Dialogue categories and Frobenius monoids

Paul-André Melliès*

Abstract. About ten years ago, Brian Day and Ross Street discovered a beautiful and unexpected connection between the notion of $*$-autonomous category in proof theory and the notion of Frobenius algebra in mathematical physics. The purpose of the present paper is to clarify the logical content of this connection by formulating a two-sided presentation of Frobenius algebras. The presentation is inspired by the idea that every logical dispute has two sides consisting of a Prover and of a Denier. This dialogical point of view leads us to a correspondence between dialogue categories and Frobenius pseudomonoids. The correspondence with dialogue categories refines Day and Street's correspondence with $*$-autonomous categories in the same way as tensorial logic refines linear logic.

Forewords

A few weeks before writing this paper, I learned that my dear friend Kohei Honda passed away in London. This sudden accident was a tremendous shock, and his disparition haunts me. Vivid memories come back of the wonderful three years we spent together in Edinburgh. Kohei and I met for the first time in early 1996. Samson Abramsky had just moved from Imperial College to the Laboratory for the Foundations of Computer Science — taking there the position of Robin Milner who had just left Edinburgh to join the University of Cambridge. Samson wanted to create a new group there and he was looking for two Research Assistants. He decided to hire Kohei and me. This was really a bold choice Samson made on that occasion because Kohei and I were coming from territories quite alien to semantics. Kohei was already recognized for his discovery of the asynchronous π-calculus with Mario Tokoro, independently and at about the same time as Gérard Boudol, see [9] for details. Kohei was absolutely fanatic about the π-calculus and he would openly declare that game semantics was only a small fragment of π — I like to think that the future will tell him right in some interesting and unexpected way. I should say that I was just as stubborn myself about rewriting theory. Back in France, Pierre-Louis Curien had strongly advised me to join Samson's group if I wanted to learn semantics — but I was so much hooked on rewriting theory when I arrived at the LFCS that it took me two long years before really working on linear logic and game semantics.

During the three years we spent together in Edinburgh, Kohei and I very soon became this slightly eccentric pair of French and Japanese researchers sharing an office on the ground floor of the JCMB building. The office was dark and cold, with

* CNRS, Laboratoire PPS, UMR 7126, Université Paris Diderot, Sorbonne Paris Cité, F-75205 Paris, France.

two narrow window panes facing a few bushes and an anonymous alley... but I spent there among the most luminous hours of my life, and I am sure that Kohei was just as enthusiastic developing his own stream of ideas.

Samson was an exquisite leader and nothing of the effervescence of the *interaction group* — this is the way we decided to call ourselves — would have been possible without his sharp understanding of logic and of semantics combined with a frenetic curiosity for the surrounding fields. Any topic could be freely discussed in the group and there was absolutely no feeling of intellectual property among us. As a matter of fact, many ideas which I have worked out in Paris in the past fifteen years were already germinating at the time. I distinctively remember Kohei and Nobuko Yoshida explaining how call-by-value programs should be interpreted by letting Player start the game rather than Opponent[1]. I also remember Samson explaining how higher-order states could be interpreted by relaxing the visibility condition on strategies[2]. And I remember Martin Wehr developping a narcotic interest in n-dimensional categories and trying to convince all of us that n-dimensional syntax would become the foundation of logic and of programming languages[3]. These are only a few illustrations coming to my mind so numerous were the ideas floating around in this small group of dedicated people.

This short period of my life in Edinburgh defines a lot about who I am today, and I am happy to dedicate the present work to Samson as a testimony of friendship and gratitude. My primary purpose here is to entertain him with a connection between two of his favorite topics of interest: game semantics and logic on the one hand, Frobenius algebras and the categorical approach to physics on the other hand.

1 Frobenius algebras and 2-dimensional cobordism

Let $n > 0$ be a positive integer. The basic idea of topological field theory is to construct a symmetric monoidal functor

$$\mathrm{Cob}(n) \qquad \longrightarrow \qquad Vect$$

from the category of n-dimensional cobordism to the category *Vect* of vector spaces on a given field k. The category $\mathrm{Cob}(n)$ is defined as follows:

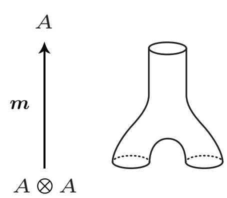
- its objects are the closed oriented $(n-1)$-dimensional manifolds,
- its morphisms $M \to N$ are the bordisms from M to N, that is, the oriented n-dimensional manifold B equipped with an orientation-preserving diffeomorphism $\partial B \cong (-M) \cup N$. Here $-M$ denotes the manifold M equipped with the opposite orientation. Two bordisms $B, B' : M \to N$ are considered equal in $\mathrm{Cob}(n)$

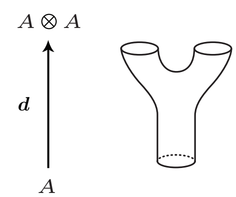
[1] A paper developing this idea was presented by Kohei and Nobuko at the ICALP 1997 conference, see [10] for details.
[2] A paper developing this idea was presented by Samson, Kohei and Guy McCusker at the LICS 1998 conference, see [1] for details.
[3] Martin presented some of his ideas on higher dimensional syntax in the CTCS 1999 conference, see [21] for details.

if there is an orientation-preserving diffeomorphism which extends the diffeomorphism $\partial B \cong (-M) \cup N \cong \partial B'$.
- For any object M in Cob(n), the identity map id_M is represented by the product bordism $B = M \times [0, 1]$,
- Composition of morphisms in Cob(n) is defined by gluing bordisms together. The operation of gluing is not canonical but the point is that it defines a unique class of manifolds modulo diffeomorphism.

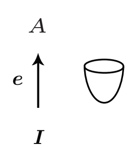
The category Cob(n) can be endowed with the structure of a symmetric monoidal category, whose tensor product $\otimes$ is given by taking the disjoint sum of two $(n-1)$-dimensional manifolds, and whose unit I is given by the empty manifold. A natural question is to understand what information is contained in a topological field theory of a given dimension n. The answer is very well known in the case of dimension $n = 2$. In that case, a topological field theory is the same thing as a commutative and cocommutative Frobenius algebra in the category *Vect*. This observation justifies the notion of *Frobenius monoid* in any monoidal category $\mathscr{V}$. A Frobenius algebra is then the same thing as a Frobenius monoid in the category *Vect*.

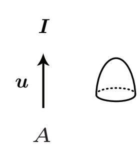
Definition 1 (Frobenius monoid). *A bimonoid A in a monoidal category $\mathscr{V}$ is an object equipped a monoid structure (A, m, e) and a comonoid structure (A, d, u). In other words, it is an object A equipped with a binary operation m and a binary co-operation d*



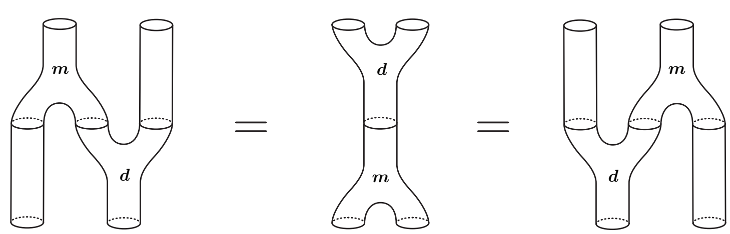


both of them associative, and equipped with a unit e and a co-unit u





A Frobenius monoid *is defined as a bimonoid A satisfying the two equalities below:*



(1)

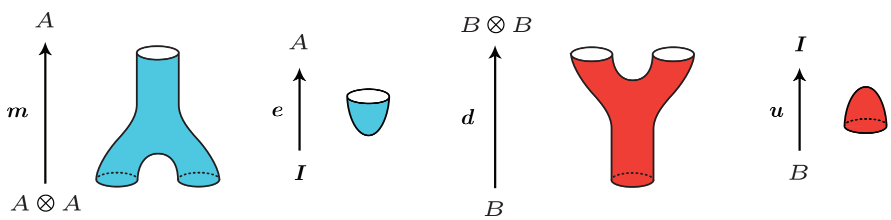
A Frobenius monoid in a symmetric monoidal category $\mathscr{V}$ is called commutative (resp. cocommutative) when its underlying monoid (resp. comonoid) is commutative (resp. cocommutative).

Note that the characterization of topological field theories of dimension 2 extends to every symmetric monoidal category $\mathscr{V}$.

Proposition 1. *A symmetric monoidal functor* Cob(2) $\rightarrow \mathscr{V}$ *into a symmetric monoidal category* $\mathscr{V}$ *is the same thing as a commutative and cocommutative Frobenius monoid in* $\mathscr{V}$.

