HOMOLOGY OF PROOF-NETS

François Métayer
Equipe de Logique
45-55, 5ème étage, Université Paris VII
2 place Jussieu 75251 PARIS Cedex 05, FRANCE
e-mail: metayer@logique.jussieu.fr

Abstract

This work defines homology groups for proof-structures in multiplicative linear logic (see [Gir1], [Gir2], [Dan]). We will show that these groups characterize proof-nets among arbitrary proof-structures, thus obtaining a new correctness criterion and of course a new polynomial algorithm for testing correctness. This homology also bears information on sequentialization. An unexpected geometrical interpretation of the linear connectives is given in the last section. This paper exclusively focuses on *abstract* proof-structures, i.e. paired-graphs. The relation with actual proofs is investigated in [Gir1], [Gir2], [Dan], [Ret] and [Tro].

1. Paired-graphs

By *graph* we mean a pair $G = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is a finite set and $\mathcal{E}$ is a set of unordered pairs of elements of $\mathcal{V}$. The elements of $\mathcal{V}$ are the *vertices* and the elements of $\mathcal{E}$ are the *edges* of the graph . If u and v are vertices, we denote the edge $\{u, v\}$ by "uv; oriented edges will be denoted by (uv) and (vu)."

Definition 1.1. A *paired-graph* is a pair $(G, \mathcal{P})$ where G is a graph and $\mathcal{P}$ a set of pairs of edges such that:

- If $\{e, f\} \in \mathcal{P}$ then e and f have a common vertex.
- If $(p, p') \in \mathcal{P}^2$ and $p \neq p'$ then $p \cap p' = \emptyset$.

◇

$(G, \mathcal{P})$ will be simply denoted by G whenever no confusion with the underlying graph is possible. We denote as well $\mathcal{P} = \mathcal{P}(G)$. A paired-graph (p.g.) G is a *tree* if $\mathcal{P}(G) = \emptyset$ and the underlying graph is connected and acyclic. Let G be a p.g. A *paired edge* e of G is an edge belonging to a pair $p \in \mathcal{P}$. We denote by e^* the unique edge such that $\{e, e^*\} \in \mathcal{P}$. The remaining edges are called *free* and the set of free edges will be denoted by $\mathcal{E}_f(G)$. An *orientation* of $(G, \mathcal{P})$ is an orientation of G such that for each pair $\{uw, vw\} \in \mathcal{P}(G)$, the corresponding pair of oriented edges is either $\{(uw), (vw)\}$ or $\{(wu), (wv)\}$. Let G and G' be p.g's . A *morphism* from G to G' is a map $g : \mathcal{V}(G) \longrightarrow \mathcal{V}(G')$ such that:

- If $uv \in \mathcal{E}_f(G)$ then $g(u)g(v) \in \mathcal{E}_f(G')$ or $g(u) = g(v)$.
- If $\{uw, vw\} \in \mathcal{P}(G)$ then $\{g(u)g(w), g(v)g(w)\} \in \mathcal{P}(G')$ or $g(u) = g(v) = g(w)$.

G is a *subgraph* of G' if and only if $\mathcal{V}(G) \subset \mathcal{V}(G')$ and the inclusion map is a morphism. If G_1 and G_2 are subgraphs of G, $G_1 \cap G_2$ defined by $\mathcal{V}(G) = \mathcal{V}(G_1) \cap \mathcal{V}(G_2)$, $\mathcal{E}(G) = \mathcal{E}(G_1) \cap \mathcal{E}(G_2)$ and $\mathcal{P}(G) = \mathcal{P}(G_1) \cap \mathcal{P}(G_2)$ is a subgraph of G.

2. Proofnets

Let us first introduce some useful notations (see figure 2.1):

- U is the p.g. defined by $\mathcal{V}(U) = \{s\}$ and $\mathcal{E}(U) = \emptyset$.
- D is the p.g. defined by $\mathcal{V}(D) = \{s, t\}$ and $\mathcal{E}(D) = \emptyset$.

- T is the p.g. defined by $\mathcal{V}(T) = \{s, t, u\}$, $\mathcal{E}(T) = \{su, tu\}$ and $\mathcal{P}(T) = \{\{su, tu\}\}$.

Let G_1 and G_2 be paired graphs. We can form their disjoint sum $G_1 \coprod G_2$. If $s^i_1, \ldots, s^i_n$ are distinct vertices of G_i for $i = 1$ and 2, the identifications $s^1_k \approx s^2_k$ in $G_1 \coprod G_2$ define a new graph $G_1 \coprod G_2/\{s^1_1 \approx s^2_1, \ldots, s^1_n \approx s^2_n\}$, provided $s^1_i s^1_j$ and $s^2_i s^2_j$ are not simultaneously edges of G_1 and G_2 respectively.

Definition 2.1. Let G_1 and G_2 be p.g's and s_1 (resp.s_2) a vertex of G_1 (resp.G_2).
We denote by $\mathrm{t}(G_1, G_2, s_1, s_2)$ the graph $G_1 \coprod G_2/\{s_1 \approx s_2\}$.
Let G_1 be a p.g. and s_1,s_2 distinct vertices of G_1.
We denote by $\mathrm{p}(G_1, s_1, s_2)$ the graph $G_1 \coprod T/\{s_1 \approx s, s_2 \approx t\}$ ⋄

By the canonical maps from G_1 and G_2 to $\mathrm{t}(G_1, G_2, s_1, s_2)$ these graphs can be seen as subgraphs of $\mathrm{t}(G_1, G_2, s_1, s_2)$, still denoted by G_1 and G_2. Likewise G_1 and T become subgraphs of $\mathrm{p}(G_1, s_1, s_2)$. Of course operations t and p correspond to the linear connectives *tensor* ($\otimes$) and *par* ($\wp$) (see figure 2.2).

Definition 2.2. Π denotes the smallest class of paired-graphs containing trees and closed by the operations t and p. Elements of Π are called *proof-nets.* ⋄

3. Homology groups of paired-graphs

Homology groups can be defined for paired-graphs in the very same manner as for ordinary graphs. To every oriented p.g. G we first associate the following complex of abelian groups

$$0 \longrightarrow C_1(G) \xrightarrow{\partial} C_0(G) \xrightarrow{\epsilon} \mathbf{Z} \longrightarrow 0$$

where $C_0(G) = \mathbf{Z}[\mathcal{V}(G)]$ and $C_1(G)$ is the subgroup of $\mathbf{Z}[\mathcal{E}(G)]$ generated by the free edges and the elements $e + e^*$ where e runs over paired edges. The elements of $C_i(G)$ are the *i-chains* of G. ∂ is the restriction to $C_1(G)$ of the boundary morphism defined by $\partial(uv) = v - u$ for each "oriented edge (uv); ϵ is the *augmentation"* morphism defined by $\epsilon(u) = 1$ for each vertex u. The elements of $\ker \partial$ (resp. $\ker \epsilon$) are the *1-cycles* (resp. *0-cycles*).

By $\epsilon\partial = 0$ we can define:

Definition 3.1. The *homology groups* of G are $H_0(G) = \ker \epsilon / \operatorname{im} \partial$ and $H_1(G) = \ker \partial$. ⋄

If $\mathcal{P}(G) = \emptyset$ we get the groups of the ordinary graph G in reduced homology. In that case $H_0(G)$ and $H_1(G)$ are free abelian groups of respective ranks r_0 and r_1 where $r_0 + 1$ is the number of connected components of G and r_1 is the maximal number of independent cycles in G.

