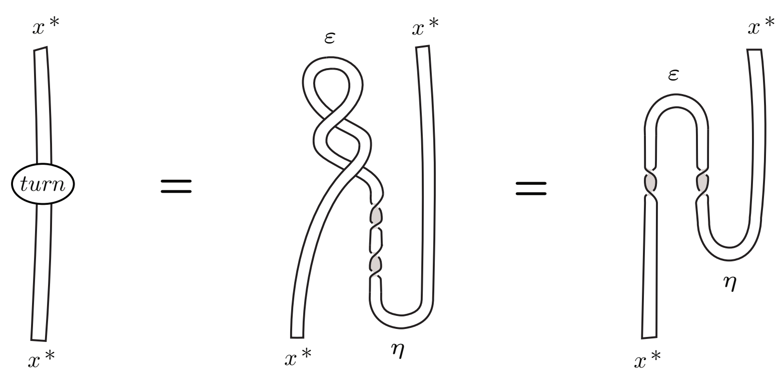
**Definition 14 (pre-ribbon category)** *A pre-ribbon category is a balanced category $\mathscr{C}$ where every object $x$ is equipped with a dual $(x, x^*, \eta_x, \varepsilon_x)$.*

So, a pre-ribbon category is like a ribbon category, except that it does not necessarily satisfy Equation (22). The point is that every pre-ribbon category $\mathscr{C}$ defines a balanced dialogue category, in the following way. The tensorial pole $\bot$ of the category $\mathscr{C}$ is defined as its tensorial unit $I$, while the left and the right negations $x \multimap \bot$ and $\bot \multimapinv x$ of an object $x$ are both defined as the dual object $x^*$. The natural bijections $\varphi$ and $\psi$ are then defined as




This enables to see the category $\mathscr{C}$ as a balanced dialogue category. An easy computation shows that, as expected, the left and right evaluation morphisms are equal to the two counits $\varepsilon$ and $\varepsilon'$ considered in § 5.4:


and that the isomorphism $\overline{turn}_x : x^* \to x^*$ is equal to the identity. On the other hand, this is not necessarily the case for the isomorphism $turn_x$ as shown by the equalities below:




This is where the notion of ribbon category plays a useful rôle: it is indeed immediate that the morphism $turn_x$ is equal to the identity when the category $\mathscr{C}$ is a ribbon category, and thus satisfies Equation (22).

**Proposition 16** *A ribbon category is the same thing as a pre-ribbon category whose tensorial pole I is utra-thin.*

## 7.2 Second illustration: ribbon categories with a distinguished object

Every ribbon category $\mathscr{C}$ equipped with a distinguished object $\bot$ defines a balanced dialogue category where the two negation functors are defined as

$$x \multimap \bot \quad = \quad x^* \otimes \bot \qquad\qquad \bot \multimapinv x \quad = \quad \bot \otimes x^*$$

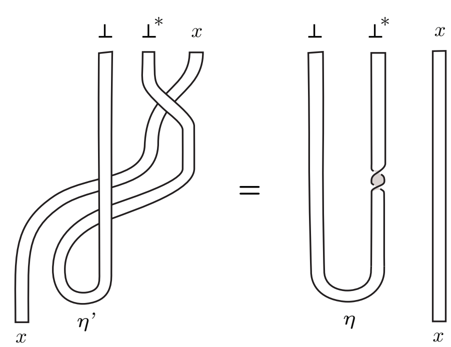
As explained in § 6.2, the balanced structure on the dialogue category $\mathscr{C}$ induces a helical structure *wheel* defined in Equation (24). The twist in Equation (24) ensures then that the associated *turn* described in § 3.3 coincides with the negative braiding

$$turn_x \quad = \quad \gamma^{-1}_{\bot, x^*} \quad : \quad x^* \otimes \bot \quad \longrightarrow \quad \bot \otimes x^*$$

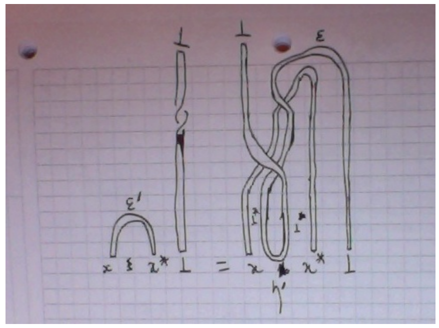
permuting the object $x^*$ under the tensorial pivot $\bot$. This also justifies the informal topological explanation for the definition of *wheel* in § 6.2. Indeed, the topological equality of Equation (25) makes sense in any balanced dialogue category coming from a ribbon category with a distinguished object $\bot$ — although the diagrams are meaningless in a general balanced dialogue category.