2 Frobenius pairs

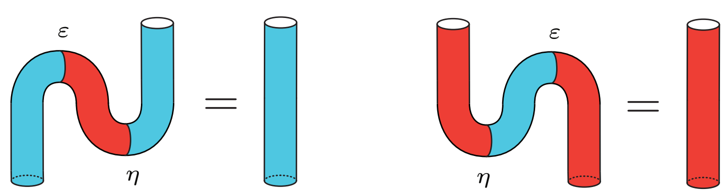
Once the notion of Frobenius algebra has been extracted from the definition of topological field theory, it makes sense to study it independently of its topological origins. In this paper, we will do something quite counterintuitive from the topological point of view, but which makes a lot of sense from the logical point of view. In the same way as a logical dispute involves a Prover and a Denier, we will decouple the monoid side (A, m, e) from the comonoid side (B, d, u) in the definition of a Frobenius monoid. Each side A and B is meant to describe an aspect of the « split personnality » of the Frobenius monoid. The operations of the monoid (A, m, e) are depicted in light blue whereas the co-operations of the comonoid (B, d, u) are depicted in dark red:



Once the notion of Frobenius monoid in a monoidal category $\mathscr{V}$ has been split in two, an interesting question is to understand how its two sides A and B are coupled inside a Frobenius monoid. The first thing to ask is that the two objects A and B are involved in an *exact pairing* $A \dashv B$ defined as a pair of morphisms

$$\eta \ : \ I \ \longrightarrow \ B \otimes A \qquad\qquad \varepsilon \ : \ A \otimes B \ \longrightarrow \ I$$

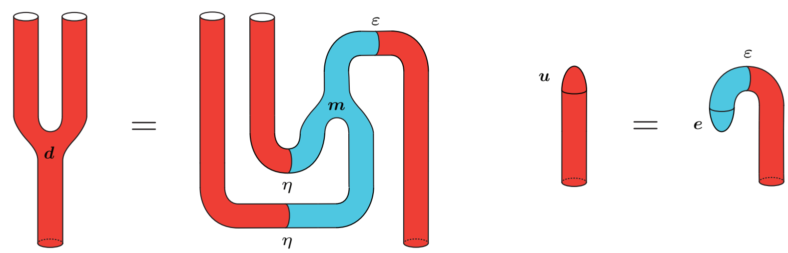
satisfying the zig-zag equalities below:



Note that when $\mathscr{V}$ is the category of k-vector spaces, one may alternatively equip the two spaces A and B with a non-degenerate binary form $\varepsilon : A \otimes B \rightarrow k$. This exact pairing $A \dashv B$ should be moreover compatible with the monoid and comonoid structures of A and B in the following sense. We define a monoid-comonoid pairing

$$(A, m, e) \quad \dashv \quad (B, d, u) \tag{2}$$

between a monoid and a comonoid as an exact pairing $A \dashv B$ between the underlying objects satisfying the two equalities:



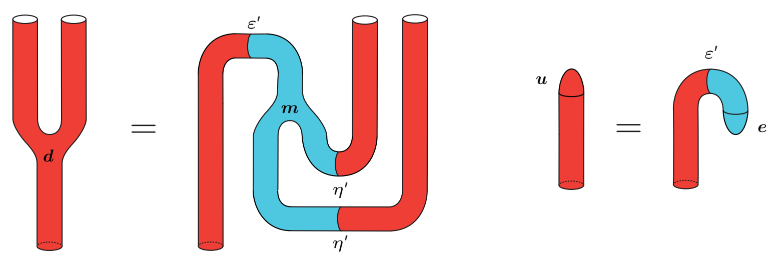
These equations mean that the comonoid structure (B, d, u) on the object B may be recovered from the monoid structure (A, m, e) on the object A, and conversely, that the monoid structure (A, m, e) on the object A may be recovered from the comonoid structure (B, d, u) on the object B. In a symmetric way, one requires the existence of a comonoid-monoid pairing

$$(B, d, u) \quad \dashv \quad (A, m, e) \tag{3}$$

defined as an exact pairing $B \dashv A$ between the underlying objects:

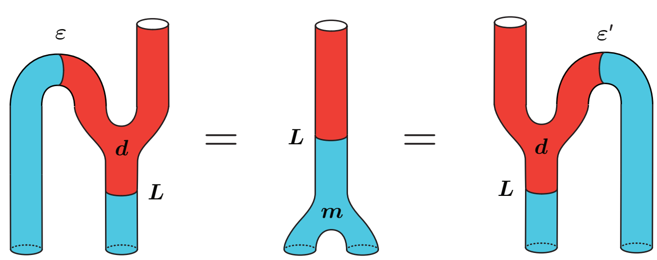
$$\eta' \ : \ I \longrightarrow B \otimes A \qquad\qquad \varepsilon' \ : \ A \otimes B \longrightarrow I$$

which moreover satisfies the two equations below:



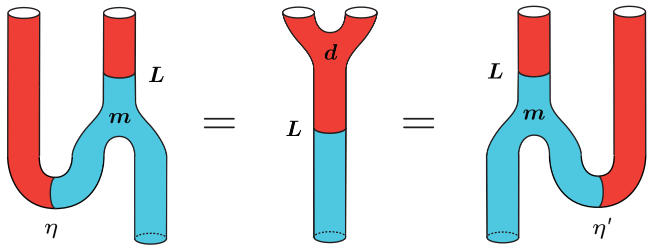
In the same way as before, the two equalities say that that the comonoid structure (B, d, u) on the object B may be recovered from the monoid structure (A, m, e) on the object A, and conversely. This leads to our definition of Frobenius pair.

Definition 2 (Frobenius pairs). *A Frobenius pair in a monoidal category $\mathscr{V}$ consists of a monoid-comonoid pairing (2) and a comonoid-monoid pairing (3) between a monoid (A, m, e) and a comonoid (B, d, u) together with an isomorphim $L : A \to B$ between the underlying objects A and B. One also requires that the equalities below are satisfied:*

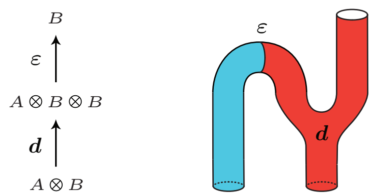


(4)

Note that the two equations (4) are equivalent to the equations below:



These equalities may be understood in the following way. The monoid-comonoid pairing $(A, m, e) \dashv (B, d, u)$ induces a left action of the monoid (A, m, e) on the object B, defined as



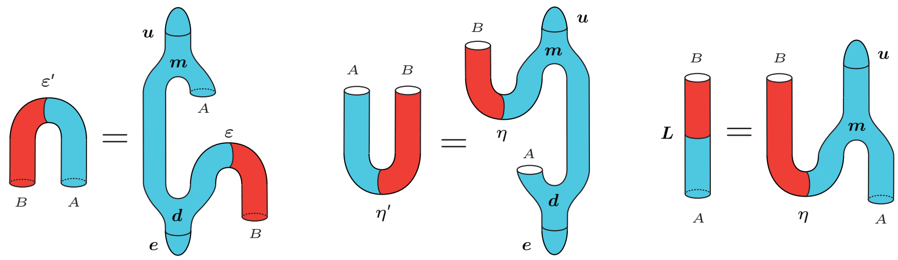
(5)

There is also a left action of the monoid A on itself, defined using the monoid structure:

Equation (4) means that the morphism L transports the left action of the monoid A on itself into the action (5) on the object B. In a symmetric way, the exact pairing $B \dashv A$ induces a right action of the monoid A on B, and the second equation (4) amounts to ask that the morphism L transports the canonical right action of the monoid A on itself in the right action of the monoid A on the object B. The ultimate justification for the notion of Frobenius pair is the following correspondence with Frobenius monoids:

Proposition 2. *A Frobenius pair* (A, B) *in a monoidal category* $\mathscr{V}$ *is the same thing as a Frobenius monoid* A *equipped with an exact pairing* $A \dashv B$.

Proof. Given a Frobenius monoid A together with an exact pairing $A \dashv B$ with unit η and counit ε, one defines the unit η' and counit ε' of the exact pairing $B \dashv A$ and the isomorphism L as follows:

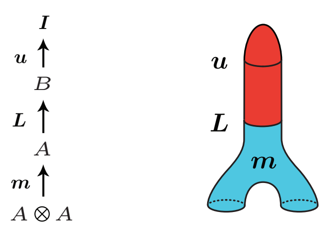


The object B inherits its comonoid structure (B, d, u) from the monoid structure (A, m, e) of the object A and the exact pairing $A \dashv B$. One checks that the resulting structure coincides with the comonoid structure on B induced from the exact pairing $B \dashv A$. This already ensures that the pair (A, m, e) and (B, d, u) satisfy the equalities (2) and (3). Finally, one easily checks that the two equalities (4) are satisfied, and that the pair (A, B) thus defines a Frobenius pair. Conversely, every Frobenius pair (A, B) defines a Frobenius monoid with monoid structure (A, m, e) and comonoid structure (A, d', u') induced from the isomorphism L with (B, d, u). A careful inspection shows that the relationship between Frobenius pairs (A, B) and Frobenius monoid A equipped with a duality $A \dashv B$ is one-to-one.

Remark. The idea of presenting Frobenius algebras as a pair consisting of a monoid A and of its canonical right dual $B = A^*$ is essentially folklore, and appears already in [5]. So, the only novelty here is that we do not ask that the object B coincides with the canonical right dual A^*. More on that specific point will be said when we move to ribbon categories in §5.

3 The Frobenius bracket

Given a Frobenius pair in a monoidal category, the following morphism



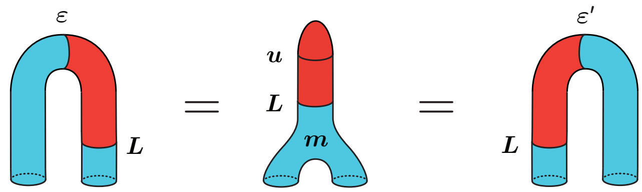
defines a bilinear form

$$\{\!|\, -, - \,|\!\} \quad : \quad A \otimes A \quad \longrightarrow \quad I$$

called the *Frobenius bracket*. The definition of Frobenius bracket together with the associativity af the product $a_1 \bullet a_2 = m(a_1, a_2)$ ensures that the following equality is satisfied:

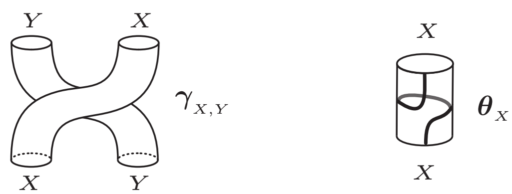
$$\{\!|\, a_1 \bullet a_2, a_3 \,|\!\} \quad = \quad \{\!|\, a_1, a_2 \bullet a_3 \,|\!\} \qquad (6)$$

This equality is generally called the *associativity* property of the Frobenius bracket, see [19] for a discussion. In addition, the defining property (4) of Frobenius pair implies that the Frobenius bracket may be alternatively formulated as:



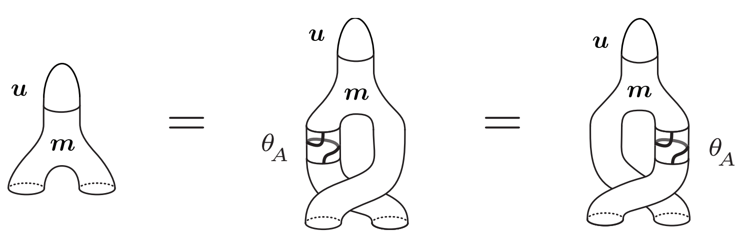
4 Helical Frobenius pairs

We suppose from now on that we work in a balanced monoidal category in the sense of [11–13] typically given by the category $\mathscr{V} = Mod(H)$ of representations of a quantum group H. The braiding γ and the twist θ of the category $\mathscr{V}$ are represented as follows:



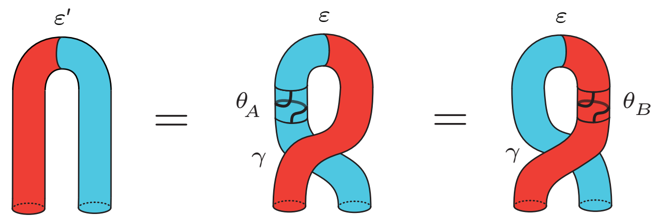
Note that the twist θ_X should be understood as the operation of applying a rotation of angle 2π on the border X of the 2-dimensional manifold. This extra structure on the category $\mathscr{V}$ enables us to formulate the following definition of *helical* Frobenius monoid.