Returning now to the general case, we denote by $\{c\}_G$ the homology class of the i-chain c in $H_i(G)$. Notice that ∂ depends on the orientation, though the groups do not. Let $\phi : G \longrightarrow G'$ be a morphism. As usual, ϕ induces morphisms ϕ^i_* from $H_i(G)$ to $H_i(G')$ such that

$$(\phi\theta)^i_* = \phi^i_*\theta^i_* \qquad \mathrm{id}_* = \mathrm{id}$$

Example 1. Define G by $\mathcal{V}(G) = \{s_1, s_2, s_3\}$, $\mathcal{E}(G) = \{s_1s_2, s_1s_3, s_2s_3\}$ and $\mathcal{P}(G) = \{\{s_1s_3, s_2s_3\}\}$. We orient G by taking $a_1 = (s_1s_3)$, $a_2 = (s_2s_3)$ and $a_3 = (s_1s_2)$ as oriented edges (see figure 3.1).

$C_0(G) \cong \mathbf{Z}^3$ with generators s_1, s_2 and s_3 and $C_1(G) \cong \mathbf{Z}^2$ with generators $a_1 + a_2$ and a_3. For each pair (m, n) of integers:

$$\partial(ma_3 + n(a_1 + a_2)) = m(s_2 - s_1) + n(2s_3 - s_1 - s_2) = 2ns_3 - (m + n)s_1 + (m - n)s_2$$

Hence ∂ is injective—that is $H_1(G) = 0$. On the other hand $\ker \epsilon \cong \mathbf{Z}^2$ generated by $c_1 = s_3 - s_1$ and $c_2 = s_3 - s_2$. But $\{c_2\}_G = -\{c_1\}_G$ because

$$c_1 + c_2 = \partial(a_1 + a_2)$$

and $2\{c_1\}_G = 0$ because

$$2c_1 = 2s_3 - 2s_1 = 2s_3 - s_1 - s_2 + s_2 - s_1 = \partial(a_1 + a_2 + a_3).$$

Now $\{c_1\}_G \neq 0$ because $c_1 \notin \operatorname{im}\partial$. Then $H_0(G)$ is generated by a unique class $\{c_1\}_G$ of order 2 and $H_0(G) \cong \mathbf{Z}/2\mathbf{Z}$. ◇

Example 2. The following table displays the homology groups of U, D and T.

	U	D	T
H_0	0	$\mathbf{Z}$	$\mathbf{Z}$
H_1	0	0	0

table 1

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Example 3. Let G be defined by $\mathcal{V}(G) = \{s, t, u, v\}$, $\mathcal{P}(G) = \{\{us, ts\}, \{tu, vu\}\}$ with oriented edges (us), (ts), (tu), (vu) and (vs) (see figure 3.2). The associated complex gives $\ker\epsilon \cong \mathbf{Z}^3$ with generators $s-u$, $s-t$ and $s-v$.

We compute the homology of G by writing down the matrix of ∂—the range being restricted to $\ker\epsilon$—in the bases $(a, b, c) = ((us) + (ts), (tu) + (vu), (vs))$ of $C_1(G)$ and $(s-u, s-t, s-v)$ of $\ker\epsilon$:

$$\begin{array}{c} \\ s-u \\ s-t \\ s-v \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

It reduces to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

by unimodular transformations. This gives the injectivity of ∂, hence $H_1(G) = 0$, and a presentation of $H_0(G)$ by generators and relations, hence $H_0(G) \cong \mathbf{Z}/3\mathbf{Z}$.

◇

Let G be a graph and K a subgraph of G. By regarding $C_i(K)$ as a subgroup of $C_i(G)$ we define a new complex

$$0 \longrightarrow C_1(G, K) \xrightarrow{\overline{\partial}} C_0(G, K) \longrightarrow 0$$

where $C_i(G, K) = C_i(G)/C_i(K)$—the group of *relative* i-chains modulo K— and $\overline{\partial} : C_1(G, K) \longrightarrow C_0(G, K)$ is induced by ∂ on factor groups. Then

Definition 3.2. The *relative* homology groups of G modulo K are $H_0(G, K) = C_0(G, K)/\operatorname{im}\overline{\partial}$ and $H_1(G, K) = \ker\overline{\partial}$. ◇

The excision theorem still holds, as well as the exact sequence of relative homology [Mun]. In order to apply the excision property we denote by $G \setminus K$ the graph obtained from G by removing all edges of K and the vertices of K not belonging to a remaining edge.

Recall that the *rank* of a free finitely generated abelian group is its dimension as a $\mathbf{Z}$-module. Now let A be a finitely generated abelian group—not necessarily free— and T its torsion subgroup. We still call *rank* of A the rank of the free group A/T. If

$$0 \longrightarrow A_p \longrightarrow \cdots \longrightarrow A_0 \longrightarrow 0$$

is a complex of abelian groups of finite type, the *Euler characteristic* of this complex is the integer

$$\chi = \sum_j (-1)^j \operatorname{rank} A_j .$$

Then

$$\chi = \sum_j (-1)^j \operatorname{rank} H_j.$$

hence $\chi = 0$ for any exact sequence.

4. The homology groups of proof-nets

We now compute the homology groups for the elements of Π.

Theorem 4.1. *Let G be a graph and $p = \operatorname{card}\mathcal{P}(G)$. If $G \in \Pi$, then $H_1(G) = 0$ and $H_0(G)$ is a finite group of order 2^p.*

This immediately follows from two lemmas:

Lemma 4.2. *Let G_1 and G_2 be graphs, s_1 (resp. s_2) a vertex of G_1 (resp. G_2). There are isomorphisms*

$$H_i(\mathrm{t}(G_1, G_2, s_1, s_2)) \cong H_i(G_1) \oplus H_i(G_2)$$

for $i = 1$ and 2.

Proof. Let G be $\mathrm{t}(G_1, G_2, s_1, s_2)$. $G_1 \cap G_2$ is isomorphic to U. We get a Mayer-Vietoris sequence

$$0 \longrightarrow H_1(U) \longrightarrow H_1(G_1) \oplus H_1(G_2) \longrightarrow H_1(G) \longrightarrow H_0(U) \longrightarrow H_0(G_1) \oplus H_0(G_2) \longrightarrow H_0(G) \longrightarrow 0$$

where $H_0(U) = H_1(U) = 0$. This gives the desired isomorphisms. ◇

Lemma 4.3. *Let G_1 be a graph and s_1, s_2 two distinct vertices of G_1.*

- $H_1(\mathrm{p}(G_1, s_1, s_2)) = H_1(G_1)$.
- *If $H_0(G_1)$ is a finite group then* $\operatorname{card} H_0(\mathrm{p}(G_1, s_1, s_2)) = 2 \operatorname{card} H_0(G_1)$.

