Parametric monads and enriched adjunctions

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Abstract

Starting from the particular case of the double negation monad in dialogue categories, we investigate what additional structure is required of an adjunction in order to give rise to a strong monad. The analysis leads to a purely combinatorial description of enriched functors, enriched natural transformations and enriched adjunctions between categorical modules.

1 Introduction

One basic principle of categorical algebra is that every monad T on a category $\mathscr{A}$ factors as $T = R \circ L$ where the functor L is left adjoint to the functor R in an adjunction of the form

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, bend left, "R"] \end{tikzcd}$$

This decomposition $T = R \circ L$ is not unique in general, and some decompositions are more illuminating than others. A typical illustration is provided by the double negation monad T in a dialogue category $\mathscr{A}$. Recall that a dialogue category $\mathscr{C}$ is a monoidal category equipped with an object $\perp$ together with two functors

$$\begin{array}{ccccccc} \mathscr{C}^{op} & \longrightarrow & \mathscr{C} & \qquad\qquad & \mathscr{C}^{op} & \longrightarrow & \mathscr{C} \\ A & \mapsto & A \multimap \bot & & A & \mapsto & \bot \multimapinv A \end{array}$$

and two families of bijections

$$\mathscr{C}(B, A \multimap \bot) \quad \cong \quad \mathscr{C}(A \otimes B, \bot) \quad \cong \quad \mathscr{C}(A, \bot \multimapinv B)$$

natural in A and B. As such, dialogue categories offer a suitable framework in order to interpret languages with linear continuations. Every dialogue category is equipped with the double negation monad defined as

$$T \quad : \quad A \quad \mapsto \quad \bot \multimapinv (A \multimap \bot).$$

It is not difficult to see that the monad T decomposes as the adjunction

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{C}^{op} \arrow[l, bend left, "R"] \end{tikzcd}$$

where L and R are defined as the two negation functors

$$L \quad : \quad A \quad \mapsto \quad A \multimap \bot, \qquad\qquad R \quad : \quad A \quad \mapsto \quad \bot \multimapinv A.$$

Now, the double negation functor T happens to be strong on the right. This means that there exists a family of morphisms

$$\sigma_{A,m} \quad : \quad T(A) \otimes m \quad \longrightarrow \quad T(A \otimes m)$$

natural in the objects A and m of the category $\mathscr{C}$, and making the expected series of coherence diagrams commute. Such a natural transformation is called a *right strength*. At this point, an important question is thus to understand what specific structure of the two functors L and R and of the adjunction $L \dashv R$ between them induces the existence of such a right strength σ.

First decomposition. One useful observation in our work on dialogue categories [5] is that the existence of the strength σ follows from the existence of a family of adjunctions

$$L(- \otimes m) \quad \dashv \quad R(m \otimes -)$$

parametrized by the objects m of the category $\mathscr{C}$. The point is that each of these adjunction amounts to the existence of a pair of families

$$\begin{array}{rcrcl} ax_m & : & L(A) & \longrightarrow & m \otimes L(A \otimes m) \\ cut_m & : & R(m \otimes A) \otimes m & \longrightarrow & R(m) \end{array}$$

natural in the object A and making the expected coherence diagrams

$$\begin{tikzcd} RL(A) \otimes m \arrow[r, "ax_m"] & R(m \otimes L(A \otimes m)) \otimes m \arrow[d, "cut_m"] \\ A \otimes m \arrow[u, "\eta_A \otimes m"] \arrow[r, "\eta_{A \otimes m}"] & RL(A \otimes m) \end{tikzcd}$$

$$\begin{tikzcd} A \otimes L(R(m \otimes A) \otimes m) \arrow[r, "cut_m"] & m \otimes LR(A) \arrow[d, "m \otimes \varepsilon_A"] \\ LR(m \otimes A) \arrow[u, "ax_m"] \arrow[r, "\varepsilon_{m \otimes A}"] & m \otimes A \end{tikzcd}$$

commute. By a parametric family of such adjunctions, we mean in addition that the two families ax_m and cut_m are dinatural in m, and that they are monoidal in the sense that the diagrams below

$$DIAGRAM$$

commute. Here, the two natural transformations η and ε denote the unit and counit of the adjunction $L \dashv R$. Now, it appears that the strength of the double negation monad T factors as:

$$\sigma_{A,m} \quad : \quad RL(A) \otimes m \xrightarrow{ax_m} R(m \otimes L(A \otimes m)) \otimes m \xrightarrow{cut_m} RL(A \otimes m)$$

This decomposition may be alternatively formulated by thinking of the tensor product of $\mathscr{C}$ as an action (on the right) noted

$$* \quad : \quad \mathscr{C} \times \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

of the monoidal category $\mathscr{C}$ on itself, and of the tensor product of $\mathscr{C}^{op}$ as an action (on the right again) noted

$$\ast\!\!- \quad : \quad \mathscr{C}^{op} \times \mathscr{C}^{op} \quad \longrightarrow \quad \mathscr{C}^{op}$$

of the monoidal category $\mathscr{C}^{op}$ on itself. See the definition of action in the next section. The two natural transformations “axiom” and “cut” are then reformulated accordingly as

$$ax_m \quad : \quad L(A) \quad \longrightarrow \quad L(A * m) \ast\!\!- m$$

$$cut_m \quad : \quad R(A \ast\!\!- m) * m \quad \longrightarrow \quad R(A)$$

together with the decomposition σ of the strength of the double negation monad $T = R \circ L$:

$$\sigma_{A,m} \quad : \quad RL(A) * m \xrightarrow{ax_m} R(L(A * m) \ast\!\!- m) * m \xrightarrow{cut_B} RL(A * m)$$

Second decomposition. This decomposition of the strength σ departs quite radically (at least apparently) from another decomposition of the same strength, based this time on the decomposition of the double negation monad $T = U \circ F$ as the canonical adjunction

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{C}_T \arrow[l, bend left, "U"] \end{tikzcd}$$

where $\mathscr{C}_T$ is the Kleisli category induced by the monad T. Observe that, in that case, the strength σ induces an action on the right

$$\otimes \quad : \quad \mathscr{C}_T \times \mathscr{C} \quad \longrightarrow \quad \mathscr{C}_T$$

of the monoidal category $\mathscr{C}$ on the Kleisli category $\mathscr{C}_T$. The situation also includes two families of functorial strengths

$$\sigma_F \quad : \quad F(A) \otimes m \quad \longrightarrow \quad F(A \otimes m)$$

$$\sigma_U \quad : \quad U(A) \otimes m \quad \longrightarrow \quad U(A \otimes m)$$

where the strength σ_U is moreover invertible. This leads to an alternative decomposition of the strength σ of the double negation monad $T = U \circ F$:

$$\sigma_{A,m} \quad : \quad UF(A) \otimes m \xrightarrow{\sigma_U} U(F(A) \otimes m) \xrightarrow{\sigma_F} UF(A \otimes m)$$

Using the notation of actions rather than of tensor products, one may rewrite the functorial strength as

$$\sigma_F \quad : \quad F(A) * m \quad \longrightarrow \quad F(A * m)$$

$$\sigma_U \quad : \quad U(A) * m \quad \longrightarrow \quad U(A * m)$$

together with the decomposition as:

$$\sigma_{A,m} \quad : \quad UF(A) * m \xrightarrow{\sigma_U} U(F(A) * m) \xrightarrow{\sigma_F} UF(A * m) \,.$$

where $*$ denotes the two actions on the right

$$* \quad : \quad \mathscr{C} \times \mathscr{C} \quad \longrightarrow \quad \mathscr{C} \qquad\qquad * \quad : \quad \mathscr{C}_T \times \mathscr{C} \quad \longrightarrow \quad \mathscr{C}_T$$

of the monoidal category $\mathscr{C}$ on the category $\mathscr{C}$ and on its Kleisli category $\mathscr{C}_T$ respectively.