Definition 3 (Helical Frobenius monoids). *A Frobenius monoid A in a balanced monoidal category is called helical when the two equalities below are satified:*

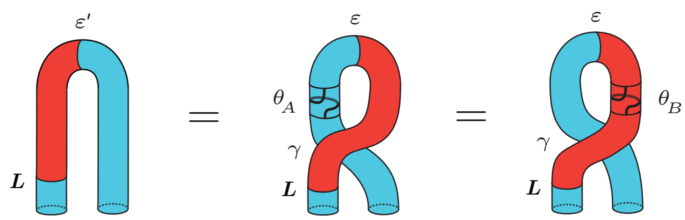


We proceed as in §2 and immediately introduce the corresponding two-sided notion of helical Frobenius pair:

Definition 4 (Helical Frobenius pairs). *A Frobenius pair in a balanced monoidal category is called helical when the two equalities below are satisfied:*

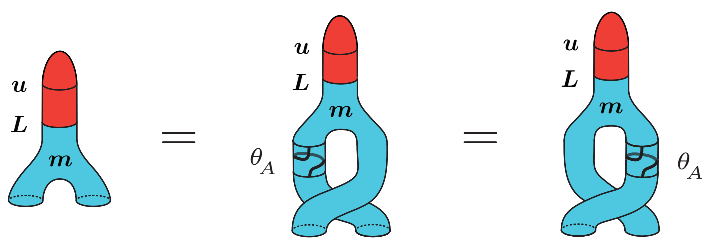


Since the morphism L is reversible in the definition of a Frobenius pair, one may replace this helicality condition by the equivalent one:



(7)

We will see that this formulation of helicality is more natural than the original one when we move one dimension up to the 2-categorical notion of Frobenius amphimonoid. It should be noted that this latter condition (7) is equivalent to asking that the Frobenius bracket is commutative in the sense that the two equalities below are satisfied:



The equivalence follows from the fact that the twist is a natural isomorphism from the identity functor into itself, and thus satisfies the equality:

The equation (7) should be understood as a commutativity property of the Frobenius bracket:

$$\{\!\!\{ a_1, a_2 \}\!\!\} \quad = \quad \{\!\!\{ a_2, a_1 \}\!\!\}. \tag{8}$$

Together with the associativity (6) the commutativity of the Frobenius bracket implies the following cyclicity property:

$$\{\!\!\{ a_1 \bullet a_2, a_3 \}\!\!\} \quad = \quad \{\!\!\{ a_3 \bullet a_1, a_2 \}\!\!\} \quad = \quad \{\!\!\{ a_2 \bullet a_3, a_1 \}\!\!\} \tag{9}$$

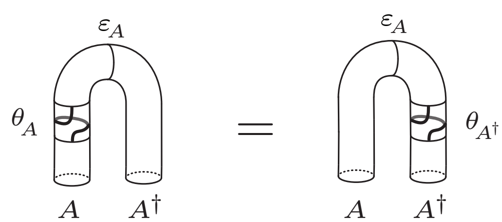
where $a_1 \bullet a_2$ is a notation for the product $m(a_1, a_2)$ of the two elements a_1 and a_2. It is worth recalling here that a symmetric monoidal category is the same thing as a balanced monoidal category whose twist θ_X is equal to the identity id_X for every object X. A helical Frobenius algebra in a symmetric monoidal category is called *symmetric*. A typical illustration is provided by matrix algebras $A \otimes A^*$ where the cyclicity equations (8–9) reflect the cyclicity of the trace functional. As expected, one needs to modulate the two equations (8–9) by a twist θ when one works in a general balanced monoidal category.

At this stage, Proposition 2 may be refined into the following correspondence between helical Frobenius monoids and helical Frobenius pairs:

Proposition 3. *A helical Frobenius pair (A, B) in a balanced monoidal category $\mathscr{V}$ is the same thing as a helical Frobenius monoid A equipped with an exact pairing $A \dashv B$.*

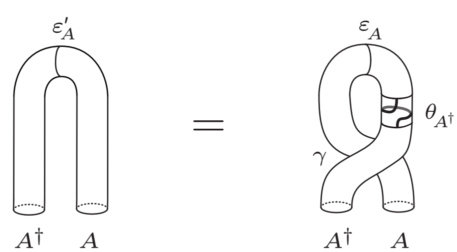
5 Frobenius pairs in ribbon categories

The two-sided formulation of a Frobenius monoid as a pair (A, B) relies on the existence of an exact pairing $A \dashv B$ between the two sides A and B of the Frobenius pair. It is thus interesting to see what happens when one embeds the notion of Frobenius pair in a monoidal category $\mathscr{V}$ which is *already* equipped with an exact pairing $A \dashv A^\dagger$ for every object A. This is precisely what happens in the case of a ribbon category like the category $\mathscr{V} = Mod_f(H)$ of *finite dimensional* representations of a quantum group H. Recall that a ribbon category[4] is defined as a balanced monoidal category $\mathscr{V}$ where every object A comes equipped with an exact pairing $A \dashv A^\dagger$ whose counit $\varepsilon_A : A \otimes A^\dagger \to I$ satisfies the equality below:



(10)

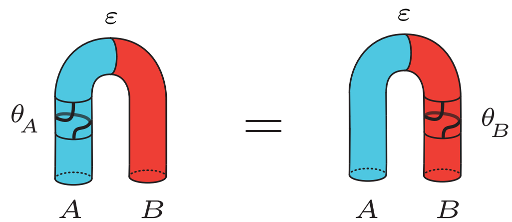
A nice consequence of the definition of ribbon category is that the right dual $A^\dagger$ is also a left dual of the object A, with counit ε'_A of the exact pairing $A^\dagger \dashv A$ defined as:



(11)

[4] The notion of ribbon category is also called tortile category in [11, 18].

Another nice property is that every Frobenius pair (A, B) in a ribbon category $\mathscr{V}$ satisfies the equality [5] below:



(12)

It is worth observing that the opposite category $\mathscr{V}^{op(0,1)}$ of a balanced monoidal category $\mathscr{V}$ is also balanced, with the same braiding and twist combinators as the original category. By $\mathscr{V}^{op(0,1)}$, we mean the monoidal category $\mathscr{V}$ where the orientation of the tensor product (of dimension 0) and of the morphisms (of dimension 1) has been reversed. The transformation $\mathscr{V} \mapsto \mathscr{V}^{op(0,1)}$ thus consists in applying a central symmetry on the string diagrams. The family of exact pairings $A \dashv A^{\dagger}$ induces a monoidal functor

$$\dagger \quad : \quad \mathscr{V} \quad \longrightarrow \quad \mathscr{V}^{op(0,1)}$$

which transports the ribbon structure of $\mathscr{V}$ to the ribbon structure of $\mathscr{V}^{op(0,1)}$ in the obvious sense. Note that one would obtain the very same functor $\dagger$ by starting from the family of exact pairings $A^{\dagger} \dashv A$ defined in (11). Now, every exact pairing $A \dashv A^{\dagger}$ in $\mathscr{V}$ induces an exact pairing $A \dashv A^{\dagger}$ in the opposite category $\mathscr{V}^{op(0,1)}$ with unit defined as the image $\varepsilon_A^{\dagger} : I \to A^{\dagger} \otimes^{op} A$ of the counit $\varepsilon_A : A \otimes A^{\dagger} \to I$ of the original exact pairing. From this follows that every ribbon structure on $\mathscr{V}$ induces a ribbon structure on $\mathscr{V}^{op(0,1)}$.

Now, suppose given a Frobenius pair (A, B) in such a ribbon category $\mathscr{V}$. The monoidal structure of the functor $\dagger$ ensures that the comonoid structure of the object B is transported to a monoid structure on the object $B^{\dagger}$. The resulting monoid structure $(B^{\dagger}, d^{\dagger}, u^{\dagger})$ may be constructed either from the exact pairing $B^{\dagger} \dashv B$ or from the exact pairing $(B^{\dagger}, d^{\dagger}, u^{\dagger})$. From this follows that the monoid $(B^{\dagger}, d^{\dagger}, u^{\dagger})$ is involved in two exact pairings with the comonoid (B, d, u), either as a left dual or as a right dual:

$$(B^{\dagger}, d^{\dagger}, u^{\dagger}) \;\dashv\; (B, d, u) \qquad\qquad (B, d, u) \;\dashv\; (B^{\dagger}, d^{\dagger}, u^{\dagger})$$

with unit η'_B and counit ε'_B in the first case, and with unit η_B and counit ε_B in the second case. These exact pairings should be compared with the exact pairings between (A, m, e) and (B, d, u) involved in the definition of the Frobenius pair:

$$(A, m, e) \;\dashv\; (B, d, u) \qquad\qquad (B, d, u) \;\dashv\; (A, m, e)$$

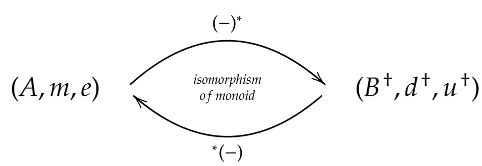
with unit η and counit ε in the first case and with unit η' and counit ε' in the second case. Each comparison induces a monoid isomorphism between the monoid (A, m, e)

[5] The equality is in fact satisfied by any bilinear form $A \otimes B \to I$ in a ribbon category.