Proof. Let G be $\mathrm{p}(G_1, s_1, s_2)$. The isomorphism $G_1 \cap T \cong D$ gives a Mayer-Vietoris exact sequence:

$$0 \longrightarrow H_1(D) \longrightarrow H_1(G_1) \oplus H_1(T) \longrightarrow H_1(G) \longrightarrow H_0(D) \longrightarrow H_0(G_1) \oplus H_0(T) \longrightarrow H_0(G) \longrightarrow 0$$

which by table 1 reduces to

$$0 \longrightarrow H_1(G_1) \xrightarrow{\phi} H_1(G) \xrightarrow{\chi} \mathbf{Z}\{s_2 - s_1\}_D \xrightarrow{\theta} H_0(G_1) \oplus \mathbf{Z}\{s_3 - s_1\}_T \longrightarrow H_0(G) \longrightarrow 0$$

where

$$\theta(\{s_2 - s_1\}_D) = (\{s_2 - s_1\}_{G_1}, -\{s_2 - s_1\}_T)$$

Since $\{s_2 - s_1\}_T$ does not vanish in $H_0(T)$, θ is injective. Hence $\operatorname{im}\chi = \ker\theta = 0$ and ϕ is an isomorphism. This proves the first statement.

Suppose now that $H_0(G_1)$ is a finite group. It has a presentation

generators	relations
$c_1, \ldots c_p$	$n_i c_i = 0$ for $i = 1, \ldots p$

There is a relation:

$$\{s_2 - s_1\}_{G_1} = \sum_{i=1}^{p} l_i c_i$$

In T, $\{s_2 - s_1\}_T = 2c_0$ where $c_0 = \{s_3 - s_1\}_T$.

On the other hand $H_0(G_1) \oplus H_0(T)$ is generated by the elements $d_0 = (0, c_0)$ and $d_i = (c_i, 0)$ for $i = 1, \ldots p$. Therefore $H_0(G)$ has a presentation :

generators	relations
$d_0, \ldots d_p$	$n_i d_i = 0$ for $i = 1, \ldots p$
	$2d_0 - \sum_{i=1}^{p} l_i d_i = 0$

by exactness in the previous sequence.

This can be put in matrix form:

$$M = \begin{pmatrix} n_1 & 0 & \dots & -l_1 \\ 0 & & & \vdots \\ \vdots & & n_p & -l_p \\ 0 & \dots & 0 & 2 \end{pmatrix}$$

Finally

$$\operatorname{card} H_0(G) = |\det M| = |2n_1 \dots n_p| = 2 \operatorname{card} H_0(G_1).$$

This proves the second statement. ⋄

In fact we shall use a slight refinement of this result, namely that the conclusion still holds if we suppose $H_0(G)$ finite instead of $H_0(G_1)$. Indeed in the exact sequence

$$0 \longrightarrow H_1(G_1) \xrightarrow{\phi} H_1(G) \xrightarrow{\chi} \mathbf{Z}\{s_2 - s_1\}_D \xrightarrow{\theta} H_0(G_1) \oplus \mathbf{Z}\{s_3 - s_1\}_T \longrightarrow H_0(G) \longrightarrow 0$$

the isomorphism between $H_1(G)$ and $H_1(G_1)$ does not depend on the finiteness of $H_0(G_1)$. Now the Euler characteristic vanishes hence $H_0(G)$ and $H_0(G_1)$ have the same rank. In particular if one of these groups is finite, so is the other.

5. Simplification lemma

Let G be a graph and a be a paired-edge. G^a denotes the graph obtained from G by removing a^* and making a free. In this section we compare the homology groups of G and G^a. Here the main result is

Lemma 5.1. *Let G be a graph such that $H_1(G) = 0$ and $H_0(G)$ is a finite group of order g. For every paired-edge a, there is a $b \in \{a, a^*\}$ such that:*

- *$H_1(G^b) = 0$.*
- *$H_0(G^b)$ is a finite group with $\operatorname{card} H_0(G^b) \geq g/2$.*

Proof. Let G be as in the lemma and a a paired-edge. We denote $a = s_1s_3$, $a^* = s_2s_3$, $G_1 = G^{a^*}$ and $G_2 = G^a$. If we see T as the subgraph of G with vertices s_1,s_2,s_3, we denote $G_0 = G \setminus T$, and $T_i = T \cap G_i$ for $i = 0$, 1 and 2 (see figure 5.1).

• $H_1(G) = H_1(T) = 0$ and $H_0(T) = \mathbf{Z}\{s_3 - s_1\}_T$ hence the exact sequence of relative homology is:

$$0 \longrightarrow H_1(G,T) \xrightarrow{\partial_*} \mathbf{Z}\{s_3 - s_1\}_T \xrightarrow{g} H_0(G) \longrightarrow H_0(G,T) \longrightarrow 0$$

But $H_0(G)$ is finite and so is $H_0(G,T)$. Then the Euler-characteristic is $\operatorname{rank} H_1(G,T) - 1 = 0$ and $H_1(G,T) \cong \mathbf{Z}$.

Let A be the set of chains c in $C_1(G_0)$ such that $\partial c \in C_0(T_0)$. $A \cong H_1(G,T) \cong \mathbf{Z}$ is generated by a unique element c_0. Then $H_1(G,T)$ is generated by

$$\gamma = \{c_0\}_{G,T}.$$

On the other hand $\epsilon\partial c_0 = 0$ and there are integers m_1 and m_2 such that

$$\partial c_0 = (m_1 + m_2)s_3 - m_1 s_1 - m_2 s_2.$$

This implies:

$$\partial_* \gamma = \{\partial c_0\}_T = (m_1 - m_2)\{s_3 - s_1\}_T = (m_2 - m_1)\{s_3 - s_2\}_T$$

Consequently

$$\operatorname{card} H_0(G) = |m_1 - m_2| \times \operatorname{card} H_0(G,T) \qquad (1)$$

Notice also that $|m_1 - m_2| > 0$ since otherwise ∂_* cannot be injective, in contradiction with exactness in $H_1(G,T)$.
• Likewise, for $i = 1$ and 2, $H_1(T_i) = 0$ and $H_0(T_i) = \mathbf{Z}\{s_3 - s_i\}_{T_i}$

Then the exact sequence becomes

$$0 \longrightarrow H_1(G_i) \longrightarrow H_1(G_i, T_i) \xrightarrow{\partial_*^i} \mathbf{Z}\{s_3 - s_i\}_{T_i} \xrightarrow{g_i} H_0(G_i) \longrightarrow H_0(G_i, T_i) \longrightarrow 0$$

Here again $\gamma_i = \{c_0\}_{G_i,T_i}$ generates $H_1(G_i, T_i)$ and by definition:

$$\partial_*^i \gamma_i = \{\partial c_0\}_{T_i} = m_i\{s_3 - s_i\}_{T_i}$$

Let $m = \max(|m_1|, |m_2|)$: since $|m_1 - m_2| > 0$, $m \neq 0$. We may suppose that $m = |m_1|$, by exchanging a and a^* if necessary. In that case ∂_*^1 is injective and exactness implies

$$H_1(G_1) = 0$$

This proves the first assertion.

Now the sequence becomes:

$$0 \longrightarrow H_1(G_1, T_1) \xrightarrow{\partial_*^1} \mathbf{Z}\{s_3 - s_1\}_{T_1} \xrightarrow{g_1} H_0(G_1) \longrightarrow H_0(G_1, T_1) \longrightarrow 0$$

By excision theorem, $H_0(G_1, T_1) \cong H_0(G,T)$ hence $H_0(G_1, T_1)$ is finite and

$$\text{card}\, H_0(G_1) = |m_1| \times \text{card}\, H_0(G,T) \tag{2}$$

Finally

$$|m_1 - m_2| \leq |m_1| + |m_2| \leq 2\,|m_1| \tag{3}$$

By (1), (2) and (3) we get

$$\text{card}\, H_0(G) \leq 2\,\text{card}\, H_0(G_1)$$

and we are done. ◇

It is now possible to bound $\text{card}\, H_0(G)$ when this group is finite and $H_1(G) = 0$.