The two decompositions of the strength σ are apparently very different. In particular, the first decomposition seems to be of an entirely new and unknown nature, since it requires a combination of actions of the monoidal category $\mathscr{C}$ as well as of its opposite category $\mathscr{C}^{op}$. The second decomposition is more familiar, since it only requires a pair of actions of the monoidal category $\mathscr{C}$ on the category $\mathscr{C}$ and on its Kleisli category $\mathscr{C}_T$. For a very long time, we believed that the two constructions were intrinsically different. We were simply wrong... Indeed, the purpose of this note is precisely to unify the two decompositions of T by recasting the two adjunctions $L \dashv R$ and $F \dashv U$ in the same enriched framework. In particular, we show that the first decomposition of the double negation monad is based on an enriched adjunction – an observation which one may find interesting for its own sake. Also, we will see that our purpose is not to translate everything in the language of enriched category theory. Quite on the contrary: our main purpose is to clarify what combinatorial structures underlie the notions of "enrichment" in a number of examples of interest where the enriched categories are coming from an action of a monoidal category $\mathscr{M}$.

Plan of the paper. Given monoidal category $\mathscr{M}$, we start by introducing in §2 the notion of positive $\mathscr{M}$-category. We explain in §3 how every such positive $\mathscr{M}$-category may be seen as an enriched category over the presheaf category $[\mathscr{M}^{op}, Set]$. We then introduce in §4 the dual notion of negative $\mathscr{M}$-category which may be also seen as an enriched category over the presheaf category $[\mathscr{M}^{op}, Set]$, but in a different way. In each positive and negative case, we characterize the enriched functors and natural transformations between these $\mathscr{M}$-categories. We complete the pictureby characterizing in §5 the enriched functors and natural transformations between a positive and a negative $\mathscr{M}$-category. This leads to the definition in §6 of the 2-category $\mathscr{M}-\mathrm{Mod}$ which embeds fully and faithfully in the 2-category of enriched categories, enriched functors and enriched natural transformations. We illustrate these ideas with a series of examples provided in §7 and conclude the paper in §8.

2 Positive $\mathscr{M}$-categories

The starting point of our analysis is provided by the notion of $\mathscr{M}$-category, defined as a categorical module $\mathscr{A}$ over a monoidal category $\mathscr{M}$. More specifically, given a monoidal category $\mathscr{M}$, we construct a 2-category $\mathscr{M}^{+}-\mathrm{Mod}$ with

- positive $\mathscr{M}$-categories as 0-cells,
- $\mathscr{M}$-functors as 1-cells,
- natural $\mathscr{M}$-transformations as 2-cells.

We will see in the next section that these data are tightly connected to the notions of enriched functors and enriched natural transformations.

Definition 1 (positive action) *A positive action of a monoidal category $(\mathscr{M}, \otimes, I)$ on a category $\mathscr{A}$ is defined as a functor*

$$* \quad : \quad \mathscr{M} \times \mathscr{A} \quad \longrightarrow \quad \mathscr{A}$$

which transports every pair of objects (m, A) to an object $m * A$ of the category $\mathscr{A}$, together with a pair of natural transformations:

$$\delta_{m,n,A} : (m \otimes n) * A \longrightarrow n * (m * A) \qquad\qquad \delta_A : I * A \longrightarrow A$$

satisfying the three expected coherence laws:

$$\begin{tikzcd}
(m \otimes n \otimes p) * A \arrow[r, "\delta"] \arrow[d, "\delta"'] & m * ((n \otimes p) * A) \arrow[d, "\delta"] \\
(m \otimes n) * (p * A) \arrow[r, "\delta"] & (m * (n * (p * A)))
\end{tikzcd}$$

$$\begin{tikzcd}
m * A \arrow[r, equal, "id"] \arrow[d] & m * A \\
(m \otimes I) * A \arrow[r, "\delta"] & m * (I * A) \arrow[u, "\delta"']
\end{tikzcd}$$

$$\begin{tikzcd}
m * A \arrow[r, equal, "id"] \arrow[d] & m * A \\
(I \otimes m) * A \arrow[r, "\delta"] & I * (m * A) \arrow[u, "\delta"']
\end{tikzcd}$$

An action is called pseudo *when the natural transformations are isomorphisms, and* strict *when the monoidal category $\mathscr{M}$ is strict and the natural transformations are equalities.*

Note that the notion of positive action generalizes the notion of comonad: indeed, a comonad in a category $\mathscr{A}$ is the same thing as positive action of the trivial monoidal category $\mathscr{M} = 1$ on the category $\mathscr{A}$.

Definition 2 (positive categories) *A positive $\mathscr{M}$, or $\mathscr{M}^+$-category, is a category $\mathscr{A}$ equipped with a positive action of the monoidal category $\mathscr{M}$.*

Definition 3 (positive functors) *An $\mathscr{M}^+$-functor*

$$(F, \sigma) \quad : \quad (\mathscr{A}, *) \quad \longrightarrow \quad (\mathscr{B}, *)$$

between two $\mathscr{M}^+$-categories is defined as a functor

$$F \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}$$

between the underlying categories, together with a natural transformation

$$\sigma_{m,A} \quad : \quad m * F(A) \quad \longrightarrow \quad F(m * A)$$

making the two coherence diagrams

$$\begin{tikzcd}
& F((m \otimes n) * A) \arrow[dr, "\delta"] & \\
(m \otimes n) * FA \arrow[ur, "\sigma_{m \otimes n}"] \arrow[d, "\delta"'] & & F(m * (n * A)) \\
m * (n * FA) \arrow[rr, "\sigma_n"] & & m * F(n * A) \arrow[u, "\sigma_m"']
\end{tikzcd} \tag{1}$$

$$
\begin{tikzcd}
I * FA \arrow[rr] \arrow[dr, "\sigma_I"'] & & FA \\
& F(I * A) \arrow[ur] &
\end{tikzcd}
\tag{2}
$$

commute.

Definition 4 (natural transformation) *A natural $\mathscr{M}^+$-transformation*

$$\theta \quad : \quad (F, \sigma) \quad \Rightarrow \quad (G, \tau) \quad : \quad (\mathscr{A}, *) \quad \longrightarrow \quad (\mathscr{B}, *)$$

is a natural transformation between the underlying functors

$$\theta \quad : \quad F \quad \Rightarrow \quad G \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}$$

such that the coherence diagram

$$
\begin{tikzcd}
m * FA \arrow[r, "\sigma_m"] \arrow[d, "m * \theta_A"'] & F(m * A) \arrow[d, "\theta_{m*A}"] \\
m * GA \arrow[r, "\tau_m"] & G(m * A)
\end{tikzcd}
$$

commutes for all objects A and m.