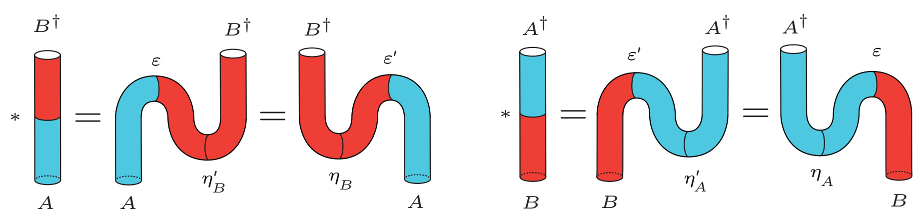
and the monoid $(B^\dagger, d^\dagger, u^\dagger)$. The two isomorphisms are respectively defined as:

(13)

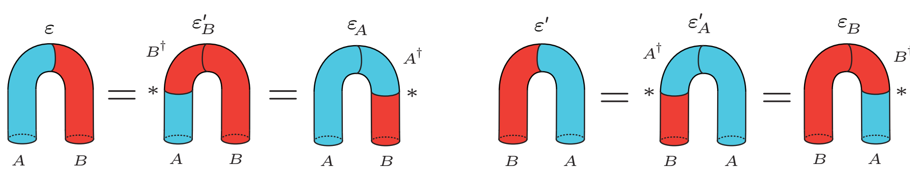
The notion of helical Frobenius pair plays an important role at this stage, and this is precisely the reason why we introduced it in §4. The point is that the two isomorphisms $A \to B^\dagger$ coincide precisely when the Frobenius pair is helical. We leave the reader check this statement starting from the observation that the equality (10) holds in any ribbon category. So, in the case of a helical Frobenius pair, one obtains an isomorphism of monoids



where the isomorphisms $(-)^* : A \to B^\dagger$ and ${}^*(-) : B \to A^\dagger$ are defined as:



From these definitions, one deduces the following equations:

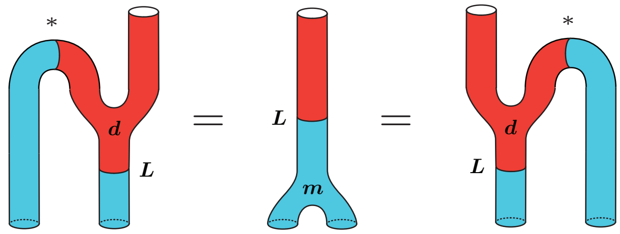


These equations lead to an alternative but equivalent formulation of helical Frobenius pairs (A, B) living in a ribbon category:

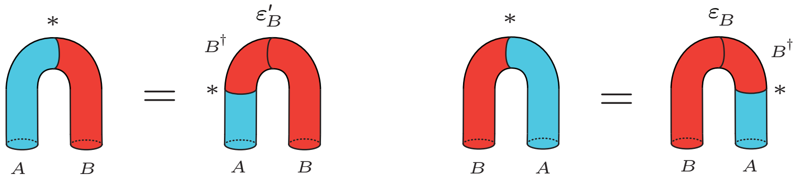
Proposition 4. *A helical Frobenius pair (A, B) in a ribbon category is the same thing as a monoid (A, m, e) and a comonoid (B, d, u) equipped with a monoid isomorphism*

$$(-)^* \quad : \quad (A, m, e) \quad \longrightarrow \quad (B^\dagger, d^\dagger, u^\dagger)$$

and an isomorphism $L : A \to B$ between the underlying objects satisfying the equalities:



Here, we use the $*$ notation for the two evaluation brackets between A and B defined as follows:



(14)

We let the reader check the statement. Starting from the alternative formulation of Proposition 4, the two evaluation brackets $\varepsilon : A \otimes B \to I$ and $\varepsilon' : B \otimes A \to I$ of the Frobenius pair (A, B) are recovered as the two operations $*$ depicted in (14).

6 Dialogue categories and chiralities

At this point, it is time to introduce the notion of dialogue category, which underlies tensorial logic in the same way as the notion of $*$-autonomous category underlies linear logic, see [4, 8] for details. Tensorial logic is a primitive logic of tensor and negation whose purpose is to circumscribe the primary ingredients of logic. Our main ambition here is to extend to dialogue categories the correspondence between $*$-autonomous categories and Frobenius algebras originally discovered by Day and Street [6][19] and then independently rediscovered a few years later by Egger [7].

Definition 5 (Dialogue categories). *A dialogue category is a monoidal category $(\mathscr{C}, \otimes, e)$ equipped with an object $\bot$ together with a family of bijections*

$$\varphi_{x,y} \quad : \quad \mathscr{C}(x \otimes y, \bot) \quad \cong \quad \mathscr{C}(y, x \multimap \bot)$$

natural in y for all objects x of the category $\mathscr{C}$, and a family of bijections

$$\psi_{x,y} \quad : \quad \mathscr{C}(x \otimes y, \bot) \quad \cong \quad \mathscr{C}(x, y \multimap \bot)$$

natural in x for all objects y of the category $\mathscr{C}$.

We will be more specifically interested in the notion of *helical dialogue category* introduced in [16, 17].

Definition 6 (Helical dialogue category). *A helical dialogue category is a dialogue category $\mathscr{C}$ equipped with a family of bijections*

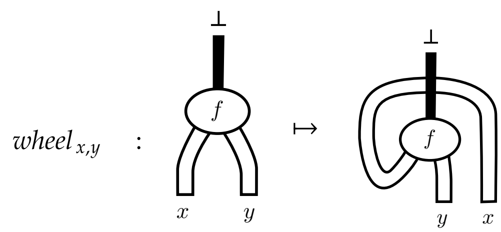
$$wheel_{x,y} \quad : \quad \mathscr{C}(x \otimes y, \bot) \quad \longrightarrow \quad \mathscr{C}(y \otimes x, \bot)$$

natural in x and y and required to make the diagram

$$\begin{array}{ccc} \mathscr{C}((y\otimes z)\otimes x,\bot) & \xrightarrow{\ associativity\ } & \mathscr{C}(y\otimes (z\otimes x),\bot) \\ \uparrow {\scriptstyle wheel_{x,y\otimes z}} & & \downarrow {\scriptstyle wheel_{y,z\otimes x}} \\ \mathscr{C}(x\otimes (y\otimes z),\bot) & & \mathscr{C}((z\otimes x)\otimes y,\bot) \\ \downarrow {\scriptstyle associativity} & & \uparrow {\scriptstyle associativity} \\ \mathscr{C}((x\otimes y)\otimes z,\bot) & \xrightarrow{\ wheel_{x\otimes y,z}\ } & \mathscr{C}(z\otimes (x\otimes y),\bot) \end{array} \tag{15}$$

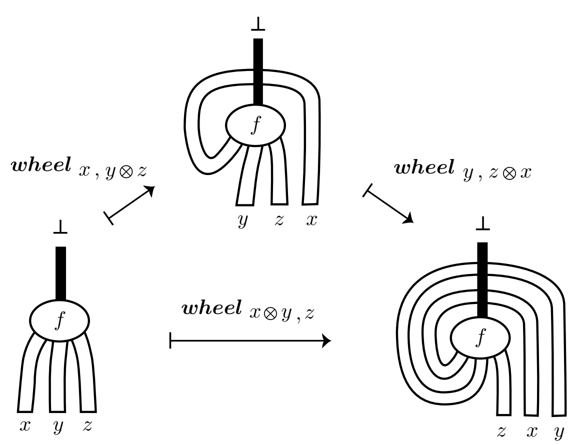
commute for all objects x, y, z of the category $\mathscr{C}$.

A useful graphical mnemonics for the *wheel* combinator is to draw it in the following way:



(16)

In that graphical formulation, the coherence diagram expresses that the diagram below commutes:

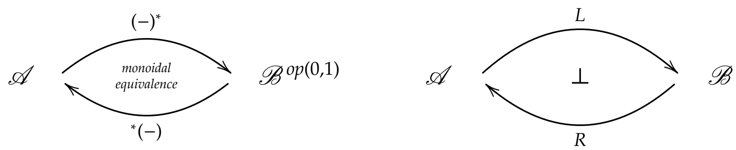


The bare notion of dialogue category is fundamental but one may argue that it does not properly reflect the symmetries of logic, and more specifically the two-sided nature of logical disputes. The equivalent notion of *dialogue chirality* was introduced for that reason, see [15] for details. We recall below the two-sided notion of helical dialogue chirality corresponding to the one-sided notion of helical dialogue category. The formal correspondence between them[6] is established in [16].