Draw the properties established in the intermezzo section.

This provides us with an opportunity to clarify the topological nature of the commutative diagrams established for every helical dialogue category in § 3. For instance, the statement of Proposition 8 boils down to the topological equality below in the case of a ribbon category $\mathscr{C}$ equipped with a distinguished object $\bot$:




In particular, the commutative diagram in Proposition 9 may be depicted as follows



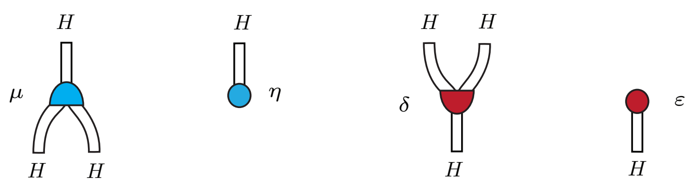
in the particular case of a helical dialogue category coming from a ribbon category $\mathscr{C}$.

### 7.3 Third illustration: representation theory

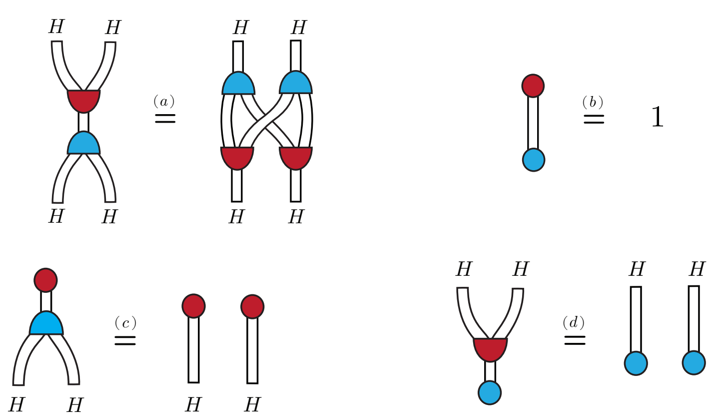
**Definition 15 (bialgebra)** *A bialgebra $H$ in a braided category $\mathscr{C}$ is an object $H$ equipped with four morphisms*

$$\mu : H \otimes H \to H \qquad \eta : I \to H \qquad \delta : H \to H \otimes H \qquad \varepsilon : H \to I$$

*depicted as*



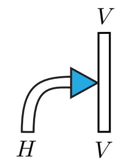

*defining a monoid $(H, \mu, \eta)$ and a comonoid $(H, \delta, \varepsilon)$ and the four diagrammatic equations below:*



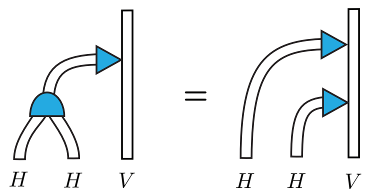

*where 1 denotes the identity of the unit $I$ of the monoidal category $\mathscr{C}$.*

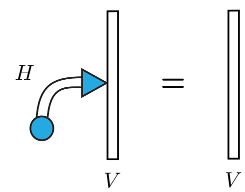
**Definition 16 (left $H$-module)** *Given a bialgebra $H$ in a braided category $\mathscr{C}$, a left $H$-module is an object $V$ equipped with a morphism*

$$\triangleright \quad : \quad H \otimes V \quad \longrightarrow \quad V \qquad\qquad \textit{depicted as}$$




*required to satisfy the two diagrammatic equations*

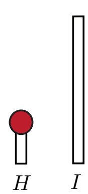


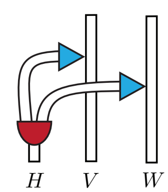




The two graphical equations reflect the familiar equations

$$(h_1 \cdot h_2) \triangleright v = h_1 \triangleright h_2 \triangleright v \qquad\qquad e \triangleright v = v$$

where $h_1 \cdot h_2 = \mu(h_1, h_2)$ and $e$ denote the multiplication and the unit of the monoid $H$. It is well-known that every bialgebra $H$ in a braided category $\mathscr{C}$ induces a monoidal category $Mod(H)$ of left $H$-modules. The action of the bialgebra $H$ on the tensorial unit $I$ of the category $\mathscr{C}$ and on the tensor product $V \otimes W$ of two $H$-modules $V$ and $W$ are defined as




(32)

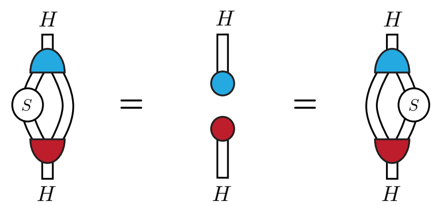
Note that our notion of bialgebra in a braided category $\mathscr{C}$ is carefully designed in order to make the category $Mod(H)$ monoidal. This requires in particular to choose the same orientation (here it is chosen positive) for the braiding defining the action of $H$ on $V \otimes W$ in Equation (32) and for the braiding defining a bialgebra in Equation $(a)$ of Definition 15. This also implies that the comultiplication $\delta : H \to H \otimes H$ is a $H$-module morphism as it should be.