Recall from the work of Anders Kock [3] that

Definition 5 *A strong monad on a $\mathscr{M}^+$-category $(\mathscr{A}, *)$ is a pair consisting of a monad T in the category $\mathscr{A}$ together with a natural transformation*

$$\sigma \quad : \quad m * T(A) \quad \longrightarrow \quad T(m * A)$$

making the familiar diagrams commute.

The following statement is essentially straightforward.

Proposition 1 *A strong monad on a $\mathscr{M}^+$-category $(\mathscr{A}, *)$ is the same thing as a monad*

$$(T, \mu, \eta) \quad : \quad (\mathscr{A}, *) \quad \longrightarrow \quad (\mathscr{A}, *)$$

in the 2-category $\mathscr{M}^+ - \mathrm{Mod}$.

3 Enrichment

We suppose from now on that the monoidal category $\mathscr{M}$ is small. At this point, we are ready to establish that

Proposition 2 *Every $\mathscr{M}^+$-category $\mathscr{A}$ may be seen as an enriched category $\iota^+\mathscr{A}$ over the presheaf category $[\mathscr{M}^{op}, Set]$ equipped with the Day tensor product on $\mathscr{M}$. The hom-space*

$$\mathscr{A}[A, B]$$

from an object A to an object B in the enriched category $\mathscr{A}$ is defined as the following presheaf:

$$\begin{array}{ccccc} \mathscr{A}[A, B] & : & \mathscr{M}^{op} & \longrightarrow & Set \\ & & m & \mapsto & \mathscr{A}(m * A, B) \end{array}$$

where $\mathscr{A}(A, B)$ denotes the hom-set from A to B in the category $\mathscr{A}$.

Recall that the Day tensor product of two presheaves is defined using the coend formula:

$$\varphi \otimes_{Day} \psi \quad : \quad m \quad \mapsto \quad \int^{m_1, m_2 \in \mathscr{M}} \mathscr{M}(m, m_1 \otimes m_2) \times \varphi(m_1) \times \psi(m_2)$$

This tensor product equips the presheaf category $[\mathscr{M}^{op}, Set]$ with the structure of a monoidal category, whose tensorial unit is provided by the representable presheaf

$$y_I \quad : \quad m \quad \mapsto \quad \mathscr{M}(m, I)$$

associated to the unit of the original monoidal category $\mathscr{M}$. Note that this construction of the enriched category generalizes the familar construction of the Kleisli category associated to a comonad in the category $\mathscr{A}$. Indeed, in the case of a trivial category $\mathscr{M} = 1$, the category $\iota^+\mathscr{A}$ is enriched over $[\mathscr{M}^{op}, Set]$ which coincides in that case with Set. This category $\iota^+\mathscr{A}$ coincides with the Kleisli category associated to the comonad $A \mapsto I * A$ where I denotes the unique object of the monoidal category $\mathscr{M} = 1$.

Proposition 3 *The construction of Proposition 2 induces a 2-functor*

$$\iota^+ \quad : \quad \mathscr{M}^+\!-\!\mathbf{Mod} \quad \longrightarrow \quad [\mathscr{M}^{op}, Set]\!-\!Cat$$

which is moreover full and faithful in the sense that

$$\mathscr{M}^+\!-\!\mathbf{Mod}((\mathscr{A}, *), (\mathscr{B}, *)) \quad \cong \quad [\mathscr{M}^{op}, Set]\!-\!Cat(\iota^+(\mathscr{A}, *), \iota^+(\mathscr{B}, *))$$

where $\cong$ means isomorphism of categories.

In other words, an $\mathscr{M}^+$-functor and a natural $\mathscr{M}^+$-transformation between two positive $\mathscr{M}$-categories $(\mathscr{A}, *)$ and $(\mathscr{B}, *)$ are the same things as an enriched functor and an enriched natural transformation between the associated enriched categories.

4 Negative $\mathscr{M}$-categories

We introduce below the notion of negative $\mathscr{M}$-category, or $\mathscr{M}^-$-category, which is dual to the notion of positive $\mathscr{M}$-category.

Definition 6 (negative action) *A negative action of a monoidal category $\mathscr{M} = (\mathscr{M}, \otimes, I)$ on a category $\mathscr{A}$ is defined as a positive action of the opposite monoidal category $\mathscr{M}^{op}$ on the opposite category $\mathscr{A}^{op}$.*

Now, recall that the opposite monoidal category $\mathscr{M}^{op(0,1)} = (\mathscr{M}^{op}, \otimes^{op}, I)$ is obtained from $\mathscr{M}$ by

- reversing the direction of the morphisms of dimension 1, this justifying the index 1,
- reversing the orientation of the tensor product of dimension 0, this justifying the index 0.

This leads to the definition of negative $\mathscr{M}$-category.

Definition 7 (negative categories) *A negative $\mathscr{M}$-category, or $\mathscr{M}^-$-category, is a category $\mathscr{A}$ equipped with a negative action of the opposite monoidal category $\mathscr{M}^{op(0,1)}$.*

The definition of negative $\mathscr{M}$-category is fine, but a bit concise, and we thus expand it in the following statement.

Proposition 4 *A negative $\mathscr{M}$-category is the same thing as a category $\mathscr{A}$ equipped with a functor*

$$\multimap\!\!* \quad : \quad \mathscr{M}^{op} \times \mathscr{A} \quad \longrightarrow \quad \mathscr{A}$$

which transports every pair of objects (m, A) to an object $m \multimap\!\!* A$ of the category $\mathscr{A}$, together with a pair of natural transformations:

$$\mu_{m,n,A} : n \multimap\!\!* (m \multimap\!\!* A) \longrightarrow (m \otimes n) \multimap\!\!* A \qquad\qquad \mu_A : A \longrightarrow I \multimap\!\!* A$$

satisfying the three coherence laws:

$$
\begin{tikzcd}
p -\!\!* (n -\!\!* (m -\!\!* A)) \arrow[r, "\mu"] \arrow[d, "\mu"'] & (n \otimes p) -\!\!* (m -\!\!* A) \arrow[d, "\mu"] \\
p -\!\!* ((m \otimes n) -\!\!* A) \arrow[r, "\mu"] & (m \otimes n \otimes p) -\!\!* A
\end{tikzcd}
$$

$$
\begin{tikzcd}
m -\!\!* A \arrow[r, equal, "id"] \arrow[d, "\mu"'] & m -\!\!* A \arrow[d] \\
I -\!\!* (m -\!\!* A) \arrow[r, "\mu"] & (m \otimes I) -\!\!* A
\end{tikzcd}
$$

$$
\begin{tikzcd}
m -\!\!* A \arrow[r, equal, "id"] \arrow[d, "\mu"'] & m -\!\!* A \arrow[d] \\
m -\!\!* (I -\!\!* A) \arrow[r, "\mu"] & (I \otimes m) -\!\!* A
\end{tikzcd}
$$