Proposition 5.2. *Let G be a graph such that $H_0(G)$ is finite and $H_1(G) = 0$. Then*

$$\text{card}\, H_0(G) \leq 2^{\text{card}\,\mathcal{P}(G)}$$

Proof. By induction on $\text{card}\,\mathcal{P}(G)$. If $\text{card}\,\mathcal{P}(G) = 0$, G is an ordinary graph and $H_0(G)$ cannot be finite unless $H_0(G) = 0$, which gives the result in this case.

Suppose that $\text{card}\,\mathcal{P}(G) = n > 0$. We can choose a paired-edge a, and (5.1) gives b with $\text{card}\, H_0(G) \leq 2\,\text{card}\, H_0(G_b)$. By induction hypothesis $\text{card}\, H_0(G_b) \leq 2^{n-1}$ hence $\text{card}\, H_0(G) \leq 2^n$, which ends the proof. ◇

Let (H) the conjunction of the following two conditions:

- $H_0(G)$ is finite, of order $2^{\text{card}\,\mathcal{P}(G)}$.
- $H_1(G) = 0$.

For graphs satisfying (H), the previous results have an important consequence:

Lemma 5.3. *If G satisfies* (H)*, then for each paired-edge a, G^a still satisfies* (H).

Proof. Let G be a graph satisfying (H) and a a paired-edge. The essential point is that the inequalities (3) in the proof of (5.1) are in fact *equalities*. Otherwise $|m_2 - m_1| < 2\,|m_1|$ and the same proof implies:

$$\text{card}\, H_0(G) < 2\,\text{card}\, H_0(G_1)$$

hence card $H_0(G_1) > 2^{\mathrm{card}\,\mathcal{P}G_1}$ in contradiction with (5.2) .

The equality in (3) then implies $|m_1| = |m_2|$ and card $H_0(G) = 2\,\mathrm{card}\, H_0(G_1) = 2\,\mathrm{card}\, H_0(G_2)$. This gives the result. ◇

Let now G be a graph with card $\mathcal{P}(G) = p$. A *switching* of G is a set $\sigma = \{a_1, \ldots, a_p\}$ obtained by the choice of one edge a_i in each pair of G. The—ordinary— graph $(((G^{a_1})^{\cdots})^{a_p})$ will be denoted by G^σ. Clearly

Proposition 5.4. *If G satisfies* (H), *then for every switching σ, G^σ is a tree.*

In other words (H) implies the correctness criterion "found by Danos and Régnier (see [Gir2] or [Dan]); thus" every graph satisfying (H) is a proof-net. We now give a direct proof of this result.

6. The sequentialization theorem

Let G be a p.g. The underlying ordinary graph (with same edges and vertices as G, and no pair) will be denoted by $|G|$. We denote by G° the graph obtained from G by removing all paired-edges. The connected components of $|G^\circ|$ are called *G-blocks.*

Let $s_1, \ldots, s_k$ be distinct vertices of G such that for every $i \in \{1, \ldots, k-1\}$ the edge $s_i s_{i+1}$ belongs to G. Then $\gamma = (s_1, \ldots, s_k)$ is called a *G-path* from s_1 to s_k. A G-path is *simple* if it contains no paired-edge, in other words, if it belongs to a G-block.

If G is oriented, we associate to each $\gamma = (s_1, \ldots, s_k)$ a chain $\sum_{i=1}^{k-1}(s_i s_{i+1})$ of $C_1(|G|)$ (by using $(st) = -(ts)$). We still denote this chain by γ, and $\partial\gamma = s_k - s_1$. For each pair (s,t) of vertices of G the two following assertions are clearly equivalent:

- s and t belong to the same block.
- There is a simple path γ such that $\partial\gamma = t - s$.

In particular, if s and t are in the same G-block, $\{t-s\}_G = 0$.

Lemma 6.1. *If $H_0(G)$ is finite, then $|G|$ is connected.*

Proof. Clearly $H_0(|G|)$ is a quotient of $H_0(G)$. In particular, if $H_0(G)$ is finite, so is $H_0(|G|)$, but since $|G|$ is an ordinary graph, $H_0(|G|) = 0$ and $|G|$ is connected. ◇

Proposition 6.2. *Let G be a graph with $\mathcal{P}(G) = \{\{s_i u_i, t_i u_i\}/i \in \{1, \ldots, p\}\}$. If $H_0(G)$ is a finite group, then it is generated by the elements $\{u_i - s_i\}_G$ for $i = 1, \ldots, p$.*

Proof. We know that $H_0(G)$ is generated by the homology classes of the $z - y$'s where (y,z) runs over the set of pairs of distinct vertices in G.

Let (y,z) be such a pair. Since $H_0(G)$ is finite, $|G|$ is connected by (6.1) and there is a path $\gamma = (z_1, \ldots, z_k)$ from y to z, for which:

$$z - y = \sum_{i=1}^{k-1} z_{i+1} - z_i$$

By taking homology classes on both sides, we notice that $\{z_{i+1} - z_i\}_G$ vanishes when $z_i z_{i+1}$ is a free edge of G. Hence $\{z-y\}_G$ is a linear combination of the $\{u-x\}_G$'s where xu runs over the set of paired edges. But for each pair $\{xu, x'u\}$, $\{u-x\}_G = -\{u-x'\}_G$ so that we can form a set of generators by choosing an edge in each pair. ◇

Recall from [Dan] that a pair is *splitting* if it cuts $|G|$ in two pieces, or more precisely:

Definition 6.3. Let G be a graph, $p = \{su, tu\} \in \mathcal{P}(G)$ and $G^* = G \setminus p$. p is *splitting* if and only if the connected component of u in $|G^*|$ does not contain s or t. ◇

Splitting pairs have a nice characterization in terms of homology as we shall see. First of all the choice of one edge in each pair p defines an element ω_p of $H_0(G)$: if $p = \{su, tu\}$ and su is the edge we choose, $\omega_p = \{u-s\}_G$. Now ω_p is called *irreducible* if and only if it does not belong to the subgroup of $H_0(G)$ generated by the $\omega_{p'}$'s for $p' \neq p$. This is clearly independent of the choice of the edges.

Proposition 6.4. *A pair p is splitting if and only if ω_p is irreducible.*

Proof. Let us choose an edge $s_q u_q$ in each pair "q of G; denote $d_q = u_q - s_q$ and $\omega_q = \{d_q\}_G$."

Suppose that $p = \{su, tu\}$ is not splitting. We get a G-path $\gamma = (z_1, \ldots, z_k)$ from s (or t) to u not containing su or tu and we may suppose that $z_1 = s$ without loss of generality.

$$u - s = \sum_{i=1}^{i=k-1} z_{i+1} - z_i.$$

By taking homology classes on both sides, terms coming from free edges vanish so that $\{u - s\}_G$ is a linear combination of the $\{z_{i+1} - z_i\}_G$'s where $z_i z_{i+1}$ is a paired-edge not su or tu. Since for each pair $q = \{s'u', t'u'\}$ $\{u' - s'\}_G = -\{u' - t'\}_G$, ω_p is a linear combination of the ω_q's for $q \neq p$ hence not irreducible.

