String diagrams

a functorial semantics of proofs and programs

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Luxembourg 17 July 2009
Connecting 2-dimensional cobordism and logic
Part 1

Categories as monads
Starting point: categories as monads

Depending on the bicategory, the point of view federates:

- **Enriched categories**
  - a bicategory of $V$-matrices
- **Internal categories**
  - a bicategory of spans
- **Algebra**
  - a bicategory of comodules

The last connection appears in Marcelo Aguiar’s PhD thesis
The bicategory $\overrightarrow{\text{Cat}(B)}$

- the 0-cells are the monads $(A, s)$ of the bicategory $B$

- the 1-cells

\[(f, \tilde{f}) : (A, s) \rightarrow (B, t)\]

are the pairs consisting of a 1-cell

\[f : A \rightarrow B\]

and of a 2-cell

\[\tilde{f} : f \otimes s \Rightarrow t \otimes f : A \rightarrow B\]
The bicategory $\overrightarrow{\text{Cat}}(\mathcal{B})$

Diagrammatically:

\[\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{s} & \Rightarrow & \downarrow{t} \\
A & \xrightarrow{f} & B
\end{array}\]

satisfying the equalities below:
The bicategory $\overrightarrow{\text{Cat}}(\mathcal{B})$

\[ \begin{align*}
A & \xrightarrow{f} B \\
\text{id} & \xRightarrow{s} \tilde{f} \xRightarrow{t} \eta \Rightarrow \tilde{f} \Rightarrow \mu \Rightarrow \\
A & \xrightarrow{f} B \\
\end{align*} \]
An alternative formulation in string diagrams
The bicategory $\xleftarrow{\text{Cat}}(\mathcal{B})$

Same definition, except that the direction of the 2-cell changes:

\[
\begin{array}{c}
A \\ s
\end{array}
\xrightarrow{f}
\begin{array}{c}
\tilde{f} \\ t
\end{array}
\begin{array}{c}
B \\ f
\end{array}
\xleftarrow{\tilde{f}}
\begin{array}{c}
A \\ f
\end{array}
\]

A kind of "fibration without its functor"
The 2-cells of the bicategory $\overrightarrow{\text{Cat}(\mathcal{B})}$

- the 2-cells

$$\theta : f \Rightarrow g : s \to t$$

are the 2-cells

$$\theta : f \Rightarrow t \otimes g : A \to B$$

such that the diagram of 2-cells commute:

$$
\begin{array}{c}
\phantom{f \otimes s} \otimes f \quad \theta \\
\theta \downarrow \quad \downarrow \mu \\
\phantom{f \otimes s} \otimes g \phantom{s} \otimes g
\end{array}
$$

Reduced form
An alternative formulation in string diagrams

Reduced form
The 2-cells of the bicategory $\overrightarrow{\text{Cat}(\mathcal{B})}$

- the 2-cells

\[ \theta : f \Rightarrow g : s \to t \]

are the 2-cells

\[ \theta : f \otimes s \Rightarrow t \otimes g : A \to B \]

making the diagram of 2-cells commute:

\[ \begin{array}{c}
    f \otimes s \otimes s \xrightarrow{\tilde{f}} t \otimes f \otimes s \xrightarrow{\theta} t \otimes t \otimes g \\
    \downarrow \theta \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \mu \\
    t \otimes g \otimes s \xrightarrow{\tilde{g}} t \otimes t \otimes g \xrightarrow{\mu} t \otimes g 
\end{array} \]

Non reduced form
Alternative formulation in string diagrams

Non reduced form
From the reduced form to the non reduced form (and conversely)
First equation

\begin{align*}
\text{Diagram 1} & \quad = \quad \text{Diagram 2} & \quad = \quad \text{Diagram 3} & \quad = \quad \text{Diagram 4}
\end{align*}
Second equation

\[ t \quad g \quad f \quad g \quad f \]

\[ = \quad \]

\[ t \quad g \quad f \quad g \quad f \]

\[ = \quad \]

\[ t \quad g \quad f \quad g \quad f \]
Property of the non reduced form
Given a category $\mathcal{C}$ with finite limits...

the bicategory $\overset{\longrightarrow}{\text{Cat}}(\text{Span})$ has

– the same 0-cells as the category $\mathcal{C}$,

– the 1-cells are the spans,

– the 2-cells are morphisms between spans.

