A NON-COMMUTATIVE PRIESTLEY DUALITY

ANDREJ BAUER, KARIN CVETKO-VAH, MAI GEHRKE, SAM VAN GOOL, AND GANNA KUDRYAVTSEVA

Abstract. We prove that the category of left-handed skew distributive lattices with zero and proper homomorphisms is dually equivalent to a category of sheaves over local Priestley spaces. Our result thus provides a non-commutative version of classical Priestley duality for distributive lattices. The result also generalizes the recent development of Stone duality for skew Boolean algebras.

1. Introduction

Skew lattices [12, 13] are a non-commutative version of lattices: algebraically, a skew lattice is a structure $(S, \vee, \wedge)$, where $\vee$ and $\wedge$ are binary operations which satisfy the associative and idempotent laws, and certain absorption laws (see 2.1 below).

Concrete classes of examples of skew lattices occur in many situations. The skew lattices in such classes of examples often have a *zero* element, and also satisfy certain additional axioms, which are called *distributivity* and *left-handedness* (see 2.3 and 2.4 below). A (proto)typical class of such examples is that of *skew lattices of partial functions*, that we will describe now. If X and Y are sets, then the collection S of partial functions from X to Y carries a natural skew lattice structure, as follows. If $f, g \in S$ are partial functions, we define $f \wedge g$ to be the *restriction* of f by g, that is, the function with domain $\mathrm{dom}(f) \cap \mathrm{dom}(g)$, where its value is defined to be equal to the value of f. We define $f \vee g$ to be the *override* of f with g, that is, the function with domain $\mathrm{dom}(f) \cup \mathrm{dom}(g)$, where its value is defined to be equal to the value of g whenever g is defined, and to the value of f otherwise. The *zero element* is the unique function with empty domain.

One consequence of the results in this paper is that *every left-handed skew distributive lattice with zero can be embedded into a skew lattice of partial functions.* This fact was first proved in [14, 3.7] as a consequence of the description of the subdirectly irreducible algebras in the variety of skew distributive lattices. Our proof will not depend on this result, and it will moreover provide a canonical choice of an enveloping skew lattice of partial functions. A related result in computer science is described in [1], where the authors give a complete axiomatisation of the structure of partial functions with the operations override and 'update', from which the 'restriction' given above can also be defined.

To convey the flavour of what follows, let us now involve topology to generalize the example of skew lattices of partial functions. Let $p : E \twoheadrightarrow X$ be a surjective continuous map of topological spaces, also called a *bundle.* Recall that a *local section* of p is a continuous function $s : U \to E$, where $U \subseteq X$ is open, and $p \circ s = \mathrm{id}_U$. Now assume that X is zero-dimensional, so that it has a basis of clopens. The set of *local sections over clopens* again carries a natural skew lattice structure, by

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defining $\vee$ to be 'override', $\wedge$ to be 'restriction', and 0 to be the empty function, as above. We call this the *skew lattice of local sections over clopens.*

One of the results that we will prove in this paper is that *every left-handed skew distributive lattice with zero is isomorphic to a skew lattice of basic local sections of a bundle*. Moreover, it will be a consequence of the duality that there is a canonical choice for the bundle and basis which represent the skew lattice. Among all representing bundles, there is one up to isomorphism unique bundle $p : E \twoheadrightarrow X$ such that p is a local homeomorphism (i.e., *étale* map) and X is a *local Priestley space* (a space whose one-point-compactification is a Priestley space, see 3.3 below). This result generalizes both Priestley duality [18] and recent results on Stone duality [19] for skew Boolean algebras [2, 7, 8].

Yet another natural non-commutative generalization of distributive lattices is a class of inverse semigroups whose idempotents form a distributive lattice. For recent work on a generalization of Stone duality to this setting, we refer the reader to [9, 10, 11].

To state our results more precisely, we will need to recall some more background on skew lattices (Section 2) and Priestley duality (Section 3). In our duality, we will make use of the well-known correspondence between étalé spaces and sheaves (Appendix A). After these preliminaries, we will be ready to state our main theorem (Theorem 3.6), that the categories of left-handed skew distributive lattices and sheaves over local Priestley spaces are dually equivalent. Starting the proof of this theorem, we first give a more formal description of the skew lattice of local sections of an étalé space, and show that it gives rise to a functor (Section 4). To show that this functor is part of a dual equivalence, we will describe how to reconstruct the étalé space from its skew lattice of local sections (Section 5), and give a general description of this process for an arbitrary left-handed skew distributive lattice (Section 6). Finally (Section 7), we will put together the results from the preceding sections to prove our main theorem.

2. The category SDL of distributive left-handed skew lattices

For an extensive introduction to the theory of skew lattices we refer the reader to [12, 13, 14, 15]. To make our exposition self-contained, we collect some definitions and basic facts of the theory.

2.1. **Skew lattices.** A *skew lattice*[1] S is an algebra $(S, \wedge, \vee, 0)$ of type $(2, 2, 0)$, such that the operations $\wedge$ and $\vee$ are associative, idempotent and satisfy the absorption identities

$$x \wedge (x \vee y) = x = x \vee (x \wedge y),$$
$$(y \vee x) \wedge x = x = (y \wedge x) \vee x,$$

and the 0 element satisfies $x \wedge 0 = 0 = 0 \wedge x$. Note that a *lattice* is a skew lattice in which $\wedge$ and $\vee$ are commutative.

The *partial order* $\leq$ on a skew lattice S is defined by

$$x \leq y \iff x \wedge y = x = y \wedge x,$$

which is equivalent to $x \vee y = y = y \vee x$, by the absorption laws. Note that 0 is the minimum element in the partial order $\leq$.

If S and T are skew lattices, we say a function $h : S \to T$ is a *homomorphism* if it preserves the operations $\wedge$, $\vee$ and the zero element. We denote by Skew_0 the category of skew lattices with zero and homomorphisms between them.