Suppose conversely that $p = \{su, tu\}$ is splitting. Let G' obtained from G by removing su and tu. The connected component of $|G'|$ containing u determines a subgraph G_2 of G. Let $G_1 = G \setminus G_2$. Since p is splitting, G_1 and G_2 have a unique vertex u in common, so that $G = \mathrm{t}(G_1, G_2, u, u)$. Let $\mathcal{P}_1 = \mathcal{P}(G_1) \setminus p$ and $\mathcal{P}_2 = \mathcal{P}(G_2)$.

If ω_p is not irreducible,

$$\{d_p\}_G = \sum_{q \in \mathcal{P}_1} \lambda_q \{d_q\}_G + \sum_{q \in \mathcal{P}_2} \lambda_q \{d_q\}_G$$

hence

$$\{d_p - \sum_{q \in \mathcal{P}_1} \lambda_q d_q\}_G = \{\sum_{q \in \mathcal{P}_2} \lambda_q d_q\}_G.$$

On the other hand

$$(\{c_1\}_{G_1}, \{c_2\}_{G_2}) \mapsto \{c_1 + c_2\}_G$$

determines an isomorphism between $H_0(G_1) \oplus H_0(G_2)$ and $H_0(G)$ such that

$$(\{d_p - \sum_{q \in \mathcal{P}_1} \lambda_q d_q\}_{G_1}, -\{\sum_{q \in \mathcal{P}_2} \lambda_q d_q\}_{G_2})$$

has image 0. Necessarily

$$\{d_p - \sum_{q \in \mathcal{P}_1} \lambda_q d_q\}_{G_1} = 0$$

and there is a 1-chain c of G_1 such that

$$d_p - \sum_{q \in \mathcal{P}_1} \lambda_q d_q = \partial c.$$

Let us compare the coefficients of u on both sides: on the left-hand side it is 1 because $d_p = u - s$ and the other terms do not contain u . On the right-hand side, the only terms containing u are $\partial \mu((su) + (tu)) = \mu(2u - s - t)$ where the coefficient is even. This is a contradiction, and ends the proof. $\diamond$

Lemma 6.5. *Let G be a graph, $p \in \mathcal{P}(G)$ and $G^* = G \setminus \{p\}$:*

$$\mathrm{rank}\, H_0(G^*) \leq \mathrm{rank}\, H_0(G) + 1.$$

Proof. $H_0(G)$ is the quotient of $H_0(G^*)$ by a single relation. $\diamond$

We are now able to prove the following lemma, which is the crucial step in sequentialization.

Lemma 6.6. *In each graph G satisfying (H) and $\mathcal{P}(G) \neq \emptyset$, there is a pair $\{su, tu\}$ such that s and t belong to the same block.*

Proof. By induction on the cardinal n of $\mathcal{P}$. If $n = 1$ let $p = \{su, tu\}$ be the unique pair. We know by (6.2) that $\{u - s\}_G$ generates $H_0(G)$, which is $\mathbf{Z}/2\mathbf{Z}$ by (H). "Consequently $\{u - s\}_G \neq 0$ and s, u belong to distinct blocks; " and the same holds for t and u. But rank $H_0(G^\circ) \leq$ rank $H_0(G) + 1 = 1$ by (6.5) hence $|G^\circ|$ has at most two connected components. Therefore s and t must belong to the same block.

Suppose now that the conclusion holds for $n \geq 1$ and let G be a graph with $n + 1$ pairs. We choose a pair $\{su, tu\}$ for which $\omega = \{u - s\}_G$ has *maximal* order 2^ν in $H_0(G)$ and denote $a = su$. By (5.3) G^a still satisfies (H), and card $\mathcal{P}(G^a) = n$ hence by induction hypothesis there is a pair $\{s'u', t'u'\}$ in G^a such that s' and t' belong to the same G^a-block.

We get a simple G^a-path γ such that

$$\partial\gamma = t' - s'.$$

If γ is already simple in G we are done. Otherwise it contains an edge which is free in G^a but not in G (see figure 6.1), that is su. Then $\gamma = \gamma_1 \pm (su)$ where γ_1 is in $C_1(G)$. Hence

$$t' - s' = \partial\gamma_1 \pm (u - s).$$

By taking homology classes we get:

$$\{t' - s'\}_G = \pm\omega.$$

On the other hand $\{t' - s'\}_G = 2\omega'$ where $\omega' = \{u' - s'\}_G$. Hence

$$\omega = \pm 2\omega'.$$

Then the order of ω' in $H_0(G)$ is $2^{\nu+1}$. This is a contradiction. ◇

We now turn to the existence of splitting pairs.

Lemma 6.7. *Each graph G satisfying (H) and $\mathcal{P}(G) \neq \emptyset$ has a splitting pair.*

Proof. By induction on the number n of pairs. "Let us first choose an edge in each pair ; if" $q = \{su, tu\}$ and su is this edge, d_q denotes the chain $u - s \in C_0(G)$ and $\omega_q^G = \{d_q\}_G$. If $n = 1$, the unique pair p is splitting: $\omega_p^G \neq 0$—because it has order 2—hence it is irreducible.

Suppose now that the property holds for $n \geq 1$ and let card $\mathcal{P}(G) = n + 1$. By (6.6) There is a pair $p = \{su, tu\}$ such that s and t belong to the same block and we get a simple G-path γ from s to t. Let G_0 be the subgraph of G reunion of p and γ. By (5.3) G^a (where $a = su$) still satisfies (H) and has a splitting pair q by induction hypothesis. Let us examine the commutative diagram

$$\begin{array}{ccccccc}
H_0(G_0) \cong \mathbf{Z}/2\mathbf{Z} & \longrightarrow & H_0(G) & \longrightarrow & H_0(G, G_0) & \longrightarrow & 0 \\
\downarrow 0 & & & & \downarrow \phi & & \\
H_0(G_0^a) \cong 0 & \longrightarrow & H_0(G^a) & \longrightarrow & H_0(G^a, G_0^a) & \longrightarrow & 0
\end{array}$$

where horizontals are exact and ϕ is an isomorphism. It defines a morphism θ "from $H_0(G)$ to $H_0(G^a)$ by $\{c\}_G \mapsto \{c\}_{G^a}$; in particular " $\theta(\omega_p^G) = 0$. If q is not splitting in G, there is a relation in $H_0(G)$

$$\omega_q^G = \alpha\omega_p^G + \sum_{r \notin \{p,q\}} \lambda_r \omega_r^G$$

that θ takes to

$$\omega_q^{G^a} = \sum_{r \notin \{p,q\}} \lambda_r \omega_r^{G^a}$$

This shows that q is not splitting in G^a: contradiction. ◇

As a consequence of the previous results we finally deduce

Theorem 6.8. *Every graph G satisfying (H) is a proof-net.*

Proof. By induction on the size of G.

Case 1. $\mathcal{P}(G) = \emptyset$.

G is an ordinary graph such that $H_0(G) = H_1(G) = 0$ hence a tree, and a proof-net.