Definition 7 (Helical chirality). *A helical chirality is a pair of monoidal categories*

$$(\mathscr{A}, \owedge, true) \qquad\qquad (\mathscr{B}, \ovee, false)$$

equipped with a monoidal equivalence and an adjunction



and with two families of bijections

$$\chi^L_{m,a,b} \quad : \quad \langle\, m \owedge a \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, a \,|\, m^* \ovee b \,\rangle$$

$$\chi^R_{m,a,b} \quad : \quad \langle\, a \owedge m \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, a \,|\, b \ovee m^* \,\rangle$$

natural in a, b and m, where the evaluation bracket is defined as

$$\langle\, - \,|\, - \,\rangle \quad := \quad \mathscr{A}(\,-\,,\, R(-)\,) \quad : \quad \mathscr{A}^{op} \times \mathscr{B} \quad \longrightarrow \quad Set$$

The currification combinators χ^L and χ^R are moreover required to make the three diagrams commute:

$$\begin{array}{ccccc}
\langle\, (m \owedge n) \owedge a \,|\, b \,\rangle & & \xrightarrow{\quad \chi^L_{m \owedge n} \quad} & & \langle\, a \,|\, (m \owedge n)^* \ovee b \,\rangle \\
\downarrow \scriptstyle{associativity} & & & & \uparrow \scriptstyle{associativity,\ monoidality\ of\ negation} \\
\langle\, m \owedge (n \owedge a) \,|\, b \,\rangle & \xrightarrow{\chi^L_m} & \langle\, n \owedge a \,|\, m^* \ovee b \,\rangle & \xrightarrow{\chi^L_n} & \langle\, a \,|\, n^* \ovee (m^* \ovee b) \,\rangle
\end{array} \qquad (17)$$

$$\begin{array}{ccccc}
\langle\, a \owedge (m \owedge n) \,|\, b \,\rangle & & \xrightarrow{\quad \chi^R_{m \owedge n} \quad} & & \langle\, a \,|\, b \ovee (m \owedge n)^* \,\rangle \\
\downarrow \scriptstyle{associativity} & & & & \uparrow \scriptstyle{associativity,\ monoidality\ of\ negation} \\
\langle\, (a \owedge m) \owedge n \,|\, b \,\rangle & \xrightarrow{\chi^R_n} & \langle\, a \owedge m \,|\, b \ovee n^* \,\rangle & \xrightarrow{\chi^R_m} & \langle\, a \,|\, (b \ovee n^*) \ovee m^* \,\rangle
\end{array} \qquad (18)$$

$$\begin{array}{ccccc}
\langle\, (m \owedge a) \owedge n \,|\, b \,\rangle & \xrightarrow{\chi^R_n} & \langle\, m \owedge a \,|\, b \ovee n^* \,\rangle & \xrightarrow{\chi^L_m} & \langle\, a \,|\, m^* \ovee (b \ovee n^*) \,\rangle \\
\Big| \scriptstyle{associativity} & & & & \Big| \scriptstyle{associativity} \\
\langle\, m \owedge (a \owedge n) \,|\, b \,\rangle & \xrightarrow{\chi^L_m} & \langle\, a \owedge n \,|\, m^* \ovee b \,\rangle & \xrightarrow{\chi^R_n} & \langle\, a \,|\, (m^* \ovee b) \ovee n^* \,\rangle
\end{array} \qquad (19)$$

[6] The notion of helical chirality described here is called "ambidextrous" in [16]. We keep the terminology "helical" here in order to stress the correspondence with helical dialogue categories.

for all objects a, m, n of the category $\mathscr{A}$ and all objects b of the category $\mathscr{B}$.

Every helical dialogue category $\mathscr{C}$ defines a helical dialogue chirality by taking $\mathscr{A} = \mathscr{C}$, $\mathscr{B} = \mathscr{C}^{op(0,1)}$, $La = a \multimap \bot$ and $Rb = \bot \multimapinv b$. The right currification combinator χ^R is simply defined using the dialogue structure of the category $\mathscr{C}$:

$$\begin{array}{ccccc} \langle\, a \oslash m \,|\, b \,\rangle & & & & \langle\, a \,|\, b \obar m^* \,\rangle \\ \| & & & & \| \\ \mathscr{C}(a \otimes m, \bot \multimapinv b) & \xrightarrow{\psi^{-1}_{a\otimes m,b}} & \mathscr{C}(a \otimes m \otimes b, \bot) & \xrightarrow{\psi_{a,m\otimes b}} & \mathscr{C}(a, \bot \multimapinv (m \otimes b)) \end{array}$$

whereas the definition of the left currification combinator χ^L is more sophisticated and requires the helical structure:

$$\begin{array}{ccccccc} \langle\, m \oslash a \,|\, b \,\rangle & & & & & & \langle\, a \,|\, m^* \obar b \,\rangle \\ \| & & & & & & \| \\ \mathscr{C}(m \otimes a, \bot \multimapinv b) & \xrightarrow{\psi^{-1}_{m\otimes a,b}} & \mathscr{C}(m \otimes a \otimes b, \bot) & \xrightarrow{wheel_{m,a\otimes b}} & \mathscr{C}(a \otimes b \otimes m, \bot) & \xrightarrow{\psi_{m\otimes b,a}} & \mathscr{C}(a, \bot \multimapinv (b \otimes m)) \end{array}$$

7 Categorical bimodules

In order to understand the connection between $*$-autonomous categories and Frobenius algebras noticed by Day and Street — and then to extend it to dialogue categories — one needs to work in a suitable bicategory of categorical bimodules or distributors (following Bénabou's original terminology). Given two categories $\mathscr{A}$ and $\mathscr{B}$, an $\mathscr{A}\mathscr{B}$-bimodule M is defined as a functor

$$M \quad : \quad \mathscr{A}^{op} \times \mathscr{B} \quad \longrightarrow \quad Set.$$

The notion of bimodule considered here is set-theoretic, but it may be easily adapted to enriched settings, where the category *Set* is typically replaced by the category *Vect* of vector spaces, see [20] for details. The bicategory (or weak 2-category) of bimodules has

– small categories as objects,
– $\mathscr{A}\mathscr{B}$-bimodules M as 1-dimensional cells $M : \mathscr{A} \to \mathscr{B}$,
– natural transformations

$$\theta \quad : \quad N \Rightarrow M \quad : \quad \mathscr{A}^{op} \times \mathscr{B} \longrightarrow Set$$

as 2-dimensional cells

$$\theta \quad : \quad M \Rightarrow N \quad : \quad \mathscr{A} \longrightarrow \mathscr{B}$$

in the weak 2-category.

Note the reverse direction of the natural transformations. This specific orientation enables to define a monoidal 2-functor:

$$\boldsymbol{Cat} \quad \to \quad \boldsymbol{BiMod}$$

which transports every functor $F : \mathscr{A} \to \mathscr{B}$ to the bimodule

$$F_\bullet \quad : \quad (a,b) \quad \mapsto \quad \mathscr{A}(Fa,b) \quad : \quad \mathscr{A}^{op} \times \mathscr{B} \quad \longrightarrow \quad Set.$$

It is possible to see ***BiMod*** as a 2-dimensional Kleisli construction on the small limit completion $\mathscr{C} \mapsto [\mathscr{C}, Set]^{op}$ of categories. One recovers the more familiar convention corresponding to the small colimit completion $\mathscr{C} \mapsto [\mathscr{C}^{op}, Set]$ of categories by taking the weak 2-category $\boldsymbol{BiMod}^{op(1,2)}$ obtained from ***BiMod*** by reversing the orientation of the 1- and 2-dimensional cells. As a matter of fact, there also exists a monoidal 2-functor

$$\boldsymbol{Cat} \quad \to \quad \boldsymbol{BiMod}^{op(1,2)}$$

defined in the following way. First of all, it is good to remember that the operation $op : \mathscr{C} \mapsto \mathscr{C}^{op}$ which transforms a category into its opposite category defines a 2-functor

$$op \quad : \quad \boldsymbol{Cat} \quad \longrightarrow \quad \boldsymbol{Cat}^{op(2)}.$$

The weak 2-category ***BiMod*** is symmetric monoidal with tensor product defined as product of categories. The underlying monoidal category is also autonomous, which simply means that it is a ribbon category with a trivial twist θ. From this follows that there exists a functor (and in fact a monoidal 2-functor)

$$\dagger \quad : \quad \boldsymbol{BiMod} \quad \longrightarrow \quad \boldsymbol{BiMod}^{op(0,1)}$$

which transports every category $\mathscr{A}$ to its dual $\mathscr{A}^{\dagger}$ in ***BiMod***. By a miracle of mathematics, this dual $\mathscr{A}^{\dagger}$ happens to coincide with the opposite category $\mathscr{A}^{op}$. Putting all this together, one obtains the monoidal 2-functor

$$\boldsymbol{Cat} \quad \xrightarrow{op} \quad \boldsymbol{Cat}^{op(2)} \quad \longrightarrow \quad \boldsymbol{BiMod}^{op(2)} \quad \xrightarrow{\dagger} \quad \boldsymbol{BiMod}^{op(0,1,2)}$$

which transports every category $\mathscr{A}$ to itself and every functor $F : \mathscr{A} \to \mathscr{B}$ to the bimodule

$$F^\bullet \quad : \quad (a,b) \quad \mapsto \quad \mathscr{A}(a,Fb) \quad : \quad \mathscr{A}^{op} \times \mathscr{B} \quad \longrightarrow \quad Set.$$

Another miracle of categorical bimodules is that for every functor $F : \mathscr{A} \to \mathscr{B}$, the bimodule $F^\bullet : \mathscr{B} \to \mathscr{A}$ is left adjoint to the bimodule $F_\bullet : \mathscr{A} \to \mathscr{B}$ in the weak 2-category ***BiMod***.

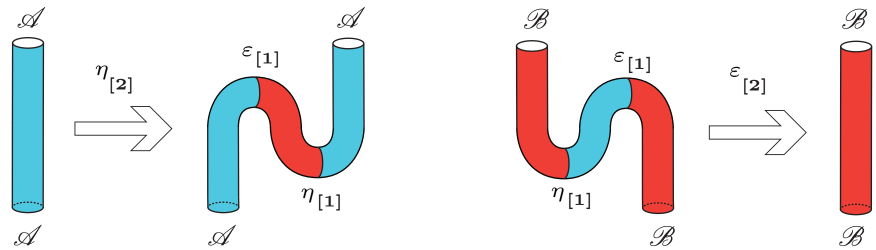
8 Frobenius pseudomonoids

Here, we introduce the notion of Frobenius pseudomonoid whose main purpose is to reflect the properties of a dialogue category $\mathscr{A}$ transported from ***Cat*** to the monoidal bicategory ***BiMod***. A preliminary step in the definition of Frobenius pseudomonoid is to adapt the notion of exact pairing to the 2-categorical setting.