[1] In this paper, all skew lattices will be assumed to have a zero element.

2.2. **Lattices form a reflective subcategory of skew lattices.** If we denote by Lat_0 the category of lattices with zero, then the full inclusion $\mathsf{Lat}_0 \to \mathsf{Skew}_0$ has a left adjoint, which can be explicitly defined using the equivalence relation $\mathcal{D}$, which is well known in semigroup theory [6]. Recall that $\mathcal{D}$ is the equivalence relation on a skew lattice S defined by $x \mathrel{\mathcal{D}} y$ if and only if $x \wedge y \wedge x = x$ and $y \wedge x \wedge y = y$, or equivalently, $x \vee y \vee x = x$ and $y \vee x \vee y = y$. The following is a version of the "first decomposition theorem for skew lattices".

Theorem 2.1 ([12], 1.7)**.** *Let S be a skew lattice. The relation $\mathcal{D}$ is a congruence, and $\alpha_S : S \to S/\mathcal{D}$ is a lattice quotient of S. For any homomorphism $h : S \to L$ where L is a lattice, there exists a unique $\overline{h} : S/\mathcal{D} \to L$ such that $\overline{h} \circ \alpha_S = h$.*

In particular, any skew lattice homomorphism $h : S \to T$ induces a homomorphism between the lattice reflections, which, by a slight abuse of notation, we will also denote by $\overline{h} : S/\mathcal{D} \to T/\mathcal{D}$, and which is defined as the unique lift of the composite $\alpha_T \circ h : S \to T/\mathcal{D}$.

Recall that a lattice homomorphism $k : L_1 \to L_2$ is called *proper* [5], provided that for any $y \in L_2$ there is some $x \in L_1$ such that $k(x) \geq y$. Note that a lattice homomorphism between *bounded* lattices is proper if, and only if, it preserves the top element. In the case of skew lattices, we need to consider algebras which may not have a largest element, so we need the 'unbounded' version of Priestley duality, where the natural morphisms are the proper homomorphisms, also see Section 3 below. We call a skew lattice homomorphism $h : S \to T$ *proper* provided that $\overline{h}$ is proper.

2.3. **Skew distributive lattices.** The object of study of this paper are *skew distributive lattices*[2], which are skew lattices satisfying the identities

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \tag{1}$$

$$(y \vee z) \wedge x = (y \wedge x) \vee (z \wedge x). \tag{2}$$

Our choice of terminology suggests that skew distributive lattices are 'skew' analogues of distributive lattices. Indeed, for S a skew distributive lattice, $S/\mathcal{D}$ is a distributive lattice. In order to state a 'converse direction' for this fact, one needs two additional properties which hold in any skew distributive lattice. A skew lattice S is called *symmetric* if $x \vee y = y \vee x$ if and only if $x \wedge y = y \wedge x$, and *normal* if each of the principal subalgebras $x \wedge S \wedge x$ forms a commutative sublattice of S. We then have the following result.

Proposition 2.2 ([14], Theorem 2.5)**.** *Let S be a skew lattice. The following are equivalent:*

(i) S is a skew distributive lattice;
(ii) S is normal and symmetric, and the lattice reflection $S/\mathcal{D}$ of S is distributive.

2.4. **Left-handed skew lattices.** For our duality, we will focus on skew distributive lattices which are left-handed. A skew lattice S is called *left-handed* if it satisfies the identity

$$x \wedge y \wedge x = x \wedge y, \text{ or, equivalently, } x \vee y \vee x = y \vee x.$$

The notion of *right-handed* skew lattices is defined dually.

[2]Note that what we call a skew distributive lattice here is termed *meet bidistributive and symmetric skew lattice* in [14]. The axioms (1) and (2) are also sometimes referred to as '$\wedge$-distributivity' in the literature. In this paper, we opt for the simpler term 'distributivity', as we do not consider other distributive axioms, so no confusion will arise.

The algebraic object of study in this paper is the category SDL whose *objects* are left-handed skew distributive lattices with zero, and whose *morphisms* are proper homomorphisms.

Left-handed skew distributive lattices have some desirable algebraic properties that we collect here, for use in what follows.

Lemma 2.3. *Let S be a left-handed skew distributive lattice, and let $a, a', b \in S$.*

(i) The semigroup $(S, \wedge)$ is left normal, i.e., $b \wedge a \wedge a' = b \wedge a' \wedge a$.

(ii) If $a, a' \leq b$ and $[a]_{\mathcal{D}} = [a']_{\mathcal{D}}$, then $a = a'$.

Proof. (i) By Proposition 2.2, S is normal. Therefore, using the definition of left-handedness, we get

$$b \wedge a \wedge a' = b \wedge a \wedge a' \wedge b = b \wedge a' \wedge a \wedge b = b \wedge a' \wedge a.$$

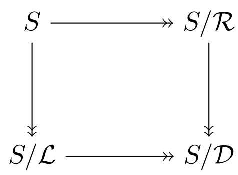
(ii) Since $a \,\mathcal{D}\, a'$, left-handedness yields $a \wedge a' = a$ and $a' \wedge a = a'$. Therefore,

$$\begin{aligned} a &= b \wedge a && (a \leq b) \\ &= b \wedge a \wedge a' && (a \,\mathcal{D}\, a') \\ &= b \wedge a' \wedge a && \text{(item (i))} \\ &= b \wedge a' && (a \,\mathcal{D}\, a') \\ &= a' && (a' \leq b). \end{aligned}$$

□

The reason we can restrict to *left-handed* skew distributive lattices without much loss of generality is the following. For a skew lattice S, we define the relation $\mathcal{R}$ on S by $x \,\mathcal{R}\, y$ iff $x \wedge y = y$ and $y \wedge x = x$. Dually, we define the relation $\mathcal{L}$ on S by $x \,\mathcal{L}\, y$ iff $x \wedge y = x$ and $y \wedge x = y$. We now have Leech's second decomposition theorem for skew lattices, which says the following.