Definition 8 (negative functors) *An $\mathscr{M}^-$-functor*

$$(F, \sigma) \quad : \quad (\mathscr{A}, -\!\!*) \quad \longrightarrow \quad (\mathscr{B}, -\!\!*)$$

between two negative $\mathscr{M}$-categories is defined as a functor

$$F \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}$$

between the underlying categories, together with a natural transformation

$$\sigma_{m,A} \quad : \quad F(m -\!\!* A) \quad \longrightarrow \quad m -\!\!* F(A)$$

making the two coherence diagrams

$$\begin{tikzcd}
& n{-\!\!*}F(m{-\!\!*}A) \arrow[dr, "\sigma_n"] & \\
F(n{-\!\!*}(m{-\!\!*}A)) \arrow[ur, "\sigma_n"] \arrow[d, "\mu"'] & & n{-\!\!*}(m{-\!\!*}FA) \arrow[d, "\mu"] \\
F((m\otimes n){-\!\!*}A) \arrow[rr, "\sigma_{m\otimes n}"] & & (m\otimes n){-\!\!*}FA
\end{tikzcd}$$

$$\begin{tikzcd}
F(I{-\!\!*}A) \arrow[rr, "\mu"] \arrow[dr, "\sigma_I"'] & & FA \\
& I{-\!\!*}FA \arrow[ur, "\mu"'] &
\end{tikzcd}$$

commute.

Definition 9 (natural transformation) *A natural $\mathscr{M}$-transformation*

$$\theta \quad : \quad (F,\sigma) \quad \Rightarrow \quad (G,\tau) \quad : \quad (\mathscr{A},{-\!\!*}) \quad \longrightarrow \quad (\mathscr{B},{-\!\!*})$$

is a natural transformation between the underlying functors

$$\theta \quad : \quad F \quad \Rightarrow \quad G \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}$$

such that the coherence diagram

$$\begin{tikzcd}
F(m{-\!\!*}A) \arrow[r, "\sigma_m"] \arrow[d, "\theta_{m{-\!\!*}A}"'] & m{-\!\!*}FA \arrow[d, "m{-\!\!*}\theta_A"] \\
G(m{-\!\!*}A) \arrow[r, "\tau_m"] & m{-\!\!*}GA
\end{tikzcd}$$

commutes for all objects A and m.

One reason for introducing the notion of negative $\mathscr{M}$-category is that it provides us with an alternative way to see a category $\mathscr{A}$ as an enriched category over the presheaf category $[\mathscr{M}^{op}, Set]$. The idea is to let the objects m of the monoidal category $\mathscr{M}$ act on the target object B rather than on the source object A of a morphism $A \to B$ of the category $\mathscr{A}$. Indeed, we establish that:

Proposition 5 *Every negative $\mathscr{M}$-category $(\mathscr{A}, \multimap\!\!*)$ may be seen as an enriched category over the presheaf category $[\mathscr{M}^{op}, Set]$ equipped with the Day tensor product. The presheaf $[A, B]$ over the category $\mathscr{M}$ is defined as follows:*

$$\mathscr{A}[A, B] \quad : \quad m \quad \mapsto \quad \mathscr{A}(A, m \multimap\!\!* B)$$

where $m \multimap\!\!* B$ denotes the result of letting the object m in $\mathscr{M}^{op}$ act on the object B of the category $\mathscr{A}$.

This very observation leads us to introduce the 2-category $\mathscr{M}^- - \mathbf{Mod}$ with

- negative $\mathscr{M}$-categories as 0-cells,
- $\mathscr{M}^-$-functors as 1-cells,
- natural $\mathscr{M}^-$-transformations as 2-cells.

Proposition 6 *The construction of Proposition 5 induces a 2-functor*

$$\iota^- \quad : \quad \mathscr{M}^- - \mathbf{Mod} \quad \longrightarrow \quad [\mathscr{M}^{op}, Set] - Cat$$

which is moreover full and faithful in the strong sense that

$$\mathscr{M}^- - \mathbf{Mod}((\mathscr{A}, *), (\mathscr{B}, *)) \quad \cong \quad [\mathscr{M}^{op}, Set] - Cat(\iota^-(\mathscr{A}, *), \iota^-(\mathscr{B}, *))$$

where $\cong$ means isomorphism of categories.

In other words, an $\mathscr{M}^-$-functor and a natural $\mathscr{M}^-$-transformation between negative $\mathscr{M}$-categories $(\mathscr{A}, \multimap\!\!*)$ and $(\mathscr{B}, \multimap\!\!*)$ are the same thing as an enriched functor and an enriched natural transformation between the associated enriched categories.

5 Transverse functors

At this point, we have constructed two full and faithful functors

$$\iota^+ : \mathscr{M}^+ - \mathbf{Mod} \longrightarrow [\mathscr{M}^{op}, Set] - Cat \qquad\qquad \iota^- : \mathscr{M}^- - \mathbf{Mod} \longrightarrow [\mathscr{M}^{op}, Set] - Cat$$

each of them characterizing the enriched functors and natural transformations between positive $\mathscr{M}$-categories and negative $\mathscr{M}^{op(0,1)}$-categories, seen as enriched categories. This leads to the question of characterizing the enriched functors and natural transformations from an object of $\mathscr{M}^+ - \mathrm{Mod}$ to an object of $\mathscr{M}^- - \mathrm{Mod}$, and conversely, from an object of $\mathscr{M}^- - \mathrm{Mod}$ to an object of $\mathscr{M}^+ - \mathrm{Mod}$. As we will see, the answer is illuminating, and clarifies the algebraic status of the adjunction $L \dashv R$ between the two negations of a dialogue category mentioned in the introduction. So, let us consider

- a positive $\mathscr{M}$-module $(\mathscr{A}, *)$,
- a negative $\mathscr{M}$-module $(\mathscr{B}, \multimap\!\!*)$.

Definition 10 (transverse functor) *A transverse $\mathscr{M}^{+-}$-functor*

$$(F, ax) \quad : \quad (\mathscr{A}, *) \quad \longrightarrow \quad (\mathscr{B}, \multimap\!\!*)$$

is a functor between the underlying categories

$$F \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}$$

together with a family of morphisms

$$ax_{m,A} \quad : \quad FA \quad \longrightarrow \quad m \multimap\!\!* F(m * A)$$

natural in A and dinatural in m, such that the coherence diagrams

$$\begin{tikzcd}
F(A) \arrow[r, "ax_{m\otimes n}"] \arrow[d, "ax_n"'] & (m \otimes n) \multimap\!\!* F((m \otimes n) * A) \arrow[dd, "\delta"] \\
n \multimap\!\!* F(n * A) \arrow[d, "ax_n"'] & \\
n \multimap\!\!* (m \multimap\!\!* F(m * (n * A))) \arrow[r, "\mu"] & (m \otimes n) \multimap\!\!* F(m * (n * A))
\end{tikzcd}$$

By dinaturality in m, we mean that the diagram

$$\begin{tikzcd}
FA \arrow[r, "ax_m"] \arrow[d, "ax_n"'] & m \multimap\!\!* F(m * A) \arrow[d, "m \multimap\!\!* F(h * A)"] \\
n \multimap\!\!* F(n * A) \arrow[r, "h \multimap\!\!* F(n * A)"] & m \multimap\!\!* F(n * A)
\end{tikzcd}$$

commutes for every morphism $h : m \to n$ in the category $\mathscr{M}$.