Case 2. $\mathcal{P}(G) \neq \emptyset$.

By (6.7) , there is a splitting pair $p = \{su, tu\}$. Let G' be the graph obtained from G by removing su and tu : the connected component of $|G'|$ containing u determines a subgraph G_1 of G. Let $G_2 = G \setminus G_1$. Since p is splitting, G_1 and G_2 have a unique common vertex u hence $G = \mathrm{t}(G_1, G_2)$.

(2.1.) G_1 reduces to u: in this case, $G = G_2 = \mathrm{p}(G_3, s, t)$ where $G_3 = G \setminus p$. By (4.3) and the subsequent remark $H_1(G_3) = H_1(G_2) = H_1(G) = 0$ and the finiteness of $H_0(G)$ implies $\mathrm{card}\, H_0(G_2) = 2\,\mathrm{card}\, H_0(G_3)$. This shows that G_3 satisfies (H) hence is a proof-net by induction hypothesis and so is G.

(2.2.) G_1 has at least one edge: in this case G_1 and G_2 are strictly smaller than G. On the other hand, by (4.2) $H_1(G_1) \oplus H_1(G_2) = H_1(G) = 0$ hence $H_1(G_1) = H_1(G_2) = 0$. Likewise $H_0(G_1) \oplus H_0(G_2) = H_0(G)$.

But $\mathrm{card}\, H_0(G) = 2^n$ hence $\mathrm{card}\, H_0(G_1) = 2^a$ and $\mathrm{card}\, H_0(G_2) = 2^b$ and we know by (5.2) that a (resp b) cannot be greater than the number of pairs of G_1 (resp. G_2). Then G_1 and G_2 satisfy (H) hence are proof-nets by induction hypothesis. So is $G = \mathrm{t}(G_1, G_2)$. $\diamond$

7. Cut-elimination

This section gives a homological proof that proofnets are closed under cut-elimination. In our abstract setting, this amounts to verify that the conditions (H) are preserved under certain substitutions of subgraphs we now describe.

Let G be a proofnet with n pairs and t, u, v, w, z five distinct vertices of G such that $\{ut, vt\}$ is a pair and wt, zt are free edges. Let K be the subgraph of G with vertices t, u, v, w, and z, free edges wt and zt and one pair $\{ut, vt\}$. Let K_1 (resp. K_2) be the graph with vertices u, v, w, et z, free edges uw and vz (resp. uz and vw) and no pair. We call G_i the graph obtained by substituting K_i for K in G: then G_i has $n - 1$ pairs.

We get exact sequences:

$$\begin{array}{cccccccccc} 0 & \longrightarrow & H_1(K) & \longrightarrow & H_1(G) & \longrightarrow & H_1(G,K) & \xrightarrow{\partial_*} & H_0(K) & \longrightarrow & \cdots \\ & & & & & \cdots & H_0(G) & \longrightarrow & H_0(G,K) & \longrightarrow & 0 \end{array} \tag{1}$$

and

$$\begin{array}{cccccccccc} 0 & \longrightarrow & H_1(K_i) & \longrightarrow & H_1(G_i) & \longrightarrow & H_1(G_i,K_i) & \xrightarrow{\partial_*} & H_0(K_i) & \longrightarrow & \cdots \\ & & & & & \cdots & H_0(G_i) & \longrightarrow & H_0(G_i,K_i) & \longrightarrow & 0 \end{array} \tag{2}$$

Clearly $H_1(K) \cong H_1(K_1) \cong H_1(K_2) \cong 0$ and $H_0(K) \cong H_0(K_1) \cong H_0(K_2) \cong \mathbf{Z}$. As G satisfies (H), (1) reduces to

$$0 \longrightarrow H_1(G,K) \xrightarrow{\partial_*} \mathbf{Z} \longrightarrow H_0(G) \longrightarrow H_0(G,K) \longrightarrow 0 \tag{3}$$

hence $H_1(G,K) \cong \mathbf{Z}$ because $H_0(G)$ is finite.

Then there is a chain c of $G \setminus K$ such that $\{c\}_{G,K}$ generates $H_1(G,K)$ and $\partial c = au + bv + cw + dz$ (hence $a + b + c + d = 0$). By excision, $\{c\}_{G_i,K_i}$ generates $H_1(G_i, K_i)$ for $i = 1$ and 2. Let us precise ∂_* in all three cases:

$$\partial_* \{c\}_{G,K} = \{au + bv + cw + dz\}_K = (a - b)\{u - w\}_K \tag{4}$$

$$\partial_* \{c\}_{G_1,K_1} = \{au + bv + cw + dz\}_{K_1} = (a + c)\{u - v\}_{K_1} \tag{5}$$

$$\partial_* \{c\}_{G_2,K_2} = \{au + bv + cw + dz\}_{K_2} = (a + d)\{u - v\}_{K_2} \tag{6}$$

where $\{u - w\}_K$, $\{u - v\}_{K_1}$ and $\{u - v\}_{K_2}$ are generators of $H_0(K)$, $H_0(K_1)$ and $H_0(K_2)$ respectively. Also

$$|a - b| = |a - b + a + b + c + d| = |a + c + a + d| \leq |a + c| + |a + d| \tag{7}$$

Now $|a - b| \neq 0$, by (4) and injectivity of ∂_* in (3). Then we may assume without loss of generality that $|a + c| \neq 0$, by (7). This together with (5) implies injectivity of ∂_* in:

$$\begin{array}{cccccccc} 0 & \longrightarrow & H_1(G_1) & \longrightarrow & H_1(G_1,K_1) & \xrightarrow{\partial_*} & H_0(K_1) & \longrightarrow \\ & \cdots & H_0(G_1) & \longrightarrow & H_0(G_1,K_1) & \longrightarrow & 0 & \end{array} \tag{8}$$

obtained from (2) in case $i = 1$. Therefore

$$\text{card}\, H_0(G_1) = \text{card}\, H_0(G_1, K_1) \times |a + c|$$

Suppose now that $|a - b| < 2\,|a + c|$. We get

$$\text{card}\, H_0(G_1) > (1/2)\,|a - b| \times \text{card}\, H_0(G, K)$$

But $|a - b| \times \text{card}\, H_0(G, K) = \text{card}\, H_0(G)$ by (1) and (4). This implies $\text{card}\, H_0(G_1) > 2^{n-1}$ in contradiction with (5.2) .

Then $2\,|a + c| \leq |a - b|$. By (7), $|a + c| \leq |a + d|$ so that we also have $|a + d| > 0$ and everything holds for $i = 2$ as well as for $i = 1$. Hence $|a - b| = 2\,|a + d| = 2\,|a + c|$ and

$$\text{card}\, H_0(G_i) = (1/2)\,|a - b| \times \text{card}\, H_0(G, K) = 2^{n-1} \tag{9}$$

for $i = 1$ and 2.

The injectivity of ∂_* in (8)—and its analogue for i=2—also yields

$$H_1(G) = H_2(G) = 0 \tag{10}$$

By (9) and (10) G_1 and G_2 still satisfy (H).
"

8. Jordan-H"older decompositions of $H_0(G)$ for proof nets "

In this section G will be a proof-net and n its number of pairs. "We shall see how the pairs of G give Jordan-H"older " decompositions of $H_0(G)$. We first choose one edge in each pair and resume the notations of (6.7) . If $X = \{p, q, r \ldots\}$ is a set of pairs of G we denote by $< p, q, r \ldots >_G$, or simply $< X >_G$ the subgroup of $H_0(G)$ generated by $\omega_p^G, \omega_q^G, \omega_r^G \ldots$. This is clearly independent of the choice of the edges. By definition, the *order* of p is the order of $< p >$ and it will be denoted by $\nu_G(p)$ or simply $\nu(p)$.