Suppose moreover that the underlying braided category $\mathscr{C}$ is equipped with an object $\perp$ together with a natural bijection

$$\psi_{x,y} \quad : \quad \mathscr{C}(x \otimes y, \perp) \quad \cong \quad \mathscr{C}(x, \perp \multimapinv y)$$

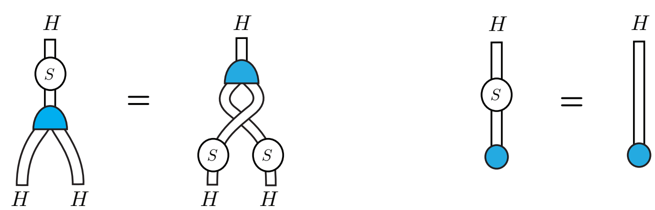
This defines what we call a right dialogue category. We would like to understand when the dialogue structure lifts to the monoidal category $HH$. A sufficient condition is that the bialgebra $H$ is equipped with an antipode $S$ defining a Hopf algebra:

**Definition 17 (Hopf algebra)** *A Hopf algebra $H$ in a braided category $\mathscr{C}$ is a bialgebra equipped with a morphism $S : H \to H$ satisfying the following equalities:*




*The morphism $S$ is called the antipode of the Hopf algebra $H$.*

**Proposition 17** *The antipode $S$ satisfies the two equalities*




*and thus defines an antihomorphism from the monoid $(H, \mu, \eta)$ to itself.*

As expected, we establish that

**Proposition 18** *Suppose that $H$ is a Hopf algebra in a braided category $\mathscr{C}$ and that $(\bot, \triangleright)$ is a left $H$-module. Then, every right dialogue structure with tensorial pole $\bot$ on the braided category $\mathscr{C}$ induces a right dialogue structure with tensorial pole $(\bot, \triangleright)$ on the monoidal category $Mod(H)$.*

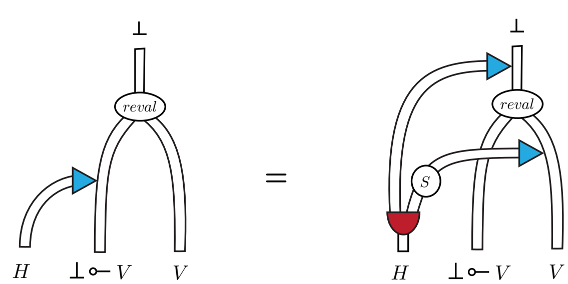
The action of the Hopf algebra on the object $\bot \multimapinv V$ is defined as

$$h \;\triangleright\; f \quad : \quad v \quad \mapsto \quad \sum \; h_{(1)} \;\triangleright\; f \;(\; S\, h_{(2)} \;\triangleright\; v \;)$$

in Sweedler's notation. It is defined pictorially as the unique morphism

$$\triangleright \quad : \quad H \otimes (\bot \multimapinv V) \quad \longrightarrow \quad \bot \multimapinv V$$

satisfying the graphical equality



**Proposition 19** *Suppose that $H$ is a Hopf algebra with an invertible antipode $S$ in a braided category $\mathscr{C}$ and that $(\bot, \triangleright)$ is a left $H$-module. Then, every left dialogue structure with tensorial pole $\bot$ on the braided category $\mathscr{C}$ induces a left dialogue structure with tensorial pole $(\bot, \triangleright)$ on the monoidal category $Mod(H)$.*

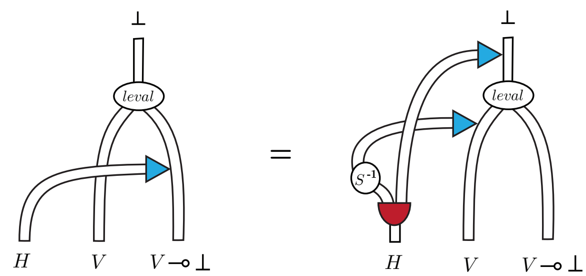
The action of $H$ on $V \multimap \bot$ is defined as

$$h \triangleright f \quad : \quad v \quad \mapsto \quad \sum \; h_{(2)} \triangleright f \; ( \; S^{-1} h_{(1)} \triangleright v \; )$$