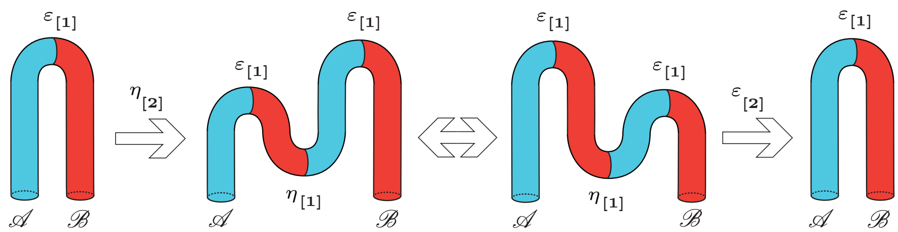
Definition 8 (Lax pairing). *A lax pairing* $\mathscr{A} \dashv \mathscr{B}$ *in a monoidal bicategory is a pair of 1-dimensional cells*

$$\eta_{[1]} : \mathscr{A} \otimes \mathscr{B} \longrightarrow I \qquad\qquad \varepsilon_{[1]} : I \longrightarrow \mathscr{B} \otimes \mathscr{A}$$

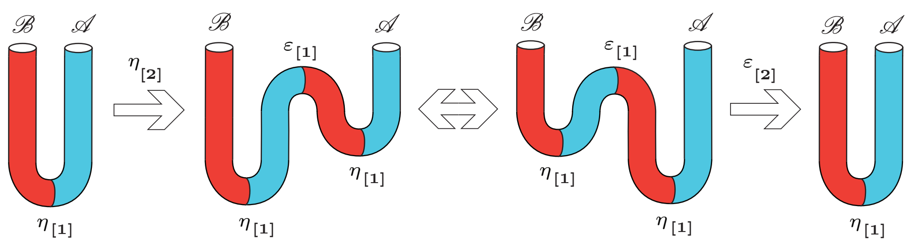
together with a pair of 2-dimensional cells



such that the composite 2-dimensional cell



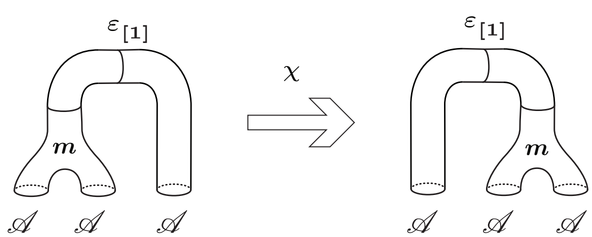
coincides with the identity on the 1-dimensional cell $\varepsilon_{[1]}$ *and symmetrically, such that the composite 2-dimensional cell*



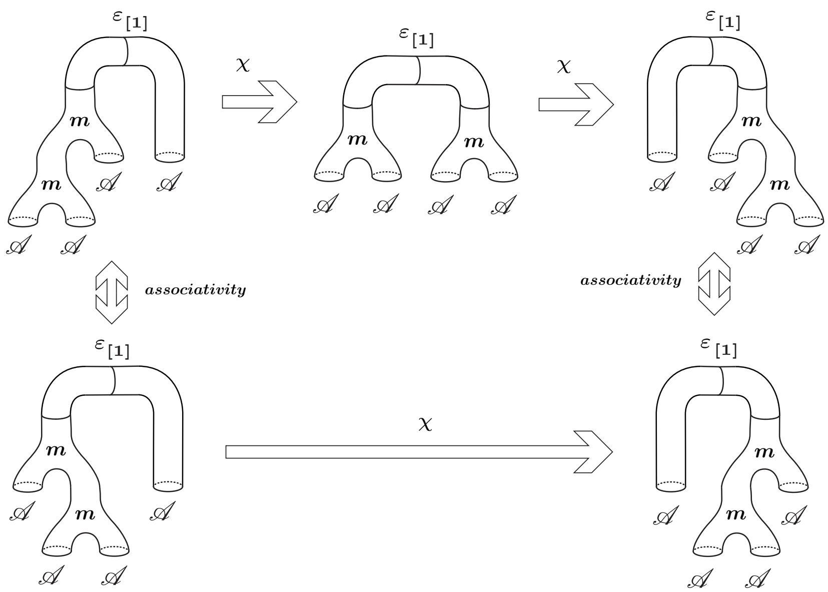
coincides with the identity on the 1-dimensional cell $\eta_{[1]}$.

At this stage, we are ready to refine the notion of *form* also introduced by Day and Street [6] in a monoidal bicategory.

Definition 9 (Frobenius form). *A Frobenius form on a pseudomonoid* $\mathscr{A}$ *in a monoidal bicategory is a lax pairing* $\mathscr{A} \dashv \mathscr{A}$ *equipped with a 2-dimensional cell*



called the associativity law *of the Frobenius form, and required to make the following variation of MacLane's pentagonal diagram commute:*



(20)

This leads us to our definition of Frobenius pseudomonoid. Note that our definition departs from the definition given by Street in [19], see the end of §9 for a comparison.

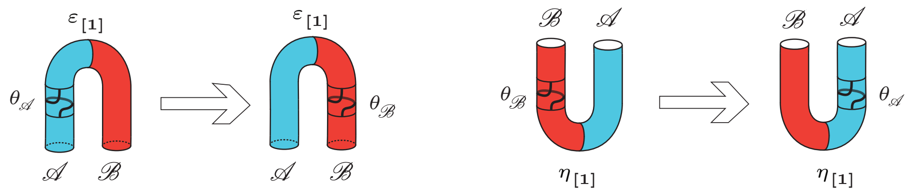
Definition 10 (Frobenius pseudomonoid).
A Frobenius pseudomonoid is a pseudomonoid $\mathscr{A}$ equipped with a Frobenius form.

Observe that once transported in the bicategory ***BiMod***, every dialogue category $\mathscr{A}$ defines such a Frobenius pseudomonoid with Frobenius form defined as

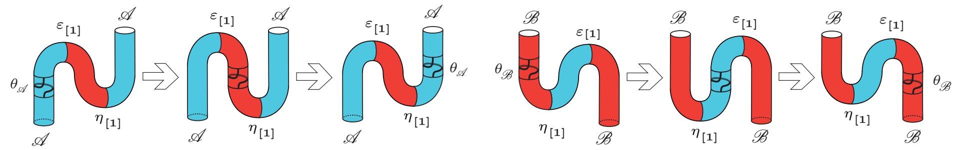
$$\varepsilon_{[1]} \;:\; (a_1, a_2) \;\mapsto\; \mathscr{A}(a_1 \otimes a_2, \bot) \qquad\qquad \eta_{[1]} \;:\; (a_1, a_2) \;\mapsto\; \mathscr{A}(\bot \multimapinv a_1, a_2)$$

and χ simply defined as the associativity law of the monoidal category $\mathscr{A}$. On the other hand, note that the notion of *Frobenius pseudomonoid* introduced above does not coincide with the notion of *Frobenius monoid* in the particular case when the underlying monoidal 2-category $\mathscr{W}$ is a monoidal category — seen as a 2-category with trivial 2-dimensional cells. The point is that nothing ensures in Definition 10 that the two comonoid structures on $\mathscr{A}$ induced from the exact pairing $\mathscr{A} \dashv \mathscr{A}$ coincide, although we require this property in our definition of Frobenius monoid. Depending on the taste of the reader, this unpleasant situation may be seen as a result of the maximalist nature of Definition 1 or as a result of the minimalist nature of Definition 10. This justifies in any case to resolve the matter by formulating a 2-categorical version of helicality. To that purpose, we need to work in a balanced monoidal bicategory, defined as a monoidal bicategory $\mathscr{W}$ equipped with a braiding and a twist compatible with the 2-dimensional structure.

Definition 11 (Lax ribbon pairing). *A ribbon structure on a lax pairing $\mathscr{A} \dashv \mathscr{B}$ in a balanced monoidal bicategory is a pair of invertible 2-cells:*

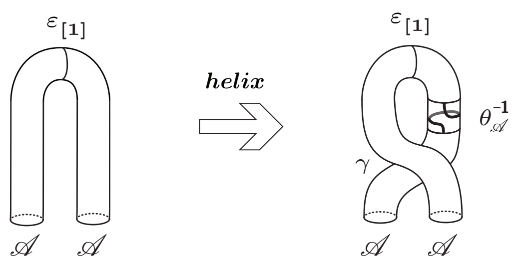


such that both composite 2-dimensional cells



coincides with the identity 2-cell[7] *A lax ribbon pairing is a lax pairing equipped with such a ribbon structure.*

Definition 12 (Helical Frobenius pseudomonoid). *A helical Frobenius pseudomonoid in a balanced monoidal bicategory is a Frobenius pseudomonoid whose lax pairing is equipped with a ribbon structure, and which is moreover equipped with an invertible 2-dimensional cell*



The 2-dimensional cell helix may be understood as a 2-dimensional cell $\{\!\{ a_2, a_1 \}\!\} \Rightarrow \{\!\{ a_1, a_2 \}\!\}$. *The helical structure is required to make the diagram commute:*

$$
\begin{array}{ccccc}
\{\!\{ a_1 \bullet a_2, a_3 \}\!\} & \xRightarrow{\chi} & \{\!\{ a_1, a_2 \bullet a_3 \}\!\} & \xRightarrow{helix} & \{\!\{ a_2 \bullet a_3, a_1 \}\!\} \\
\Big\Downarrow {\scriptstyle helix} & & & & \Big\Downarrow {\scriptstyle \chi} \\
\{\!\{ a_3, a_1 \bullet a_2 \}\!\} & \xLeftarrow{\chi} & \{\!\{ a_3 \bullet a_1, a_2 \}\!\} & \xLeftarrow{helix} & \{\!\{ a_2, a_3 \bullet a_1 \}\!\}
\end{array}
\qquad (21)
$$

Every helical dialogue category $\mathscr{A}$ induces a helical Frobenius pseudomonoid in ***BiMod*** with *helix* simply defined as the natural transformation

$$
wheel^{-1}_{a_1,a_2} \quad : \quad \mathscr{A}(a_2 \otimes a_1, \bot) \quad \Rightarrow \quad \mathscr{A}(a_1 \otimes a_2, \bot).
$$

[7] The composite 2-cell is required to coincide with the 2-dimensional coercion of θ when one defines the twist of a balanced monoidal bicategory as a pseudonatural (rather than natural) transformation, which we do not do here.