Let us first examine more closely the pairs $q = \{su, tu\}$ such that s and t belong to the same block. We call such a pair *initial* and Q the set of initial pairs. Let $q = \{su, tu\}$ be initial, and $a = su$. In the proof of (6.7) , we have defined a morphism θ from $H_0(G)$ to $H_0(G^a)$ by $\{c\}_G \mapsto \{c\}_{G^a}$, for which the following diagram commutes

$$\begin{array}{ccccccc}
H_0(G_0) \cong \mathbf{Z}/2\mathbf{Z} & \xrightarrow{i_*} & H_0(G) & \longrightarrow & H_0(G, G_0) & \longrightarrow & 0 \\
\downarrow 0 & & \downarrow \theta & & \downarrow \phi & & \\
H_0(G_0^a) \cong 0 & \longrightarrow & H_0(G^a) & \longrightarrow & H_0(G^a, G_0^a) & \longrightarrow & 0
\end{array}$$

where G_0 is any subgraph of G obtained by the reunion of q and "a simple path from s to t; ϕ is the isomorphism given" by excision. i_* is non-zero otherwise $\text{card}\, H_0(G^a) = \text{card}\, H_0(G)$ in contradiction with (5.2) and we get an exact sequence:

$$0 \longrightarrow H_0(G_0) \xrightarrow{i_*} H_0(G) \xrightarrow{\theta} H_0(G^a) \longrightarrow 0$$

In particular $\text{im}\, i_* =< q >_G$ hence $\nu(q) = 2$. More generally,

Lemma 8.1. *For each pair p, $\nu(p) \geq 2$.*

Proof. By induction on n . If $n = 0$, there is nothing to prove. Otherwise (6.6) gives an initial pair $q = \{a, a^*\}$, of order 2 as we have just seen and G^a satisfies the induction hypothesis. Let p be a pair. If $p = q$ we are done. If not, let $\theta : H_0(G) \longrightarrow H_0(G^a)$ as before, $\theta(< p >_G) =< p >_{G^a}$ which is non-zero by induction hypothesis and $\nu_G(p) \geq 2$. ◇

We can now prove the

Lemma 8.2. *Each pair of order 2 is initial.*

Proof. By induction on n. If $n = 0$, there is nothing to prove. Otherwise let $q = \{a, a^*\}$ an initial pair—by (6.6) —and $q' = \{s'u', t'u'\}$ a pair of order 2. If $q' = q$ we are done, otherwise $\theta(< q' >_G) =< q' >_{G^a}$ hence $\nu(q')_{G^a} \leq 2$ and by (8.1) , $\nu(q')_{G^a} \geq 2$. Thus q' has order 2 *in* G^a, and it is initial by the induction hypothesis.

Let γ be a simple G^a-path from s' to t': it cannot contain a otherwise

$$2\omega^G_{q'} = \{t' - s'\}_G = \omega^G_q \neq 0.$$

But this is a contradiction with $\nu_G(q') = 2$. ◇

Lemma 8.3. *For each subset $X \neq \emptyset$ of Q, $\sum_{r \in X} \omega^G_r \neq 0$.*

Proof. By induction on n. If $n = 0$, there is nothing to prove. Otherwise let $q = \{a, a^*\}$ be initial and θ as above. Notice that the set of initial pairs of G^a is exactly $Q \setminus \{q\}$, by the previous lemmas. Suppose that there is a relation

$$\sum_{r \in X} \omega^G_r = 0$$

for a subset $X \neq \emptyset$ of Q. By (8.1) we cannot have $X = \{q\}$ and on the other hand θ takes the above relation to

$$\sum_{r \in X \setminus \{q\}} \omega^{G^a}_r = 0$$

where $X \setminus \{q\}$ is a non void subset of $Q \setminus \{q\}$, in contradiction with the induction hypothesis. ◇

As an easy consequence of this result we get card $< Q >= 2^{\text{card}\, Q}$. (Indeed the lemma asserts that the canonical projector $\oplus_{q \in Q} < q > \longrightarrow < G >$ is injective.) Consider now a certain ordering $q_1, q_2, \ldots q_\ell$ of initial pairs. We denote $q_i = \{a_i, a_i^*\}$ and for every $i \in \{1, \ldots \ell\}$, $G_i = (((G^{a_1})^{a_2})^{\cdots})^{a_i}$ with $G = G_0$. We define as above a morphism $\theta_i : H_0(G_{i-1}) \longrightarrow H_0(G_i)$ by $\theta(\{c\}_{G_{i-1}}) = \{c\}_{G_i}$. Let $\overline{\theta} = \theta_\ell \circ \cdots \circ \theta_1$ from $H_0(G)$ to $H_0(G_\ell)$. Clearly $\overline{\theta}$ is surjective and $< Q >_G$ lies in the kernel of $\overline{\theta}$. But G_ℓ is a proof-net with $n - \ell$ pairs hence card $H_0(G_\ell) = 2^{n-\ell}$. Thus we get an exact sequence

$$0 \longrightarrow < Q >_G \longrightarrow H_0(G) \stackrel{\overline{\theta}}{\longrightarrow} H_0(G_\ell) \longrightarrow 0$$

We finally notice that the intersection of $< p >$ ($p \in \mathcal{P}(G)$) with $< Q >$ is never trivial. Precisely

Lemma 8.4. *For every pair p,* $\text{card}(< p > \cap < Q >) = 2$.

Proof. It suffices to prove that the intersection is not 0 because the elements of $< Q >$ have order 2 and the cyclic group $< p >$, of order 2^x, contains exactly one element of order 2. We prove this by induction on n. If $n = 0$, there is nothing to prove. Otherwise let $q = \{a, a^*\}$ and p be pairs, with q initial. If $< q >_G \subset < p >_G$ we are done. Otherwise we get an exact sequence

$$0 \longrightarrow < q >_G \stackrel{i_*}{\longrightarrow} H_0(G) \stackrel{\theta}{\longrightarrow} H_0(G^a) \longrightarrow 0$$

By induction hypothesis there is a non-zero τ^a in $< p >_{G^a} \cap < Q \setminus \{q\} >_{G^a}$. Then there is a non-zero τ in $< p >_G$ such that $\theta(\tau) = \tau^a$. On the other hand, $\theta(< Q >_G) =< Q \setminus \{q\} >$ and $< Q >_G$ contains $\ker \theta$, so that $\theta^{-1}(< Q \setminus \{q\} >_{G^a}) = < Q >_G$ and $\tau \in < Q >_G$. ◇

The structure of the subgroups generated by sets of pairs can now be made precise. The key result is

Proposition 8.5. *Let X be a set of pairs and $p \notin X$.*
If for every $q \in X$, $\nu(q) \leq \nu(p)$, then $< p > \not\subset < X >$.