Definition 11 (natural transformations) *A natural $\mathscr{M}^{+-}$-transformation*

$$\theta \quad : \quad (F, ax) \quad \Rightarrow \quad (G, ax) \quad : \quad (\mathscr{A}, *) \quad \longrightarrow \quad (\mathscr{B}, \multimap\!\!*)$$

is a natural transformation between the underlying functors

$$\theta \quad : \quad F \quad \Rightarrow \quad G \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}$$

making the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "ax_m"] \arrow[d, "\theta_A"'] & m \multimap\!\!* F(m * A) \arrow[d, "m \multimap\!\!* \theta_{m*A}"] \\
G(A) \arrow[r, "ax_m"] & m \multimap\!\!* G(m * A)
\end{tikzcd}$$

commute for all objects A and m.

The definition of transverse functor and natural transformations are justified by the following proposition.

Proposition 7 *A transverse $\mathscr{M}^{+-}$-functor*

$$(\mathscr{A}, *) \quad \longrightarrow \quad (\mathscr{B}, \multimap\!\!*)$$

is the same thing as a functor

$$\iota^{+}(\mathscr{A}, *) \quad \longrightarrow \quad \iota^{-}(\mathscr{B}, \multimap\!\!*)$$

enriched over the presheaf category $[\mathscr{M}^{op}, Set]$. Similarly, a natural $\mathscr{M}^{+-}$-transformation

$$\theta \quad : \quad (F, \sigma) \quad \Rightarrow \quad (G, \tau) \quad : \quad (\mathscr{A}, *) \quad \longrightarrow \quad (\mathscr{B}, \multimap\!\!*)$$

is the same thing as an enriched natural transformation between the underlying enriched functors.

Similarly, the enriched functors and natural transformations from a negative $\mathscr{M}$-category to a positive $\mathscr{M}$-category are characterized by the following definitions. Suppose given

- a negative $\mathscr{M}$-module $(\mathscr{A}, \multimap\!\!*)$,
- a positive $\mathscr{M}$-module $(\mathscr{B}, *)$.

Definition 12 (transverse functor) *A transverse $\mathscr{M}^{-+}$-functor*

$$(F, cut) \quad : \quad (\mathscr{A}, \twoheadrightarrow\!\!*) \quad \longrightarrow \quad (\mathscr{B}, *)$$

is a functor between the underlying categories

$$F \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}$$

together with a family of morphisms

$$cut_{m,A} \quad : \quad m * F(m \multimap\!\!* A) \quad \longrightarrow \quad FA$$

natural in A and dinatural in m, such that the coherence diagrams

$$\begin{tikzcd}
(m \otimes n) * F(n \multimap\!\!*(m \multimap\!\!* A)) \arrow[r, "\delta"] \arrow[dd, "\mu"'] & m * (n * F(n \multimap\!\!*(m \multimap\!\!* A))) \arrow[d, "cut_n"] \\
& m * F(m \multimap\!\!* A))) \arrow[d, "cut_m"] \\
(m \otimes n) * F((m \otimes n) \multimap\!\!* A) \arrow[r, "cut_{m \otimes n}"] & FA
\end{tikzcd}$$

By dinaturality in m, we mean that the diagram

$$\begin{tikzcd}
m * F(n \multimap\!\!* A) \arrow[r, "m * F(h \multimap\!\!* A)"] \arrow[d, "h * F(n \multimap\!\!* A)"'] & m * F(m \multimap\!\!* A) \arrow[d, "cut_m"] \\
n * F(n \multimap\!\!* A) \arrow[r, "cut_n"] & FA
\end{tikzcd}$$

commutes for every morphism $h : m \to n$ in the category $\mathscr{M}$.

Definition 13 (natural transformations) *A natural $\mathscr{M}^{-+}$-transformation*

$$\theta \quad : \quad (F, cut) \quad \Rightarrow \quad (G, cut) \quad : \quad (\mathscr{A}, \multimap\!\!*) \quad \longrightarrow \quad (\mathscr{B}, *)$$

is a natural transformation between the underlying functors

$$\theta \quad : \quad F \quad \Rightarrow \quad G \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}$$

making the diagram

$$\begin{tikzcd} m * F(m \multimap A) \arrow[r, "cut_m"] \arrow[d, "m * \theta_{m \multimap A}"'] & FA \arrow[d, "\theta_A"] \\ m * G(m \multimap A) \arrow[r, "cut_m"] & GA \end{tikzcd}$$

commute for all objects m and A.

6 General $\mathscr{M}$-categories

At this point, we are ready to construct the 2-category $\mathscr{M}-\mathbf{Mod}$ with

- the positive as well as the negative $\mathscr{M}$-categories as 0-cells,
- the positive, negative or transverse $\mathscr{M}$-functors as 1-cells,
- the positive, negative or transverse natural $\mathscr{M}$-transformations as 2-cells.

As we explained in the introduction, the 2-category $\mathscr{M}-\mathbf{Mod}$ provides a purely combinatorial description of the 2-category of enriched categories over $[\mathscr{M}^{op}, Set]$ "restricted" to the positive and negative $\mathscr{M}$-categories. This informal statement is formalized as follows.

Proposition 8 *Put together, the constructions ι^+ and ι^- induce a 2-functor*

$$\mathscr{M}-\mathbf{Mod} \quad \longrightarrow \quad [\mathscr{M}^{op}, Set]-Cat$$

which is full and faithful in the sense that

$$\mathscr{M}-\mathbf{Mod}(\mathscr{A}, \mathscr{B}) \quad \cong \quad [\mathscr{M}^{op}, Set]-Cat(\iota\mathscr{A}, \iota\mathscr{B})$$

for every positive or negative $\mathscr{M}$-categories $\mathscr{A}$ and $\mathscr{B}$, where $\cong$ means isomorphism of categories.

A technical benefit of this combinatorial reformulation of enriched functors and natural transformations is that one may relax the condition that the monoidal category $\mathscr{M}$ is small.

7 Illustrations

In this section, we review a series of examples of strong monads $T = R \circ L$ on $\mathscr{M}$-categories arising from an enriched adjunction $L \dashv R$. Our starting point is provided by the two different decompositions of the double negation monad in a dialogue category.