It is defined pictorially as the unique morphism

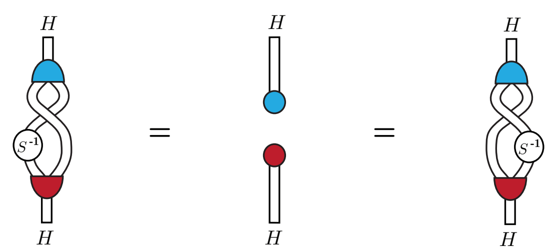
$$H \otimes (V \multimap \bot) \quad \longrightarrow \quad V \multimap \bot$$

satisfying the equality:




The statement of Proposition 19 is established using the following property of the inverse of the antipode.

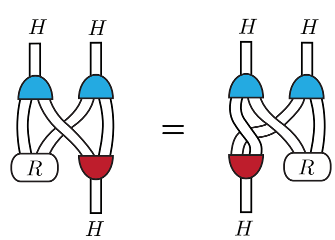
**Proposition 20** *The inverse $S^{-1}$ of the antipode $S$ of a Hopf algebra satisfies the equality*

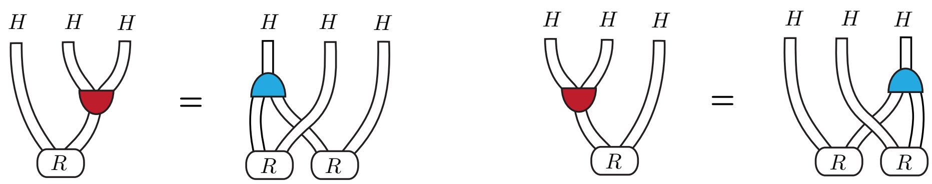



**Definition 18 (braiding)** *A braiding on a bialgebra $H$ in a braided monoidal category $\mathscr{C}$ is defined as a morphism*

$$R \quad : \quad I \quad \longrightarrow \quad H \otimes H$$

*invertible in the monoid $(H \otimes H, (\mu \otimes \mu) \circ (H \otimes \gamma, H), \eta \otimes \eta)$ and satisfying the pictorial equalities:*






*A Hopf algebra equipped with a braiding is called braided or quasi-triangular.*

We suppose from now on that the Hopf algebra $H$ is thin.

**Definition 19 (thin)** *A Hopf algebra $H$ in a balanced monoidal category $\mathscr{C}$ is thin when*

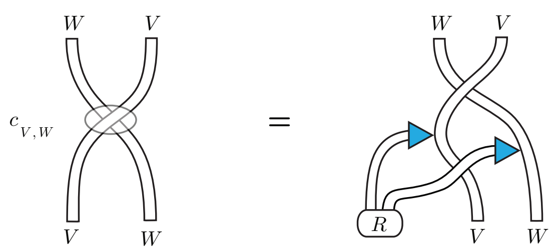
$$\theta_{H\otimes V} = H \otimes \theta_V$$

*for all the objects of the category $\mathscr{C}$.*

In that case, every quasitriangular structure on the Hopf algebra $H$ induces a braiding on the category $Mod(H)$

$$c_{V,W} \quad : \quad V \otimes W \quad \longrightarrow \quad W \otimes V$$

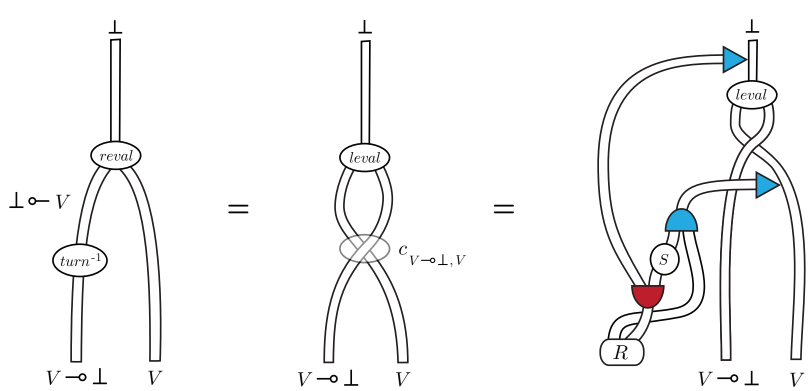
defined as




The braiding in the category $Mod(H)$ of left $H$-modules induces a natural family of morphisms of $H$-modules:

$$turn_V \quad : \quad V \multimap \bot \quad \longrightarrow \quad \bot \circ\!\!- V$$