Proof. Notice that the result is obvious if we suppose a *strict* inequality $\nu(q) < \nu(p)$. The proof is now by induction on n. If $n = 0$, there is nothing to prove. Otherwise the set Q of initial pairs is not empty and we get as shown above an exact sequence

$$0 \longrightarrow < Q >_G \longrightarrow H_0(G) \stackrel{\overline{\theta}}{\longrightarrow} H_0(G_\ell) \longrightarrow 0\,.$$

Let X be a set of pairs and $p \notin X$ a pair such that for every $r \in X$, $\nu_G(r) \leq \nu_G(p)$.
Case 1. $p \in Q$.

All elements of X have order 2 then belong to Q by (8.2) and $< p > \not\subset < Q >$ by (8.3) .

Case 2. $p \notin Q$.

For every pair q, card$(< q >_G \cap < Q >_G) = 2$ by (8.4) . In particular for every $r \notin Q$,

$$< r >_{G_\ell} \cong \frac{< r >_G}{< r >_G \cap < Q >_G}$$

hence $\nu_{G_\ell}(r) = \nu_G(r)/2$ so that $\nu_{G_\ell}(r) \leq \nu_{G_\ell}(p)$ for every $r \in X \setminus Q$. Then the induction hypothesis holds for G_ℓ with the pair p "and the set $X \setminus Q$; therefore" $< p >_{G_\ell} \not\subset < X \setminus Q >_{G_\ell}$.

But $\overline{\theta}(< p >_G) = < p >_{G_\ell}$ and $\overline{\theta}(< X >_G) = < X \setminus Q >_{G_\ell}$ thus $< p >_G \not\subset < X >_G$. ◇

Recall that the index of a subgroup B of a group A, which is denoted by $[A : B]$ is card(A/B).

Theorem 8.6. *Let $p_1, \ldots, p_n$ an ordering of the pairs of G such that $\nu(p_1) \leq \nu(p_2) \leq \ldots \leq \nu(p_n)$. The subgroups $F_0 = 0$ and $F_i = < p_1, \ldots p_i >$ for "$i = 1, \ldots n$ build a Jordan-H"older sequence for" $H_0(G)$, in other words $[F_i : F_{i-1}] = 2$ for every $i \in \{1, \ldots n\}$.*

Proof. By (8.5) , $[F_i : F_{i-1}] \geq 2$ and on the other hand

$$\mathrm{card}(H_0(G)) = 2^n = [F_n : F_0] = \prod_{i=1}^{n} [F_i : F_{i-1}].$$

◇

9. A geometrical interpretation.

Let us point out a rather unsatisfactory feature of our $H_0(G)$: it is finite and non trivial for proof-nets with at least one pair. This of course never happens in topology. Nevertheless there is a geometrical reason for this: indeed $H_0(G)$ and $H_1(G)$ will be seen respectively as $H_1(X)$ and $H_2(X)$ for a certain topological space X which can be naturally associated with G.

Remark. All topological spaces we shall mention are CW-complexes: the homology of such a complex K will be computed by using the complex of the $C_i(K)$'s—the sets of formal linear combinations of i-cells with integer coefficients.

Let G be a paired-graph with n pairs and V_n a wedge of n circles (i.e. n copies of S_1 "glued together at a single point v.); to each pair p we associate " one to one a circle S_1^p of this wedge. We now consider a map f from $|G|$ to V_n, having the following properties:

- If x is any point not interior to a paired edge, $f(x) = v$.
- if $p = \{su, tu\} \in \mathcal{P}(G)$, f maps one to one—and of course continuously—the interior of su (resp.tu) onto $S_1^p \setminus \{v\}$.
- $f(x)$ describes S_1^p in opposite senses according as x goes from s to u or from t to u.

Then f induces morphisms $f_\#^i$ from $C_i(|G|)$ to $C_i(V_n)$. and we get an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{G} \longrightarrow \mathcal{V} \longrightarrow 0$$

of complexes:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker f_\#^1 & \xrightarrow{\partial_\mathcal{K}} & \ker f_\#^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_1(|G|) & \xrightarrow{\partial} & C_0(|G|) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_1(V_n) & \xrightarrow{\partial} & C_0(V_n) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

We easily see that $C_1(\mathcal{K}) = \ker f^1_\#$ is exactly the subgroup of $C_1(|G|)$ generated by the free edges and the $(su)+(tu)$'s where $p = \{su, tu\}$ runs over $\mathcal{P}(G)$, in other words the group we previously called $C_1(G)$ (not $C_1(|G|)$!). On the other hand $C_0(V_n) \cong \mathbf{Z}$ hence $C_0(\mathcal{K}) = \ker f^0_\#$ is $\ker\epsilon$ by resuming the notations of section 4.

Finally $\partial_\mathcal{K}$ is the boundary morphism of the complex we have associated to G. Consequently the homology groups $H_1(\mathcal{K})$ and $H_0(\mathcal{K})$ of $\mathcal{K}$ are respectively $H_1(G)$ and $H_0(G)$. The above diagram gives a long exact sequence

$$0 \longrightarrow H_1(\mathcal{K}) \longrightarrow H_1(\mathcal{G}) \longrightarrow H_1(\mathcal{V}) \longrightarrow H_0(\mathcal{K}) \longrightarrow H_0(\mathcal{G}) \longrightarrow H_0(\mathcal{V}) \longrightarrow 0 \qquad (1)$$

Let now X_G be the *cone* of f, which is the quotient space of the disjoint sum of $I \times |G|$—where $I = [0,1]$—and V_n by the identifications $(0,x) \sim (0,y)$ and $(1,x) \sim f(x)$ for all x, y in $|G|$.

We know that the homology groups of $|G|$, V_n and X_G fit into an exact sequence:

$$\cdots \longrightarrow H_i(|G|) \xrightarrow{f_*} H_i(V_n) \longrightarrow H_i(X_G) \longrightarrow H_{i-1}(|G|) \longrightarrow \cdots \qquad (2)$$

Comparing (1) and (2) yields first

$$H_2(X_G) \cong \ker f^1_* \cong H_1(G)$$

On the other hand, $H_0(V_n) = 0$ so that we get two short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(V_n)/\operatorname{im} f^1_* & \longrightarrow & H_0(G) & \longrightarrow & H_0(|G|) & \longrightarrow & 0 \\ 0 & \longrightarrow & H_1(V_n)/\operatorname{im} f^1_* & \longrightarrow & H_1(X_G) & \longrightarrow & H_0(|G|) & \longrightarrow & 0 \end{array}$$

Because $H_0(|G|)$ is free abelian, both sequences split and

$$H_1(X_G) \cong H_0(G)$$

This proves

Theorem 9.1. *$H_2(X_G) = H_1(G)$ and $H_1(X_G) = H_0(G)$.*

Remark. There are of course many f's satisfying the required properties. Yet the topology of the cone of f is uniquely determined.

X_G can be defined more intuitively by the 2-dimensional CW-complex K we now describe. There are two 0-cells (vertices) α and β. For each vertex u of G there is an open 1-cell η_u. For each free edge $a = uv$ (resp. each pair p) there is an open 2-cell σ_a (resp. τ_p). These cells fit together as pictured on figure 9.1 and figure 9.2. The topological space we get is in fact homeomorphic to X_G.

Let G be for instance the proof net of the example 1 of section 4, X_G is homeomorphic to the projective plane $P^2(\mathbf{R})$ (see figure 9.3).

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Summary