First decomposition. The adjunction

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{C}^{op} \arrow[l, bend left, "R"] \end{tikzcd}$$

where L and R are defined as the two negation functors

$$L \quad : \quad A \quad \mapsto \quad A \multimap \bot, \qquad\qquad R \quad : \quad A \quad \mapsto \quad \bot \multimapinv A.$$

happens to define an adjunction in the 2-category $\mathscr{M}-\mathrm{Mod}$ where $\mathscr{M}$ is defined as $\mathscr{C}^{op(1)} = (\mathscr{C}, \otimes^{op}, I)$. The fact that one needs to take the opposite tensor product is just a matter of convention in the definition of our $\mathscr{M}$-modules. The important point to observe is that the monoidal category $\mathscr{M}$ acts *positively* on the category $\mathscr{C}$ by the tensor product:

$$\begin{array}{ccccc} * & : & \mathscr{M} \times \mathscr{C} & \longrightarrow & \mathscr{C} \\ & & (m, A) & \mapsto & A \otimes m \end{array}$$

and that its opposite monoidal category $\mathscr{M}^{op(0,1)} = \mathscr{C}^{op}$ acts *negatively* on the category $\mathscr{C}^{op}$ by the same tensor product:

$$\begin{array}{ccccc} \multimap\!\!* & : & \mathscr{M}^{op(0,1)} \times \mathscr{C}^{op} & \longrightarrow & \mathscr{C}^{op} \\ & & (m, A) & \mapsto & m \otimes A. \end{array}$$

considered this time in the opposite category $\mathscr{C}^{op}$. This means that the category $\mathscr{C}$ should be considered as a positive $\mathscr{M}$-category $(\mathscr{C}, *)$ whereas its opposite category $\mathscr{C}^{op}$ defines a negative $\mathscr{M}$-category $(\mathscr{C}^{op}, \multimap\!\!*)$. From this, it is not difficult to see that the adjunction $L \dashv R$ defines an adjunction in the 2-category $\mathscr{M}-\mathrm{Mod}$. The axiom and cut are given by the natural transformations

$$ax_m \quad : \quad L(X) \quad \longrightarrow \quad m \otimes L(X \otimes m)$$

$$cut_m \quad : \quad R(m \otimes X) \otimes m \quad \longrightarrow \quad R(X).$$

From this follows by Proposition 8 that the canonical adjunction $L \dashv R$ of dialogue categories is enriched over the category $[\mathscr{C}^{op(0,1)}, Set]$ equipped with its Day tensor product. More generally, one can show that

Proposition 9 *An adjunction*

$$\begin{tikzcd} (\mathscr{A}, *) \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & (\mathscr{B}, -\!\!*) \arrow[ll, bend left, "R"] \end{tikzcd}$$

between a positive $\mathscr{M}$-category $\mathscr{A}$ and a negative $\mathscr{M}$-category $\mathscr{B}$ is the same thing as a family of categorical adjunctions

$$L(m * -) \quad \dashv \quad R(m -\!\!* -)$$

natural in the object m of the category $\mathscr{M}$.

Second decomposition. The following statement clarifies the algebraic status of our second decomposition of the double negation monad as $T = U \circ F$.

Proposition 10 *Every monad T in the 2-category $\mathscr{M}^+ - \mathrm{Mod}$ induces an adjunction*

$$\begin{tikzcd} \mathscr{C} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & \mathscr{C}_T \arrow[ll, bend left, "U"] \end{tikzcd}$$

living in the 2-category $\mathscr{M}^+ - \mathrm{Mod}$, where $\mathscr{C}_T$ is the Kleisli category of the monad T equipped with the action

$$\circledast \quad : \quad \mathscr{M} \times \mathscr{C}_T \quad \longrightarrow \quad \mathscr{C}_T$$

defined on two morphisms

$$f \quad : \quad A \quad \longrightarrow \quad T(B) \qquad\qquad h \quad : \quad m \quad \longrightarrow \quad n$$

as follows:

$$h \circledast f \quad : \quad m * A \xrightarrow{h*f} n * T(B) \xrightarrow{\sigma_n} T(n * B)\,.$$

This is precisely the way the double negation monad T factors as $T = U \circ F$ in the introduction.

The state monad. Another example of adjunction in $\mathscr{M}^+-\mathrm{Mod}$ where $\mathscr{M} = \mathscr{C}$ is provided by the state monad

$$T \quad : \quad A \quad \mapsto \quad S \multimap (S \otimes A)$$

in any monoidal closed category $\mathscr{C}$, where S is a specific object of $\mathscr{C}$, describing a space of states for the system. In that case, the monad T factors as the adjunction

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{C} \arrow[l, bend left, "R"] \end{tikzcd}$$

where $L : A \mapsto S \otimes A$ and $R : A \mapsto S \multimap A$. It is not difficult to see that the two functors L and R are $\mathscr{M}$-functors between the positive $\mathscr{M}$-module $\mathscr{C}$ and itself – as testified by the existence of the natural transformations

$$L(A) \otimes m = (S \otimes A) \otimes m \quad \cong \quad S \otimes (A \otimes m) = L(A \otimes m)$$

$$R(A) \otimes m = (S \multimap A) \otimes m \quad \longrightarrow \quad S \multimap (A \otimes m) = R(A \otimes m)$$

As a matter of fact, one easily checks that the adjunction lives in the category $\mathscr{M}^+-\mathrm{Mod}$. This provides a conceptual way to understand why the state monad is strong.

The Eilenberg-Moore adjunction. Suppose given a strong monad T on a monoidal closed category $\mathscr{C}$. By monoidal closed, we mean that the functor

$$A \quad \mapsto \quad m \otimes A \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

has a right adjoint denoted as

$$A \quad \mapsto \quad m \multimap A \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

for every object m of the monoidal category $\mathscr{M} = \mathscr{C}$. In that case, the category of algebras $\mathscr{C}^T$ defines a negative $\mathscr{M}$-module $(\mathscr{C}^T, \multimap)$. The action of an object m of $\mathscr{C}$ on a T-algebra

$$a \quad : \quad TA \quad \longrightarrow \quad A$$

is defined as the T-algebra

$$m \multimap a \quad : \quad T(m \multimap A) \xrightarrow{\bar{\sigma}} m \multimap T(A) \longrightarrow m \multimap A.$$

where the natural transformation

$$\overline{\sigma}_{m,A} \quad : \quad T(m \multimap A) \quad \longrightarrow \quad m \multimap T(A)$$

follows from the strength of the monad T. Hence, the category $\mathscr{C}$ defines a positive $\mathscr{M}$-category whereas its category of T-algebras $\mathscr{C}^T$ defines a negative category. It is not difficult to check then that the adjunction

$$\begin{tikzcd} \mathscr{C} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & \mathscr{C}^T \arrow[ll, bend left, "U"] \end{tikzcd}$$

is an adjunction living inside the 2-category $\mathscr{M}-\mathrm{Mod}$ of positive and negative $\mathscr{M}$-categories. In other words, it is an adjunction $F \dashv U$ together with a family of adjunctions

$$F(m * -) \quad \dashv \quad U(m \twoheadrightarrow\!\!* -)$$

parametrized by the object m of the category $\mathscr{C}$. This enables to decompose the strength σ of the monad T in the same way as in the case of the double negation monad:

$$\sigma_{A,m} \quad : \quad m \otimes UF(A) \xrightarrow{ax_m} m \otimes U(m \multimap F(m \otimes A)) \xrightarrow{cut_m} UF(m \otimes A)$$

using the two natural transformations

$$ax_m \quad : \quad F(A) \quad \longrightarrow \quad m \multimap F(m \otimes A)$$

$$cut_m \quad : \quad m \otimes U(m \multimap A) \quad \longrightarrow \quad U(A).$$