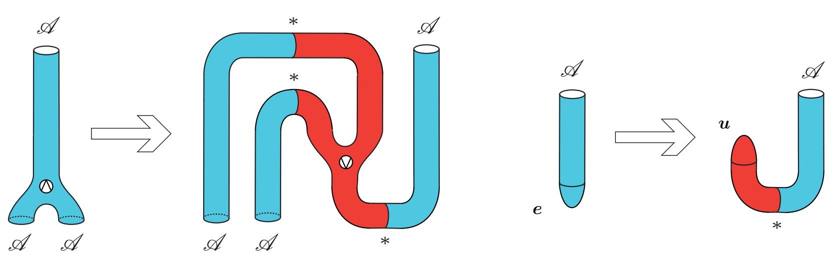
which goes in the reverse direction in the bicategory ***BiMod***. We will see in the next section that a helical dialogue category is the same thing as a helical Frobenius pseudomonoid in ***BiMod*** whose bimodules $\varepsilon_{[1]}$ and $\eta_{[1]}$ are represented by functors L and R in the appropriate sense. We could establish the statement directly, but we find clarifying to reformulate first the notion of helical Frobenius pseudomonoid in a two-sided fashion. This is precisely the way the correspondence between helical dialogue categories and helical Frobenius pseumonoids originally emerged in our work.

9 Frobenius amphimonoids

Here, we reformulate in a two-sided fashion the notion of lax helical Frobenius monoid... in just the same way as we did in §2 for Frobenius monoids. To that purpose, we start by relaxing the notion of *monoid-comonoid pairing* between a monoid and a comonoid, and introduce the corresponding 2-dimensional notion of *amphimonoid*.

Definition 13 (Biexact pairing). *A biexact pairing $\mathscr{A} \dashv \mathscr{B}$ is a lax pairing whose 2-dimensional cells $\eta_{[2]}$ and $\varepsilon_{[2]}$ are reversible. A biexact ribbon pairing is a biexact pairing equipped with a ribbon structure.*

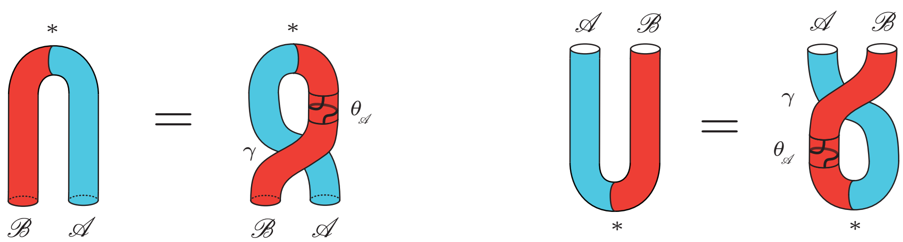
Definition 14 (Amphimonoid). *An amphimonoid $(\mathscr{A}, \mathscr{B})$ in a balanced monoidal bicategory $\mathscr{W}$ is defined as a pseudomonoid $(\mathscr{A}, \owedge, true)$ and a pseudocomonoid $(\mathscr{B}, \ovee, false)$ equipped with a biexact ribbon pairing $\mathscr{A} \dashv \mathscr{B}$ (noted $*$ in the picture) and with a pair of invertible 2-dimensional cells*



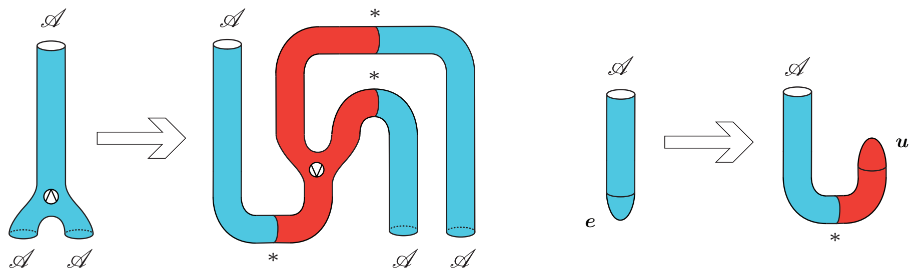
(22)

defining a pseudomonoid equivalence between $(\mathscr{A}, \owedge, true)$ and the pseudomonoid structure on $\mathscr{B}$ deduced from the biexact pairing.

An important point about the definition is that every amphimonoid induces a biexact ribbon pairing $\mathscr{B} \dashv \mathscr{A}$ defined as follows:

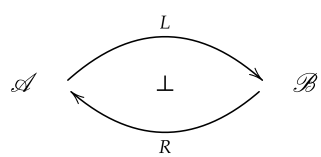


together with a pair of invertible 2-dimensional cells

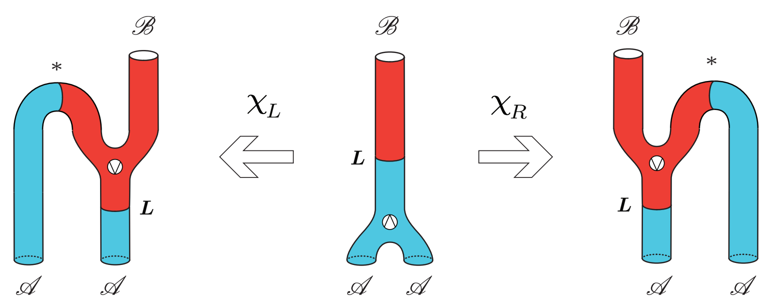


defining a pseudomonoid equivalence between ($\mathscr{A}$, $\oslash$, *true*) and the pseudomonoid structure on $\mathscr{B}$ deduced from the biexact pairing applied in the opposite direction. We are ready now to introduce our two-sided notion of Frobenius pseudomonoid:

Definition 15 (Frobenius amphimonoid). *A Frobenius amphimonoid* $(\mathscr{A}, \mathscr{B}, L, R)$ *consists of an amphimonoid* $(\mathscr{A}, \mathscr{B})$ *equipped with an adjunction*



and two invertible 2-dimensional cells:



(23)

The 1-dimensional cell $L : \mathscr{A} \to \mathscr{B}$ *may be understood as defining a bracket* $\langle\, a \,|\, b \,\rangle$ *between the objects* $\mathscr{A}$ *and* $\mathscr{B}$ *of the bicategory* $\mathscr{V}$. *Each side of Equation (23) may be thus seen as implementing a* currification *step:*

$$\chi_L : \langle\, a_1 \oslash a_2 \,|\, b \,\rangle \Rightarrow \langle\, a_2 \,|\, a_1^* \owedge b \,\rangle \qquad\qquad \chi_R : \langle\, a_1 \oslash a_2 \,|\, b \,\rangle \Rightarrow \langle\, a_1 \,|\, b \owedge a_2^* \,\rangle$$

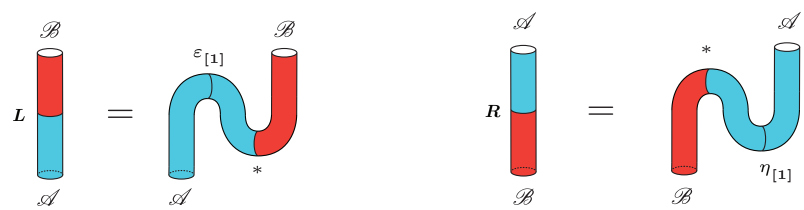
In the definition of a Frobenius amphimonoid, we require that the two combinators χ_L *and* χ_R *make the three coherence diagrams of Equations (17), (18) and (19) commute. We leave the reader depict each coherence diagram as the relevant string diagram.*

Every helical dialogue category defines a Frobenius amphimonoid in ***BiMod***, by taking $La = a \multimap \perp$ and $Rb = \perp \multimapinv b$.

Proposition 5. *Given an amphimonoid $(\mathscr{A}, \mathscr{B})$ in a balanced monoidal bicategory $\mathscr{W}$, there is a back-and-forth translation between the two following data:*

- *the helical Frobenius structures on the pseudomonoid $(\mathscr{A}, \oslash, true)$,*
- *the Frobenius structures (L, R, χ_L, χ_R) on the amphimonoid $(\mathscr{A}, \mathscr{B})$.*

Proof. The correspondence between the two Frobenius structures works as follows. Given an amphimonoid $(\mathscr{A}, \mathscr{B})$ whose $\mathscr{A}$-side is a Frobenius pseudomonoid $(\mathscr{A}, \oslash, true)$ with Frobenius form noted $\{\!\{ -, - \}\!\}$, one defines the 1-dimensional cells L and R of the Frobenius amphimonoid $(\mathscr{A}, \mathscr{B})$ in the following way:



(24)

The definition of a lax pairing ensures that $L \dashv R$ defines an adjunction in the bicategory $\mathscr{W}$. The 2-dimensional cell χ_R is defined as

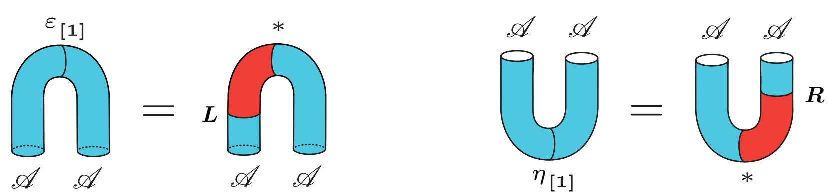
$$\{\!\{ a_1 \oslash m, a_2 \}\!\} \xRightarrow{\ \chi\ } \{\!\{ a_1, m \oslash a_2 \}\!\}$$

while the 2-dimensional cell χ_L is defined as the composite

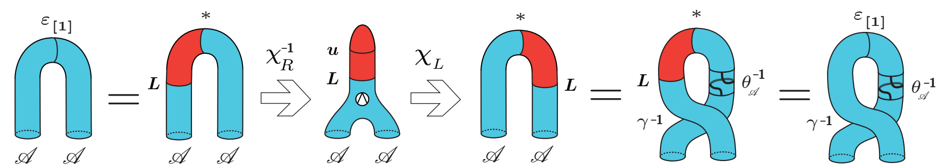
$$\{\!\{ m \oslash a_1, a_2 \}\!\} \xRightarrow{\ helix\ } \{\!\{ a_2, m \oslash a_1 \}\!\} \xRightarrow{\ \chi^{-1}\ } \{\!\{ a_2 \oslash m, a_1 \}\!\} \xRightarrow{\ helix^{-1}\ } \{\!\{ a_1, a_2 \oslash m \}\!\}$$

each of them appropriately composed with the coercion (22) and the 2-dimensional structure of the biexact pairing $\mathscr{A} \dashv \mathscr{B}$ in order to obtain the expected currification diagrams (23). A careful check establishes that the two combinators χ_L and χ_R just constructed make the three coherence diagrams of Equations (17), (18) and (19) commute. This establishes that $(\mathscr{A}, \mathscr{B})$ together with the adjunction $L \dashv R$ defines a Frobenius amphimonoid in the balanced monoidal bicategory $\mathscr{W}$.