Theorem 2.4 ([12], Theorem 1.15)**.** *The relations $\mathcal{L}$ and $\mathcal{R}$ are congruences for any skew lattice S. Moreover, $S/\mathcal{L}$ is the maximal right-handed image of S, $S/\mathcal{R}$ is the maximal left-handed image of S, and the following diagram is a pullback:*



2.5. **Primitive skew lattices.** In what follows, primitive skew lattices will play an important role. A skew lattice S is called *primitive* if it has only one non-zero $\mathcal{D}$-class, or, equivalently, if $S/\mathcal{D}$ is the bounded distributive lattice $\mathbf{2} = \{0, 1\}$. Primitive skew lattices are distributive.

If T is any set, there is an up to isomorphism unique primitive left-handed skew lattice with T as its only non-zero $\mathcal{D}$-class (see figure 1). The operations inside this $\mathcal{D}$-class are determined by lefthandedness: $t \wedge t' = t$ and $t \vee t' = t'$, for any $t, t' \in T$.

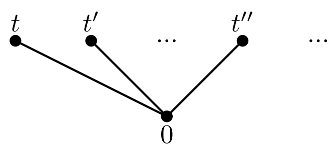


FIGURE 1. A primitive skew lattice

3. Sheaves over Priestley spaces

In this section we first outline a slight modification of classical Priestley duality for bounded distributive lattices to distributive lattices which may not have a largest element. For an extensive introduction to bounded distributive lattices and Priestley duality, we refer the reader to [17, 18, 4]. We then define the category of sheaves over local Priestley spaces, and state our main theorem.

3.1. **The category DL_0 of distributive lattices with zero.** The *objects* of the category DL_0 are *distributive lattices (with a zero element)*. The *morphisms* of the category DL_0 are the proper lattice homomorphisms (see 2.2 above).

3.2. **The category LPS of local Priestley spaces.** Recall that a *Boolean space* [19] is a compact Hausdorff space in which the clopen sets form a basis.

A subset E of a partially ordered set (poset) X is called *increasing* or an *upset* provided that for any $x \in E$ and $y \geq x$ we have $y \in E$. *Decreasing sets* or *downsets* of X are defined order-dually. A map $f : X \to Y$ between partially ordered sets is called *monotone* if $x_1 \leq x_2$ in X implies $f(x_1) \leq f(x_2)$ in Y.

We call a triple $(X, \tau, \leq)$ a *partially ordered topological space* if (X, τ) is a topological space and $(X, \leq)$ is a poset. Let $\tau^{\uparrow}$ be the set of all increasing open subsets of X and $\tau^{\downarrow}$ be the set of all decreasing open subsets of X. It is easy to see that $\tau^{\uparrow}$ and $\tau^{\downarrow}$ are topologies on X. A partially ordered topological space (X, τ) is called *totally order-disconnected* [17] provided that for any $x, y \in X$ such that $x \not\leq y$, there exist disjoint clopen sets $U \in \tau^{\uparrow}$ and $V \in \tau^{\downarrow}$ such that $x \in U$ and $y \in V$. A *Priestley space* $(X, \tau, \leq)$ is a partially ordered topological space which is compact and totally order-disconnected. The topological reduct (X, τ) of a Priestley space $(X, \tau, \leq)$ is a Boolean space.

Priestley duality is a dual categorical equivalence between the category of bounded distributive lattices DL_{01} and the category PS of Priestley spaces with continuous monotone maps, also see below for more details. In order to generalize Priestley duality to the category DL_0, we will now introduce local Priestley spaces [18].

Recall that for any topological space (X, τ), its *one-point-compactification* $(\widehat{X}, \widehat{\tau})$ is defined by $\widehat{X} = X \cup \{*\}$, where $* \notin X$, and $U \in \widehat{\tau}$ iff either $U \in \tau$, or $* \in U$ and $X \setminus U$ is compact for τ. For an ordered space $(X, \tau, \leq)$, we define the *ordered one-point-compactification* $(\widehat{X}, \widehat{\tau}, \widehat{\leq})$ by letting $(\widehat{X}, \widehat{\tau})$ be the (topological) one-point-compactification, and $\widehat{\leq}$ the extension of $\leq$ by adding $*$ as a maximum point.

We say $(X, \tau, \leq)$ is a *local Priestley space* if its ordered one-point-compactification $(\widehat{X}, \widehat{\tau}, \widehat{\leq})$ is a Priestley space. We define *the category* LPS *of local Priestley spaces*, in which a morphism $f : (X, \tau_X, \leq_X) \to (Y, \tau_Y, \leq_Y)$ is the restriction of a continuous monotone map between the one-point-compactifications $f : \widehat{X} \to \widehat{Y}$ for which $f^{-1}(*_Y) = \{*_X\}$.

Remark 3.1. It is possible to give an equivalent definition of this category without referring to the ordered one-point-compactification: local Priestley spaces are exactly the totally order disconnected spaces for which the space $(X, \tau^{\downarrow})$ has a basis consisting of τ-compact open downsets, and LPS-morphisms $(X, \tau_X, \leq_X) \to (Y, \tau_Y, \leq_Y)$ are equivalently described as continuous monotone maps with the further property that the inverse image of a τ_Y-compact set is τ_X-compact.

3.3. **Priestley duality.** Let D be a bounded distributive lattice. The *spectrum* of D is the Priestley space $\mathcal{S}(D) := (X, \tau, \leq)$, defined as follows. The points of X are the prime filters of D, τ is the topology defined by taking as a subbasis for the open sets the collection $\{\widehat{a}, \widehat{a}^c \mid a \in D\}$, where $\widehat{a} := \{p \in X \mid a \in p\}$, and $\leq$ is the reverse inclusion order on prime filters. A homomorphism $h : D_1 \to D_2$ of bounded distributive lattices yields a continuous monotone function $\mathcal{S}(h) : \mathcal{S}(D_2) \to \mathcal{S}(D_1)$ by sending $p \in \mathcal{S}(D_2)$ to $h^{-1}(p) \in \mathcal{S}(D_1)$.