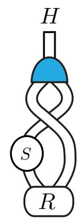
whose inverse is characterized by the equality



Observe that when the return object $\perp$ is defined as the tensorial unit $I$ equipped with the trivial $H$ module structure, then

$$turn_V^{-1} \quad : \quad f \quad \mapsto \quad v \mapsto f(u \rhd v)$$

where the morphism $u : I \to H$ is itself defined as

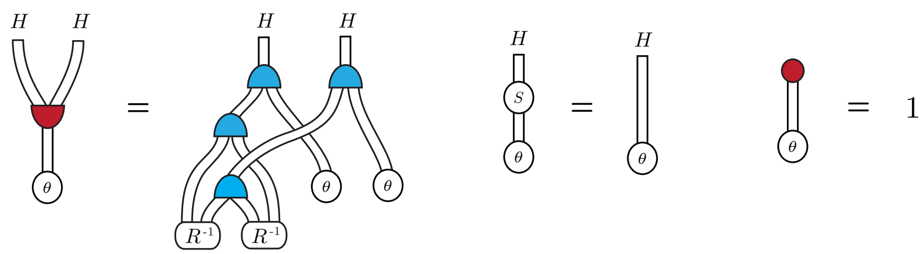



Note that one recovers the element $u \in H$ defined by Kassel in ... The problem is that $u$ is not group-like. This is precisely the reason why $turn$ does not define a cyclic structure. This motivates the following definition.

**Definition 20 (ribbon algebra)** *A ribbon Hopf algebra $H$ is a braided Hopf algebra $H$ equipped with a morphism*

$$\theta \quad : \quad I \quad \longrightarrow \quad H$$

*central and invertible in the monoid $(H, \mu, \eta)$ and satisfying the three equations*



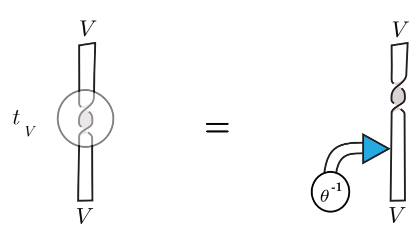

*where $1$ denotes the identity $\mathrm{id}_I$ of the tensorial unit, which should be seen here as the neutral element of the commutative monoid $\mathscr{C}(I, I)$ of scalars of the monoidal category $\mathscr{C}$.*

**Proposition 21** *Suppose that the ribbon Hopf algebra $H$ is moreover thin in the balanced category $\mathscr{C}$. Then, the category $Mod(H)$ is balanced.*

Every ribbon structure on the Hopf algebra $H$ induces a twist on the braided category $Mod(H)$

$$t_V \quad : \quad V \quad \longrightarrow \quad V$$

defined as



In particular,

**Proposition 22** *The element*

$$\sigma \quad = \quad \theta^{-1} \cdot u \quad \in \quad H$$

*is group-like:*

$$\Delta(\sigma) = \sigma \otimes \sigma.$$

We may summarize the content of this subsection as follows:

**Proposition 23** *Suppose given a ribbon Hopf algebra $H$ in a balanced dialogue category $\mathscr{C}$ together with a left $H$-module structure $\triangleright$ on the tensorial pole $\perp$. Suppose moreover that $H$ is thin in $\mathscr{C}$ and has an invertible antipode. Then, the category $Mod(H)$ of left $H$-modules is a balanced dialogue category with tensorial pole the left $H$-module $(\perp, \triangleright)$.*

**Proposition 24** *The subcategory of $H$-modules whose underlying object has a left dual... is a ribbon category.*

## 7.4 Fourth illustration: $*$-autonomous categories

The notion of cyclic $*$-autonomous category was originally introduced by Rosenthal in [25]. A slightly stronger notion of cyclic $*$-autonomous category was then considered by Blute, Lamarche and Ruet in [6]. We advise the interested reader to have a look at [9, 21] where Egger and McCurdy develop a careful comparison of the various possible notions of cyclic $*$-autonomous categories. The notions of helical and cyclic dialogue categories specialize as expected to $*$-autonomous categories.

**Definition 21** *A $*$-autonomous category is called helical (resp. cyclic) when the underlying dialogue category is helical (resp. cyclic).*

This leads to the following dictionary with the existing notions of cyclic categories in the literature.

**Proposition 25**

- *The notion of helical $*$-autonomous category coincides with the notion of cyclic $*$-autonomous category in the sense of Rosenthal in [25].*
- *The notion of cyclic $*$-autonomous category coincides with the notion of cyclic $*$-autonomous category in the sense of Blute, Lamarche, Ruet in [6].*

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