Conversely, given a Frobenius amphimonoid $(\mathscr{A}, \mathscr{B}, L, R)$ in a balanced monoidal bicategory $\mathscr{W}$, one defines a Frobenius form on the pseudomonoid $\mathscr{A}$ in the following way:



The associativity law χ of the Frobenius form is defined using χ_R together with the coercion (22) and the 2-dimensional structure of the biexact pairing $\mathscr{A} \dashv \mathscr{B}$. One obtains in this way a Frobenius pseudomonoid ($\mathscr{A}, \oslash, true$) whose helical structure is then defined using the currification combinators χ_L and χ_R:



One needs then to check carefully that the helical structure makes the coherence diagram (21) does indeed commute. This establishes that ($\mathscr{A}, \oslash, true$) equipped with the structure above defines a helical Frobenius pseudomonoid. This concludes the proof.

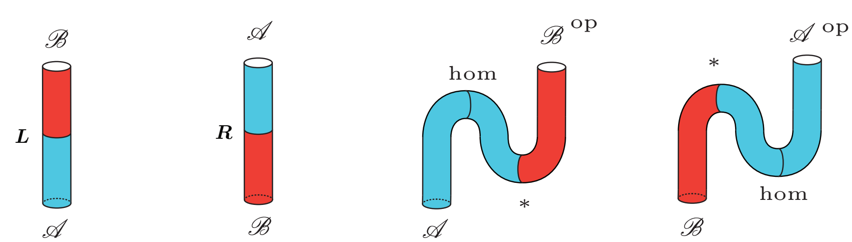
We like to think of the two-sided notion of Frobenius amphimonoid as *logical* since it is based on the currification combinators χ_L and χ_R whereas the original one-sided formulation of helical Frobenius pseudomonoid would be rather *algebraic* or *topological*. Although the correspondence exhibited in Proposition 5 does not define a one-to-one relationship, it conveys the idea that the notions of helical Frobenius pseudomonoid and of Frobenius amphimonoid should be considered as morally equivalent. This statement is informal but it could be made rigorous by constructing a 2-dimensional equivalence between bicategories corresponding to each notions, in the same way as was done for dialogue categories and chiralities, see [15] for details.

At this point, we are ready to apply Proposition 5 to the specific monoidal bicategory ***BiMod***. Every category $\mathscr{A}$ comes equipped with a biexact pairing $\mathscr{A} \dashv \mathscr{A}^{op}$ whose unit and counit are defined as the bimodule:

$$\text{hom} \quad : \quad (a_1, a_2) \quad \mapsto \quad \mathscr{A}(a_1, a_2) \quad : \quad \mathscr{A}^{op} \times \mathscr{A} \quad \longrightarrow \quad Set$$

This leads us to the two main results of the paper:

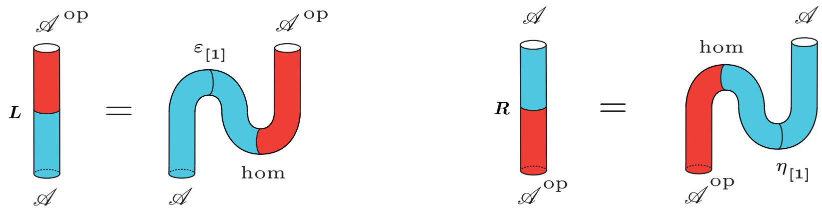
Theorem (First correspondence theorem) *A helical chirality is the same thing as a Frobenius amphimonoid in the bicategory **BiMod** whose 1-dimensional cells*



are representable, that is, images of functors along the 2-dimensional functor $(-)_\bullet$: ***Cat*** $\rightarrow$ ***BiMod***.

The proof is based on a direct comparison between the definition of helical chirality (Definition 7) and the definition of Frobenius amphimonoid (Definition 15). The second main result of the paper follows then from this result and Proposition 5.

Theorem (Second correspondence theorem) *A helical dialogue category is the same thing as a helical Frobenius pseudomonoid in the bicategory* ***BiMod*** *whose 1-dimensional cells*



are representable, that is, images of functors along the 2-dimensional functor $(-)_\bullet$: ***Cat*** $\rightarrow$ ***BiMod****.*

10 Epilogue: a comparison with Day and Street

One may recover here the correspondence between $*$-autonomous categories and Frobenius pseudomonoids drawn by Day and Street in [6, 19].

Definition 16 ($*$-autonomous pseudomonoid). *A $*$-autonomous pseudomonoid is a Frobenius pseudomonoid whose Frobenius form is based on a biexact pairing $\mathscr{A} \dashv \mathscr{A}$.*

The definition coincides with the original definition of $*$-autonomous pseudomonoid given by Street in [19] except that we add the requirement that the coherence diagram (20) commutes. In particular, we may establish the following property, which adapts to our notion of Frobenius pseudomonoid the Proposition 3.2 stated by Ross Street in [19] for $*$-autonomous pseudomonoids.

Proposition 6. *A pseudomonoid $(\mathscr{A}, m, e)$ is Frobenius if and only if it is equipped with a 1-dimensional cell*

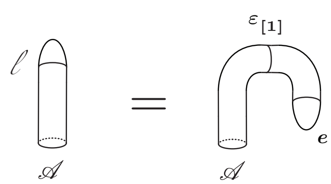
$$\ell \quad : \quad \mathscr{A} \quad \longrightarrow \quad I$$

such that

$$\mathscr{A} \otimes \mathscr{A} \xrightarrow{\quad m \quad} \mathscr{A} \xrightarrow{\quad \ell \quad} I \tag{25}$$

defines the unit $\varepsilon_{[1]}$ of a lax pairing $\mathscr{A} \dashv \mathscr{A}$.

Proof. Given a Frobenius pseudomonoid, the 1-cell ℓ is defined as



Conversely, given a pseudomonoid equipped with such a 1-dimensional cell ℓ, one defines the Frobenius form as in Equation (25) with coercion $\chi : \{\!|\, a_1 \bullet a_2, a_3 \,|\!\} \longrightarrow \{\!|\, a_1, a_2 \bullet a_3 \,|\!\}$ induced from the associativity law of the binary product m. Note that Abramsky and Heunen recently characterized the orthonormal basis on a Hilbert space A as a possibly nonunital Frobenius algebra structure on the space A, see [3] for details. The relaxation of unitality is fundamental here because every unital Frobenius algebra A is isomorphic to its dual A^*. The relaxation is also connected to the theory nuclear and trace ideals, see [2] for a categorical account by Abramsky, Blute and Panangaden. It would be interesting to know whether this characterization may be performed at the 2-categorical level, with adapted notions of Hilbert spaces and orthonormal basis.

11 Conclusion

The mathematical style of the paper should not distract the reader from the main idea conveyed here, which is that the primitive mechanisms of reasoning are of a purely topological nature — with encouraging and somewhat surprising affinities to cobordism.

This geometric conception of logic is likely to appear awkward and even disturbing to the unprepared reader. The reason is that we logicians (and non-logicians) are traditionally reluctant to think of language as a *material phenomenon* embedded in space and time. Even worse, we have learned along the years to treat reasoning as a purely desincarnated and formal activity living in the ether of symbolic logic. However, this formalist inclination of the field is probably temporary... and we like to think that the destiny of logic is to become a « geometry of mind » in the same way as physics has become a « geometry of nature ». The purpose of this geometry will not be to explain the « mind » as a whole — the idea would be ridiculous — but rather to shed light on some of its most elementary and fundamental mechanisms, in the same way as physics does with « nature ». And then to investigate in a reflexive (and somewhat ethnographic) turn how these micrological mechanisms interact with the macrological (or foundational) program originally assigned to logic at the beginning of the 20th century.

The dream of « geometrizing logic » is far from accomplished today, but the novelty is that it does not seem entirely inaccessible anymore. In particular, the recent advances of contemporary mathematics — at the crossroad of algebra, topology and physics — provide us with a series of very nice conceptual tools for trying the adventure. By analogy with physics, a tentative starting point in the exploration of this evanescent « geometry of logic » is offered by the study of the configuration space of n logical players (or computer programs) conversing in time on a specific formula. The present paper is a very preliminary attempt to substantiate these geometric intuitions in the specific case $n = 2$ where one benefits from the perfect adequation between tensorial logic and dialogue games.

For lack of space, we have only scratched the surface of the connection between tensorial logic and 2-dimensional cobordism. In particular, we did not include any description of the interplay between the topological flow of negation defining the proofs of tensorial logic (and thus the innocent strategies in dialogue games, see [14] for details) and the lax 2-dimensional cobordism describing the formulas of the logic (and thus the dialogue games themselves). An interesting issue for the connection between logic and physics is probably to understand whether the lax and two-sided account of cobordism developed here in dimension 2 still makes sense in higher dimensions, and whether it is supported by any appropriate physical (or at least geometric) intuition.

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