Conversely, if $(X, \tau, \leq)$ is a Priestley space, we let $\mathcal{L}(X, \tau, \leq)$ be the bounded distributive lattice of clopen decreasing sets with the set-theoretic operations. A continuous monotone map $f : (X, \tau_X, \leq_X) \to (Y, \tau_Y, \leq_Y)$ yields a lattice homomorphism $\mathcal{L}(f) : \mathcal{L}(Y, \tau_Y, \leq_Y) \to \mathcal{L}(X, \tau_X, \leq_X)$ by sending a clopen downset $U \in \mathcal{L}(Y, \tau_Y, \leq_Y)$ to $f^{-1}(U) \in \mathcal{L}(X, \tau_X, \leq_X)$.

Theorem 3.2 (Classical Priestley duality)**.** *The contravariant functors $\mathcal{S} : \mathsf{DL}_{01} \to \mathsf{PS}$ and $\mathcal{L} : \mathsf{PS} \to \mathsf{DL}_{01}$ establish a dual equivalence between the categories DL_{01} and PS. The natural isomorphisms $\alpha : 1_{\mathsf{DL}_{01}} \to \mathcal{LS}$ and $\beta : 1_{\mathsf{PS}} \to \mathcal{SL}$ are given by*

$$\alpha_D(a) := \widehat{a} = \{p \in \mathcal{S}(D) : a \in p\},$$
$$\beta_X(p) := N_p = \{U \in \mathcal{L}(X, \tau, \leq) : p \in U\}.$$

Remark 3.3. Priestley duality is a *natural duality*, in the sense that the functors $\mathcal{S}$ and $\mathcal{L}$ are naturally isomorphic to Hom-functors into a so-called schizophrenic object, as follows. Let $\mathbf{2}$ denote both the unique 2-element distributive lattice in the category DL_{01}, and the unique 2-element Priestley space with non-trivial order in the category PS. A prime filter p in a bounded distributive lattice D then corresponds to the lattice homomorphism $h_p : D \to \mathbf{2}$ for which $p = h_p^{-1}(1)$, and a clopen decreasing set U of a Priestley space $(X, \tau, \leq)$ corresponds to the continuous monotone function $\chi_U : X \to \mathbf{2}$ for which $U = \chi_U^{-1}(0)$.

Priestley duality can be generalized to a duality between the categories DL_0 and LPS, as follows. If D is a distributive lattice with 0, let $D^\top \in \mathsf{DL}_{01}$ be the lattice $D + \{\top\}$, where $\top$ is a (new) largest element. Notice that $* := \{\top\}$ is the maximum point in the order on $\mathcal{S}(D^\top)$, because $\{\top\}$ is the minimum prime filter of $D^\top$. Now let $\mathcal{S}(D)$ be the local Priestley space $(\mathcal{S}(D^\top) \setminus \{*\}, \tau, \leq)$; its ordered one-point-compactification is exactly $\mathcal{S}(D^\top)$. Conversely, if $(X, \tau, \leq)$ is a local Priestley space, let $\mathcal{L}(X, \tau, \leq)$ be the distributive lattice of clopen decreasing proper subsets of the one-point-compactification $(\widehat{X}, \widehat{\tau}, \widehat{\leq})$, or equivalently, compact open decreasing subsets of $(X, \tau, \leq)$. The assignments $\mathcal{S} : \mathsf{DL}_0 \to \mathsf{LPS}$ and $\mathcal{L} : \mathsf{LPS} \to \mathsf{DL}_0$ can be extended to functors. We then have the following corollary to Priestley duality:

Corollary 3.4. *The contravariant functors $\mathcal{S} : \mathsf{DL}_0 \to \mathsf{LPS}$ and $\mathcal{L} : \mathsf{LPS} \to \mathsf{DL}_0$ establish a dual equivalence between the categories DL_0 and LPS.*

Remark 3.5. The duality stated in this corollary is not a natural duality with respect to the schizophrenic object $\mathbf{2}$ as in Remark 3.3 above. However, it is still true that the set underlying $\mathcal{S}(D)$ is in a bijective correspondence with $\mathsf{DL}_0(D, \mathbf{2})$, for any $D \in \mathsf{DL}_0$. Under this correspondence, a basic open $\widehat{a}$ gets sent to the set $\{h \in \mathsf{DL}_0(D, \mathbf{2}) : h(a) = 1\}$.

3.4. **The category of sheaves over local Priestley spaces.** We refer the reader to the Appendix for our notation and some preliminaries on sheaf theory and étalé spaces.

If E is a sheaf on a topological space X and $f : X \to Y$ is a continuous map, recall that the direct image sheaf $f_* E$ on $\Omega(Y)$ is defined on objects by $(f_* E)(V) := E(f^{-1}(V))$.

We will denote by $\mathsf{Sh}(\mathsf{LPS})$ the category of *sheaves over local Priestley spaces*: an *object* is $(X, \tau, \leq, E)$, where $(X, \tau, \leq)$ is a local Priestley space, and E is a sheaf[3] on the topological space (X, τ).

[3]In this paper, we adopt the convention that the word 'sheaf' means 'sheaf with *global support*', i.e., for every $x \in X$ there exists an open neighbourhood U around x such that $E(U) \neq \emptyset$, or equivalently, the corresponding bundle is surjective.

A *morphism* from $(X, \tau, \leq, E)$ to $(Y, \tau, \leq, F)$ is a pair (f, λ), where the function $f : (X, \tau, \leq) \to (Y, \tau, \leq)$ is a morphism in LPS, and $\lambda : F \Rightarrow f_*E$ is a natural transformation, see the diagram in Figure 2. If $(f, \lambda) : (X, E) \to (Y, F)$ and $(g, \mu) : (Y, F) \to (Z, G)$ are morphisms in $\mathsf{Sh}(\mathsf{LPS})$, their composition is defined by (gf, σ), where $\sigma_U := \lambda_{g^{-1}(U)} \circ \mu_U$.