Note that there is another way to decompose the strength of the monad T, which is to think of the original category $\mathscr{C}$ as a negative $\mathscr{M}$-category $\mathscr{C}$ thanks to the action

$$\multimap \quad : \quad \mathscr{M}^{op} \times \mathscr{C} \quad \longrightarrow \quad \mathscr{C}.$$

In that acception, the strength of T is decomposed in a more expected way:

$$\overline{\sigma}_{m,A} \quad : \quad UF(m \multimap A) \xrightarrow{\overline{\sigma}_F} U(m \multimap FA) \xrightarrow{\overline{\sigma}_U} m \multimap UFA.$$

The effect calculus. In a series of recent papers, Jeff Egger, Rasmus Møgelberg, Alex Simpson introduce a basic effect-calculus (EC) together with a semantic notion of model for the calculus, see [1] and [2]. Here, we follow [2] and consider an EC model based on a cartesian category $\mathscr{M}$ rather than on a cartesian closed category as in the preliminary version of the paper [1]. We recall the definition of EC model below.

Definition 14 ([2]) *An effect-calculus model is an $[\mathscr{M}^{op}, Set]$-enriched adjunction*

$$\begin{tikzcd} \mathscr{M} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & \mathscr{A} \arrow[ll, bend left, "U"] \end{tikzcd}$$

between a cartesian category $\mathscr{M}$ and a $[\mathscr{M}^{op}, Set]$-enriched category $\mathscr{A}$. One requires moreover that the category $\mathscr{A}$ has $[\mathscr{M}^{op}, Set]$-cotensors indexed by representables, and finite $[\mathscr{M}^{op}, Set]$-enriched products.

We relate this definition of effect-calculus model to the notion of $\mathscr{M}$-category using the following elementary observation. Suppose given a monoidal category $\mathscr{M}$. In that case:

Proposition 11 *A category $\mathscr{A}$ enriched over $[\mathscr{M}^{op}, Set]$ which is moreover equipped with $[\mathscr{M}^{op}, Set]$-cotensors indexed by representables $y_m = \mathscr{M}(-,-)$ is the same thing as a negative $\mathscr{M}$-category $(A, \multimap\!\!*)$ with invertible coercions*

$$\mu_{m,n,A} : n \multimap\!\!* (m \multimap\!\!* A) \longrightarrow (m \otimes n) \multimap\!\!* A, \qquad\qquad \mu_A : A \longrightarrow I \multimap\!\!* A.$$

This correspondence enables to reformulate the effect-calculus models in a purely combinatorial way, where the underlying notion of enrichment is not mentioned anymore.

Proposition 12 *An effect-calculus model on a cartesian category $\mathscr{M}$ is the same thing as a categorical adjunction*

$$\begin{tikzcd} \mathscr{M} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & \mathscr{A} \arrow[ll, bend left, "U"] \end{tikzcd}$$

together with a pseudo-action

$$\Rightarrow \quad : \quad \mathscr{M}^{op} \times \mathscr{A} \quad \longrightarrow \quad \mathscr{A}$$

of the cartesian category $\mathscr{M}$ *on the category* $\mathscr{A}$*, and a family of adjunctions*

$$F(m \times -) \quad \vdash \quad U(m \Rightarrow -)$$

parametrized by m*. Finally, one requires that the category* $\mathscr{A}$ *has finite products, and that the canonical morphisms*

$$m \Rightarrow (A \times B) \quad \longrightarrow \quad (m \Rightarrow A) \times (m \Rightarrow B) \qquad\qquad m \Rightarrow I \quad \longrightarrow \quad I$$

are invertible for all objects m *of the category* $\mathscr{M}$ *and all objects* A *of the category* $\mathscr{A}$*.*

Here, it should be noticed that the family of adjunctions

$$F(m \times -) \quad \vdash \quad U(m \Rightarrow -)$$

may be understood as a form of closure property, providing a natural family of bijections

$$\varphi_{m,A,B} \quad : \quad \mathscr{M}(m \times A, UB) \quad \cong \quad \mathscr{M}(A, U(m \Rightarrow B)).$$

Note that this situation is precisely what is called an exponential ideal

$$U \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{M}$$

on the cartesian category $\mathscr{M}$. In a preliminary version of their journal paper [2], the cartesian category $\mathscr{M}$ is also required to be closed, see [1] for details. In that case, another formulation based on implication in $\mathscr{M}$ rather than product becomes possible.

Proposition 13 *An effect-calculus model on a cartesian closed category* $\mathscr{M}$ *is the same thing as a categorical adjunction*

$$\begin{tikzcd} \mathscr{M} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & \mathscr{A} \arrow[ll, bend left, "U"] \end{tikzcd}$$

together with a pseudo-action

$$\Rightarrow \quad : \quad \mathscr{M}^{\mathrm{op}} \times \mathscr{A} \quad \longrightarrow \quad \mathscr{A}$$

of the cartesian category $\mathscr{M}$ on the category $\mathscr{A}$, and a family of isomorphisms

$$\sigma_{m,A} \quad : \quad U(m \Rightarrow A) \quad \longrightarrow \quad m \Rightarrow U(A)$$

natural in m and A, making the two coherence diagrams

$$\begin{tikzcd}
& n \Rightarrow U(m \Rightarrow A) \arrow[dr, "\sigma_n"] & \\
U(n \Rightarrow (m \Rightarrow A)) \arrow[ur, "\sigma_n"] \arrow[d, "\mu"'] & & n \Rightarrow (m \Rightarrow UA) \arrow[d, "\mu"] \\
U((m \otimes n) \Rightarrow A) \arrow[rr, "\sigma_{m \otimes n}"] & & (m \otimes n) \Rightarrow U(A)
\end{tikzcd}$$

$$\begin{tikzcd}
U(I \Rightarrow A) \arrow[rr, "\mu"] \arrow[dr, "\sigma_I"'] & & U(A) \\
& I \Rightarrow U(A) \arrow[ur, "\mu"'] &
\end{tikzcd}$$

commute for all object m of the category $\mathscr{M}$ and all objects A of the category $\mathscr{A}$. Finally, one requires that the category $\mathscr{A}$ has finite products, and that the canonical morphisms

$$m \Rightarrow (A \times B) \quad \longrightarrow \quad (m \Rightarrow A) \times (m \Rightarrow B) \qquad\qquad m \Rightarrow I \quad \longrightarrow \quad I$$

are invertible for all objects m of the category $\mathscr{M}$ and all objects A of the category $\mathscr{A}$.

Note that a enriched calculus model where the category $\mathscr{A}$ has finite sums such that the

$$(m \multimap\!\!* A) + (m \multimap\!\!* B) \quad \longrightarrow \quad m \multimap\!\!* (A + B) \qquad\qquad 0 \quad \longrightarrow \quad m \multimap\!\!* 0$$

is equivalent to the model of Paul Blain Levy's call-by-push-value in [**?**].

Linear effect systems. This leads to a tentative definition of linear effect system.