Fact: a monad in $\text{Span}$ is an internal category in $\mathcal{C}$
Part 2

Modules between categories

categories seen as rings with several objects
Representation principle

Every monad (in the bicategorical sense)

\[ t : A \rightarrow A \]

induces a monad (in the categorical sense)

\[ \mathcal{B}(X, t) : \mathcal{B}(X, A) \rightarrow \mathcal{B}(X, A) \]

defined by post-composition

\[ X \xrightarrow{f} A \quad \Rightarrow \quad X \xrightarrow{f} A \xrightarrow{t} A \]

for every 0-cell \(X\) of the bicategory \(\mathcal{B}\).
Representation principle (dual)

Dually, every monad (in the bicategorical sense)

\[ t : A \rightarrow A \]

induces a monad (in the categorical sense)

\[ \mathcal{B}(t, X) : \mathcal{B}(A, X) \rightarrow \mathcal{B}(A, X) \]

defined this time by pre-composition:

\[ A \xrightarrow{f} X \quad \mapsto \quad A \xrightarrow{t} A \xrightarrow{f} X \]

for every 0-cell \( X \) of the bicategory \( \mathcal{B} \).
Representation principle (on both sides)

Every pair of monads (in the bicategorical sense)

\[ s : A \to A \quad t : B \to B \]

induces a monad (in the categorical sense)

\[ \mathcal{B}(s, t) : \mathcal{B}(A, B) \to \mathcal{B}(A, B) \]

defined by pre- and post- composition:

\[
\begin{align*}
A \xrightarrow{f} B & \quad \mapsto \quad A \xrightarrow{f} B \\
& \quad s \downarrow \quad t \\
A & \quad B
\end{align*}
\]

for every 0-cell \( X \) of the bicategory \( \mathcal{B} \).
The bicategory \textbf{Module}(\mathcal{B})

- the \textbf{0-cells} are the monads \((A, s)\) of the bicategory \(\mathcal{B}\)

- the \textbf{1-cells}

\[
(A, s) \rightarrow (B, t)
\]

are the algebras of the monad

\[
\mathcal{B}(s, t) : \mathcal{B}(A, B) \rightarrow \mathcal{B}(A, B).
\]

So, they are pairs \((f, \phi)\) consisting of a 1-cell

\[
A \xrightarrow{f} B
\]
and a 2-cell

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow s & & \downarrow t \\
A & \xrightarrow{f} & B \\
\end{array}
\]

\[\Downarrow \phi\]

in the bicategory $\mathcal{B}$, satisfying the coherence diagrams:
The bicategory $\text{Module}(\mathcal{B})$
The bicategory $\textbf{Module}(\mathcal{B})$

– the 2-cells

are the morphisms of $\mathcal{B}(s, t)$-algèbres in the category $\mathcal{B}(A, B)$. 
The bicategory \textbf{Module}(\mathcal{B})

In other words, a 2-cell

\[
\begin{array}{c}
(A, s) \\
\downarrow \theta \\
(g, \psi)
\end{array}
\quad \xrightarrow{(f, \phi)}
\begin{array}{c}
(B, t)
\end{array}
\]

is a 2-cell

\[
\begin{array}{c}
A \\
\downarrow \theta \\
B
\end{array}
\quad \xrightarrow{f}
\begin{array}{c}
A \\
g
\end{array}
\]
satisfying the equality:
The bicategory **Module(\mathcal{B})**

The composite of the two 1-cells

\[(A, s) \xrightarrow{(f, \phi)} (B, t) \xrightarrow{(g, \psi)} (C, u)\]

is defined as the co-equalizer of the 2-cells described by the two different ways to compose:
We make the hypothesis
(1) that the category $\mathcal{B}(A, B)$ has coequalizers, for every $A, B$,
(2) the horizontal composition $\otimes$ in $\mathcal{B}$ preserves these co-equalizers.
Part 3

Game cobordism
Dialogue categories

A symmetric monoidal category $\mathcal{C}$ equipped with a functor

$$\neg : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$$

and a natural bijection

$$\varphi_{A,B,C} : \mathcal{C}(A \otimes B, \neg C) \cong \mathcal{C}(A, \neg (B \otimes C))$$
Cobordism
Frobenius objects

A Frobenius object $F$ is a monoid and a comonoid satisfying

\[
\begin{align*}
md &= \qquad = \qquad = \\
md &= \qquad = \\
md &=
\end{align*}
\]

an alternative formulation of cobordism
Frobenius objects

Equivalently, a Frobenius object $F$ is a monoid with an isomorphism

$$S : F \longrightarrow F^*$$

to its dual object $F^*$ such that
Lax dualities in a 3-dimensional category
satisfying two coherence properties \((a)\)

is the identity 3-cell on the unit \(\eta\) of the 2-adjunction \(L \dashv R\).
satisfying two coherence properties (b)

is the identity 3-cell on the counit $\varepsilon$ of the 2-adjunction $L \dashv R$. 
Pseudo Frobenius objects

A pseudo Frobenius object in the bicategory of modules is the same thing as a \(*\)-autonomous category... when the two modules \(m\) and \(e\) are functors.

An observation by Brian Day and Ross Street (2003)
Lax Frobenius objects

Relax the self-duality equivalence

\[ \mathcal{C} \cong \mathcal{C}^{\text{op}} \]

into an adjunction

\[ \mathcal{C} \dashv \bot \dashv \mathcal{C}^{\text{op}} \]

this connects game semantics and quantum groups
Game semantics in ribbon diagrams

Idea: replace the elementary particles by the game boards
Game semantics in ribbon diagrams

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Conclusion

Logic = Data Structure + Duality

This point of view is accessible thanks to 2-dimensional algebra