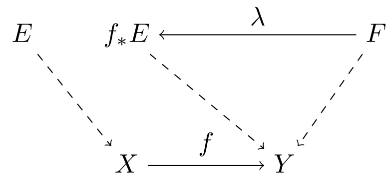


FIGURE 2. A morphism in the category $\mathsf{Sh}(\mathsf{LPS})$.

3.5. **Statement of the main theorem.** We are now ready to state our main theorem.

Theorem 3.6. *The category* SDL *of left-handed skew distributive lattices is dually equivalent to the category* $\mathsf{Sh}(\mathsf{LPS})$ *of sheaves over local Priestley spaces.*

The proof of this theorem will take up the rest of this paper, which is organized as follows. In Section 4, we will define a left-handed skew distributive lattice from a sheaf over a local Priestley space, and extend this assignment to a functor. We will then show how to retrieve the original sheaf from this in Section 5. This will lead to the right way to associate a sheaf over a local Priestley space to a left-handed skew distributive lattice in Section 6. In Section 7 we will put all of this together to prove Theorem 3.6.

4. From an étalé space to a skew lattice

In this section, let $X = (X, \tau, \leq)$ be a local Priestley space and let $p : E \twoheadrightarrow X$ be an étalé space over X. We denote the corresponding sheaf of local sections by E as well. From these data, we will now construct a left-handed skew distributive lattice S.

Let us denote by $L := \mathcal{L}(X)$ the distributive lattice of compact open downsets of X. We define the underlying set of S to be $\bigsqcup_{U \in L} E(U)$, that is, the set of all local sections over all compact open downsets of X. We now define operations $\vee$ and $\wedge$ on S that will make it into a left-handed skew distributive lattice.

Let $U, V \in L$ and $a \in E(U)$, $b \in E(V)$. We define the *override* $a \vee b$ to be the local section over $U \cup V$ given by

$$(a \vee b)(x) := \begin{cases} b(x), & \text{if } x \in V, \\ a(x), & \text{if } x \in U \setminus V. \end{cases} \tag{3}$$

Note that this indeed defines a continuous map $U \cup V \to E$, so $a \vee b \in E(U \cup V)$. Viewing E as a sheaf over X, note that $a \vee b$ is the patch of the compatible family consisting of the two elements $a|_{U \setminus V}$ and $b|_V$, that is, $a \vee b := a|_{U \setminus V} \vee b|_V$.

We define the *restriction* $a \wedge b$ to be the section in $E(U \cap V)$ given by

$$(a \wedge b)(x) := a(x) \text{ for all } x \in U \cap V. \tag{4}$$

Viewing E as a sheaf, $a \wedge b$ is simply the restriction $a|_{U \cap V}$.

Finally, we let the *zero element*, 0, be the unique element of $E(\emptyset)$.

In the following proposition, we collect some basic properties of the algebra S that we constructed here.

Proposition 4.1. *Let $p : E \twoheadrightarrow X$ be an étalé space over a local Priestley space, $L := \mathcal{L}(X)$ and $(S, \wedge, \vee, 0)$ the algebra on $S = \bigsqcup_{U \in L} E(U)$ defined in* (3) *and* (4)*. Then the following hold.*

(i) The algebra S is a left-handed skew distributive lattice.
(ii) The lattice reflection $S/\mathcal{D}$ of S is isomorphic to L.
(iii) The order on S is given by $a \leq b$ if and only if a is a restriction of b.

Proof. (i) It is known [14] and easy to check that the skew lattice $\mathcal{P}(X, E)$ of all partial maps from X to E is a left-handed skew distributive lattice. It is easy to verify that S is a subalgebra of $\mathcal{P}(X, E)$, and therefore it is also a left-handed skew distributive lattice.

(ii) Note that the relation $\mathcal{D}$ on S is given by $a \,\mathcal{D}\, b$ if and only if $\mathrm{dom}(a) = \mathrm{dom}(b)$ (recall that the notation $\mathrm{dom}(a)$ denotes the domain of the function a). Hence, $S/\mathcal{D}$ is indeed isomorphic to the lattice of domains, L.

(iii) By definition of $\leq$, we have $a \leq b$ if and only if $a \wedge b = a = b \wedge a$. The statement now follows from the definition of $\wedge$. □

Let us call the left-handed skew distributive lattice S the *dual algebra* of the étalé space $p : E \twoheadrightarrow X$. We will sometimes denote S by $E^\star$ or $(E, p, X)^\star$, to emphasize that it is constructed from the étalé space (E, p, X). We now use the above construction to define a contravariant functor from $\mathsf{Sh}(\mathsf{LPS})$ to SDL, which will be one of the equivalence functors of the duality in Theorem 3.6.

Let E and F be sheaves over local Priestley spaces $(X, \tau, \leq)$ and $(Y, \tau, \leq)$, respectively. The naturally associated étalé spaces $E \twoheadrightarrow X$ and $F \twoheadrightarrow Y$ yield dual algebras $E^\star$ and $F^\star$. Suppose (f, λ) is a morphism from E to F, as in Figure 2 in 3.4. We will define a skew lattice morphism $(f, \lambda)^\star : F^\star \to E^\star$.

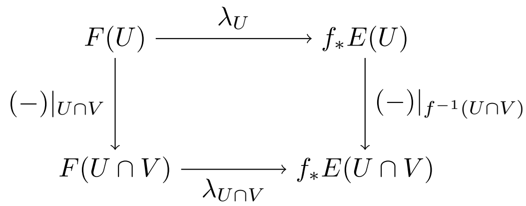
Let $a \in F^\star$, so $a \in F(U)$ for some $U \in \mathcal{L}(Y)$. By classical Priestley duality, we have $f^{-1}(U) \in \mathcal{L}(X)$. We now define

$$(f, \lambda)^\star(a) := \lambda_U(a),$$

which is an element of $f_* E(U) = E(f^{-1}(U)) \subseteq E^\star$.