Definition 15 *A linear effect system is an adjunction*

$$\begin{tikzcd} \mathscr{L} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{A} \arrow[l, bend left, "R"] \end{tikzcd}$$

between a monoidal category $\mathscr{L}$ and a $\mathscr{L}$-category $\mathscr{A}$ (either positive or negative). The adjunction is moreover required to be enriched over the presheaf category $[\mathscr{L}^{op}, Set]$.

It is possible to give a purely combinatorial definition of linear effect systems based on our reformulation of enriched adjunctions in the 2-category $\mathscr{L}-\mathrm{Mod}$. Quite obviously, there are two cases to treat, depending whether the $\mathscr{L}$-category $\mathscr{A}$ is positive or negative. When the $\mathscr{L}$-category $(\mathscr{A}, *)$ is positive, the enrichment requirement on $L \dashv R$ boils down to the existence of a family of isomorphisms

$$\sigma_{\ell,A} \quad : \quad \ell \otimes L(A) \quad \longrightarrow \quad L(\ell * A)$$

natural in the objects ℓ and A of the category $\mathscr{L}$, and satisfying the coherence diagrams (1) and (2) of a positive functor. When the $\mathscr{L}$-category $(\mathscr{A}, -\!\!*)$ is negative, the enrichment requirement on $L \dashv R$ boils down to the existence of a family of adjunctions

$$L(\ell * -) \quad \dashv \quad R(\ell -\!\!* -)$$

parametrized by the objects ℓ of the category $\mathscr{L}$. Suppose now that we have a resource modality on the category $\mathscr{L}$, in the sense of linear logic. In its modern formulation, such a modality is described as a lax monoidal adjunction

$$\begin{tikzcd} \mathscr{M} \arrow[r, bend left, "Lin"] \arrow[r, phantom, "\perp"] & \mathscr{L} \arrow[l, bend left, "Mult"] \end{tikzcd}$$

between a cartesian category $\mathscr{M}$ and the monoidal category $\mathscr{L}$. Recall that such a lax monoidal adjunction $Lin \dashv Mult$ is the same thing as a categorical adjunction where the left adjoint functor Lin is strongly monoidal, this meaning that the monoidal coercions

$$Lin(A) \otimes Lin(B) \quad \longrightarrow \quad Lin(A \times B) \qquad\qquad I \quad \longrightarrow \quad Lin(I)$$

are isomorphisms. From this follows that the functor

$$\mathscr{M} \times \mathscr{L} \quad \xrightarrow{Lin} \quad \mathscr{L} \times \mathscr{L} \quad \xrightarrow{\otimes} \quad \mathscr{L}$$

defines an action of the category $\mathscr{M}$ on the category $\mathscr{L}$. Similarly, the functor

$$\mathscr{M} \times \mathscr{A} \quad \xrightarrow{Lin} \quad \mathscr{L} \times \mathscr{A} \quad \longrightarrow \quad \mathscr{A}$$

transfers the positive (resp. negative) action of the $\mathscr{L}$ to a positive (resp. negative) action of $\mathscr{M}$ on the category $\mathscr{A}$. This enables to see the adjunction $L \dashv R$ as an adjunction in the 2-category $\mathscr{L}-\mathrm{Mod}$. Now, the two adjunctions live in $\mathscr{L}-\mathrm{Mod}$ and may be composed in the following way:

$$\begin{tikzcd}
\mathscr{M} \arrow[r, bend left, "Lin"] \arrow[r, phantom, "\perp"] & \mathscr{L} \arrow[l, bend left, "Mult"] \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{A} \arrow[l, bend left, "R"]
\end{tikzcd}$$

From this follows that the adjunction defines

$$\mathscr{M} \times \mathscr{A} \quad \xrightarrow{Lin} \quad \mathscr{L} \times \mathscr{A} \quad \longrightarrow \quad \mathscr{A}$$

transfers the positive (resp. negative) action of the $\mathscr{L}$ to a positive (resp. negative) action of $\mathscr{M}$ on the category $\mathscr{A}$. From this follows that

Proposition 14 *Suppose given a linear effect system between a monoidal category $\mathscr{L}$ and a category $\mathscr{A}$ with finite products commuting to the action of $\mathscr{L}$, in the sense that the canonical morphisms*

$$\ell \multimap\!\!* (A \times B) \quad \longrightarrow \quad (\ell \multimap\!\!* A) \times (\ell \multimap\!\!* B) \qquad\qquad \ell \multimap\!\!* A \quad \longrightarrow \quad 1$$

are invertible. In that case, every resource modality between a cartesian category $\mathscr{M}$ and the category $\mathscr{L}$ induces an adjunction

$$\begin{tikzcd}
\mathscr{M} \arrow[r, bend left, "L \circ Lin"] \arrow[r, phantom, "\perp"] & \mathscr{A} \arrow[l, bend left, "Mult \circ R"]
\end{tikzcd}$$

which defines an effect calculus model. The resulting strong monad T on the cartesian category $\mathscr{M}$ is equal to

$$T \quad = \quad Mult \circ R \circ L \circ Lin \quad : \quad \mathscr{M} \quad \longrightarrow \quad \mathscr{M}.$$

A typical illustration is provided by the case of a dialogue category $\mathscr{L}$ which always defines a linear effect system with its opposite category $\mathscr{L}^{op}$ seen as a negative $\mathscr{L}$-category. Every resource modality on $\mathscr{L}$ thus induces an effect calculus model between the cartesian category $\mathscr{M}$ involved in the resource modality and the category $\mathscr{L}^{op}$ seen this time as a negative $\mathscr{M}$-category. In that case, the proposition enables to transfer the double negation monad of the dialogue category $\mathscr{L}$ based on linear continuations to a double negation monad living in the cartesian category $\mathscr{M}$, and thus based on general continuations.

8 Conclusion

This paper may be seen as a step towards a more combinatorial and unified understanding of side effects in concrete case studies. We started the paper in that spirit by describing two apparently different ways to recover the strength σ of the double negation monad T from its decomposition into an adjunction $L \dashv R$ or $F \dashv U$. We then showed that each of the two recipes to recover σ may be understood as a procedure to lift the original adjunction $L \dashv R$ or $F \dashv U$ living in the 2-category Cat to an adjunction living in the 2-category of enriched categories over the presheaf category $[\mathscr{M}^{op}, Set]$. The statement is established by reformulating the traditional notions of enriched functors and enriched natural transformations in a purely combinatorial way in the particular case of functors and natural transformations between $\mathscr{M}$-categories. The combinatorial reformulation in itself is based on functorial strengths, costrengths, axioms and cuts. All this leads to the definition of a 2-category $\mathscr{M}-\mathrm{Mod}$ of positive and negative $\mathscr{M}$-categories where these specific adjunctions decomposing strong monads may be formulated and studied. The combinatorial reformulation is then applied in order to clarify the structure of categories equipped with computational effects. In particular, we take this opportunity to reformulate in that style the notion of model of effect calculus described by Jeff Egger, Rasmus Møgelberg and Alex Simpson in a series of recent papers. A natural question is to understand in a similar fashion other effect systems considered in the literature, and more specifically the enriched version of the effect calculus [2]. We leave that point for future work.

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