Lemma 4.2. *The function $(f, \lambda)^\star : F^\star \to E^\star$ is a morphism in SDL for which $\overline{(f, \lambda)^\star} = \mathcal{L}(f)$.*

Proof. Let us write h for the function $(f, \lambda)^\star$. We show in detail that h preserves the operation $\wedge$, and leave it to the reader to verify that h preserves $\vee$ and 0, since the proofs are similar. Let $a \in F(U)$, $b \in F(V)$. By definition of $\wedge$, we have $h(a) \wedge h(b) = h(a)|_{f^{-1}(U) \cap f^{-1}(V)}$. By naturality of λ, the following diagram commutes:



In particular, we get

$$\begin{aligned} h(a \wedge b) = \lambda_{U \cap V}(a \wedge b) = \lambda_{U \cap V}(a|_{U \cap V}) &= \lambda_U(a)|_{f^{-1}(U \cap V)} \\ &= h(a)|_{f^{-1}(U) \cap f^{-1}(V)} = h(a) \wedge h(b). \end{aligned}$$

Further note that $\bar{h} : F^\star/\mathcal{D} \to E^\star/\mathcal{D}$ is exactly the proper homomorphism $\mathcal{L}(f) = f^{-1}$ dual to f in classical Priestley duality. Hence, h is a morphism in SDL. □

We can now conclude:

Proposition 4.3. *The assignments* $(E, p, X) \mapsto (E, p, X)^\star$ *and* $(f, \tau) \mapsto (f, \tau)^\star$ *define a contravariant functor* $(-)^\star$ *from* $\mathsf{Sh}(\mathsf{LPS})$ *to* SDL.

Proof. By Proposition 4.1(i) and Lemma 4.2, the assignments are well-defined. We leave functoriality to the reader. □

5. Reconstructing an étalé space from its dual algebra

In this section, we show how a sheaf E over a local Priestley space X can be reconstructed (up to homeomorphism) from its dual algebra $E^\star$, defined in the previous section. This will be the main motivation for the construction leading to the definition of a contravariant functor $(-)_\star : \mathsf{SDL} \to \mathsf{Sh}(\mathsf{LPS})$ in the next section.

In the remainder of this section, let E be a sheaf over a local Priestley space X, and let $p : E \twoheadrightarrow X$ be the étalé space associated to the sheaf. Let $E^\star$ be the dual algebra of E, and $L := E^\star/\mathcal{D}$ its lattice reflection.

5.1. **Reconstructing the base space.** We first note that we can reconstruct the base space X from the left-handed skew distributive lattice $E^\star$. By Proposition 4.1(i), L is isomorphic to $\mathcal{L}(X)$. Hence, X is homeomorphic to the space $\mathcal{S}(L)$, by classical Priestley duality. A point of $\mathcal{S}(L)$ can be concretely given by a morphism $L \to \mathbf{2}$ in DL_0, by Remark 3.5. By Theorem 2.1, the hom-set $\mathsf{DL}_0(L, \mathbf{2}) = \mathsf{DL}_0(E^\star/\mathcal{D}, \mathbf{2})$ is naturally isomorphic to the hom-set $\mathsf{SDL}(E^\star, \mathbf{2})$, because $\mathbf{2}$ is a lattice. In summary, we obtain

$$X \cong \mathcal{S}(L) \cong \mathsf{DL}_0(L, \mathbf{2}) \cong \mathsf{SDL}(E^\star, \mathbf{2}), \tag{5}$$

where the topology on $\mathsf{SDL}(E^\star, \mathbf{2})$ is given by taking as a basis the sets of the form $\{h \in \mathsf{SDL}(E^\star, \mathbf{2}) : h(a) = 1\}$ and their complements, where a ranges over $E^\star$.

5.2. **Reconstructing the stalks.** We will now reconstruct, for any $x \in X$, the stalk E_x above it. Fix $x \in X$. Let P_x be the primitive skew lattice whose non-zero $\mathcal{D}$-class is the set E_x. Then we have a natural evaluation homomorphism $\mathsf{ev}_x : E^\star \to P_x$, defined by

$$\mathsf{ev}_x(a) := \begin{cases} a(x) & \text{if } x \in \mathrm{dom}(a) \\ 0 & \text{otherwise.} \end{cases}$$

Note that the composition $\alpha \circ \mathsf{ev}_x : E^\star \to \mathbf{2}$ is exactly the map h_x naturally associated to x in (5): it sends $a \in E^\star$ to 1 iff $x \in \mathrm{dom}(a)$. We can now characterize the kernel of ev_x by an algebraic property which only refers to $\wedge$, $\vee$, 0 and the map h_x, as follows.

Lemma 5.1. *Let* $x \in X$*. For any* $a, b \in E^\star$*, the following are equivalent:*

(i) $\mathsf{ev}_x(a) = \mathsf{ev}_x(b)$*;*

(ii) There exist $c, d \in E^\star$ *such that* $h_x(c) = 0$*,* $h_x(d) = 1$*, and* $(a \wedge d) \vee c = (b \wedge d) \vee c$*.*

Proof. (i) ⇒ (ii). If $\mathsf{ev}_x(a) = 0$, then, since h_x is proper, pick some d with $h_x(d) = 1$. Put $c := a \vee b$. Note that $h_x(c) = 0$ since $x \notin \mathrm{dom}(c) = \mathrm{dom}(a) \cup \mathrm{dom}(b)$. Now $(a \wedge d) \vee c$ and $(b \wedge d) \vee c$ are both equal to $a \vee b$, as required.

If $\mathsf{ev}_x(a) \neq 0$, then $x \in \mathrm{dom}(a) \cap \mathrm{dom}(b)$. Since a and b are continuous sections, their equalizer $\|a = b\| = \{x \in \mathrm{dom}(a) \cap \mathrm{dom}(b) \mid a(x) = b(x)\}$ is open in X, and it contains x, so there exist compact open downsets U and V of X such that $x \in U \cap V^c \subseteq \|a = b\|$. Pick some $d \in E(U)$ and $c \in E(V)$. Then $x \notin \mathrm{dom}(c) = V$, so $h_x(c) = 0$, and $x \in \mathrm{dom}(d) = U$, so $h_x(d) = 1$. It is clear from the definitions of $\wedge$ and $\vee$ that $\mathrm{dom}((a \wedge d) \vee c) = U \cup V = \mathrm{dom}((b \wedge d) \vee c)$, and that the values of $(a \wedge d) \vee c$ and $(b \wedge d) \vee c$ are equal, since a and b are equal on $U \cap V^c$ by construction.

(ii) $\Rightarrow$ (i). Note that (ii) implies $h_x(a) = h_x(b)$, since h_x is a homomorphism. Hence, we have either $x \notin \text{dom}(a)$ and $x \notin \text{dom}(b)$, or $x \in \text{dom}(a)$ and $x \in \text{dom}(b)$. In the first case, (i) clearly holds and we are done. If $x \in \text{dom}(a) \cap \text{dom}(b)$, we show that $a(x) = b(x)$. Pick $c, d \in E^\star$ such that $h_x(c) = 0$, $h_x(d) = 1$ and $(a \wedge d) \vee c = (b \wedge d) \vee c$. Since $x \notin \text{dom}(c)$ and $x \in \text{dom}(d)$, we get from the definitions of $\wedge$ and $\vee$ that $((a \wedge d) \vee c)(x) = a(x)$ and $((b \wedge d) \vee c)(x) = b(x)$, so $a(x) = b(x)$, as required. □

Hence, given a point $x \in X$, we define a relation $\sim_x$ on $E^\star$ by

$$a \sim_x b \iff \exists c, d \in S : h_x(c) = 0, h_x(d) = 1, \text{ and } (a \wedge d) \vee c = (b \wedge d) \vee c,$$

and we immediately obtain:

Proposition 5.2. *Let $x \in X$. The relation $\sim_x$ is a skew lattice congruence on $E^\star$, and there is an isomorphism between $E^\star/{\sim_x}$ and P_x, which takes the quotient map $E^\star \twoheadrightarrow E^\star/{\sim_x}$ to the evaluation map $\mathsf{ev}_x : E^\star \twoheadrightarrow P_x$.*

Proof. The preceding lemma exactly shows that $\sim_x$ is the kernel of the morphism ev_x. The result now follows from the first isomorphism theorem of universal algebra. □

5.3. **Reconstructing the étalé space.** For a primitive skew lattice P, we denote by P^1 the unique non-zero $\mathcal{D}$-class of P, considered as a set.

Corollary 5.3. *The étalé space $p : E \twoheadrightarrow X$ is isomorphic to $q : (E^\star)_\star \twoheadrightarrow X$, where*

- *The set underlying the space $(E^\star)_\star$ is*
$$\bigsqcup_{x \in X} (E^\star/{\sim_x})^1 := \{(x, [a]_{\sim_x}) : x \in X, [a]_{\sim_x} \in (E^\star/\sim_x)^1\},$$
- *The function $q : (E^\star)_\star \twoheadrightarrow X$ sends an element of the disjoint union to its index $x \in X$,*
- *The topology on $(E^\star)_\star$ is given by taking as a basis of open sets the sets of the form*
$$\widehat{a} := \{(x, [a]_{\sim_x}) \mid x \in \text{dom}(a)\},$$
where a ranges over the elements of $E^\star$.

Proof. Define a map $\psi : E \to (E^\star)_\star$ by sending $e \in E_x$ to $(x, [a]_{\sim_x})$, where a is any local section for which $a(x) = e$; such a section exists because p is an étale map, and the value of $\psi(e)$ does not depend on the choice of a because of Lemma 5.1. By Proposition 5.2, ψ is a bijection. It is not hard to see from the definition of the topologies on E and $(E^\star)_\star$ that ψ is open and continuous. Hence, ψ is a homeomorphism, which clearly commutes with the étale maps. □

6. From a left-handed skew distributive lattice to an étalé space

In this section, we generalize the construction from the previous section to an *arbitrary* left-handed skew distributive lattice S. This is the main contribution of this paper, and it is the key to the proof that the functor $(-)^\star$ defined in Section 4 is part of a contravariant equivalence of categories.

Let S be a left-handed skew distributive lattice. We will define an étalé space $q : S_\star \to X$ over a local Priestley space.

6.1. **The base space X.** Recall from Proposition 2.2 that $S/\mathcal{D}$ is a distributive lattice with 0. By Remark 3.5 and Theorem 2.1, the set underlying the local Priestley space $\mathcal{S}(S/\mathcal{D})$ is in a bijection with the set $\mathsf{SDL}(S, \mathbf{2})$. A topology on $\mathsf{SDL}(S, \mathbf{2})$ is given by taking as a basis the sets of the form $\widehat{a} = \{h : S \to \mathbf{2} \mid h(a) = 1\}$ and their complements, where a ranges over S. With this topology, $\mathsf{SDL}(S, \mathbf{2})$ is homeomorphic to the local Priestley space $\mathcal{S}(S/\mathcal{D})$. We will denote this space by X, and we will define an étalé space over X.

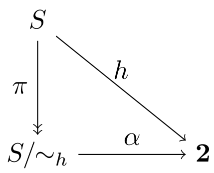
6.2. **A maximal primitive quotient.** Inspired by the results in the previous section, for $h \in \mathsf{SDL}(S, \mathbf{2})$, we define the relation $\sim_h$ as follows:

$$(6)\; a \sim_h b \iff \exists c, d \in S : h(c) = 0, h(d) = 1, \text{ and } (a \wedge d) \vee c = (b \wedge d) \vee c.$$

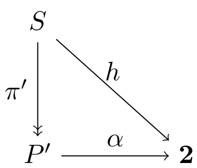
The following proposition is now the central technical result we need to construct $S_\star$.

Proposition 6.1. *Let S be a left-handed skew distributive lattice, and $h \in \mathsf{SDL}(S, \mathbf{2})$. The following properties hold:*

(i) The relation $\sim_h$ is a skew lattice congruence on S which refines $\ker(h)$.
(ii) The quotient skew lattice $S/{\sim_h}$ is primitive and the diagram



commutes.
(iii) For any commuting diagram in SDL *of the form*



where P' is primitive, there is a unique factorization $t : S/{\sim_h} \to P'$ such that $t \circ \pi = \pi'$.

Proof. (i) It is clear that $\sim_h$ is reflexive and symmetric. For transitivity, if $a \sim_h f \sim_h b$, pick $c, c', d, d' \in S$ are such that $h(c) = 0 = h(c')$, $h(d) = 1 = h(d')$, $(a \wedge d) \vee c = (f \wedge d) \vee c$, and $(b \wedge d') \vee c' = (f \wedge d') \vee c'$. Put $c'' := c \vee c'$ and $d'' := d \wedge d'$, then $h(c'') = 0$ and $h(d'') = 1$ since h is a homomorphism. One may now check that the elements $(a \wedge d'') \vee c''$ and $(b \wedge d'') \vee c''$ are in the same $\mathcal{D}$-class, and that both are below $f \vee c''$. Therefore, by Lemma 2.3(ii), $(a \wedge d'') \vee c'' = (b \wedge d'') \vee c''$, and we obtain $a \sim_h b$.

Suppose $a \sim_h a'$, and let $b \in S$. We first show that $a \vee b \sim_h a' \vee b$ and $b \vee a \sim_h b \vee a'$. Pick $c, d \in S$ such that $h(c) = 0$, $h(d) = 1$ and $(a \wedge d) \vee c = (a' \wedge d) \vee c$. We use distributivity and left-handedness to show that $((a \vee b) \wedge d) \vee c = ((a' \vee b) \wedge d) \vee c$:

$$\begin{aligned}((a \vee b) \wedge d) \vee c &= (a \wedge d) \vee (b \wedge d) \vee c && \text{(distributivity)}\\ &= (a \wedge d) \vee c \vee (b \wedge d) \vee c && \text{(left-handedness)}\\ &= (a' \wedge d) \vee c \vee (b \wedge d) \vee c && \text{(assumption)}\\ &= (a' \wedge d) \vee (b \wedge d) \vee c && \text{(left-handedness)}\\ &= ((a' \vee b) \wedge d) \vee c. && \text{(distributivity)}\end{aligned}$$

The proof that $((b \vee a) \wedge d) = ((b \vee a') \wedge d)$ is similar, but slightly simpler.

The proof that $\sim_h$ is also a congruence for the operation $\wedge$ on both sides proceeds along similar lines, using left normality (Lemma 2.3(i)), and is left for the reader to check.

To see that $\sim_h \subseteq \ker(h)$, suppose $a \sim_h b$ and pick $c, d \in S$ as in the definition of $\sim_h$. Then

$$h(a) = (h(a) \wedge h(d)) \vee h(c) = h((a \wedge d) \vee c) = h((b \wedge d) \vee c)) = h(b).$$

(ii) We will show that the $\mathcal{D}$-classes of the skew lattice $S/{\sim_h}$ are exactly $h^{-1}(0)$ and $h^{-1}(1)$, which is clearly enough for this item. Since h is proper, fix $a \in S$ such that $h(a) = 1$. We first claim that the $\mathcal{D}$-class of $[0]_{\sim_h}$ is $h^{-1}(0)$. If $b \sim_h 0$ then $h(b) = h(0) = 0$. Conversely, if $h(b) = 0$, one may prove that (6) holds by taking $c := b$ and $d := a$, concluding the proof of the claim. We will now show that the $\mathcal{D}$-class of $[a]_{\sim_h}$ is $h^{-1}(1)$. Suppose $b \in S$ is such that $h(b) = 1$. We claim that $[a]_{\sim_h} \mathcal{D} [b]_{\sim_h}$. By definition of $\mathcal{D}$, we need to show that $[a \wedge b]_{\sim_h} = [a]_{\sim_h}$ and $[b \wedge a]_{\sim_h} = [b]_{\sim_h}$. Both of these equalities hold indeed, because we can take $c := 0$ and $d := a \wedge b$ to prove that (6) holds.

(iii) Suppose that $\pi' : S \twoheadrightarrow P'$ is a primitive quotient of S such that $\alpha \circ \pi' = h$. If $t : S/{\sim_h} \to P'$ is a factorization such that $t \circ \pi = \pi'$, then for any $a \in S$ we must have $t([a]_{\sim_h}) = \pi'(a)$, proving that t is unique if it exists.

We now show that the assignment $[a]_{\sim_h} \mapsto \pi'(a)$ does not depend on the choice of representative for the class $[a]_{\sim_h}$. Suppose $a \sim_h a'$. If $h(a) = 0 = h(a')$, then $[\pi'(a)]_{\mathcal{D}} = h(a) = 0$ so $\pi'(a) = 0$ since the $\mathcal{D}$-class of 0 only contains 0 itself, and similarly $\pi'(a') = 0$. Otherwise, we have $h(a) = 1 = h(a')$. Pick $c, d \in S$ such that $h(c) = 0$, $h(d) = 1$ and $(a \wedge d) \vee c = (a' \wedge d) \vee c$. As before, since $h(c) = 0$, we have $\pi'(c) = 0$. Since P' is primitive, we have, for any non-zero $x, y \in P'$, that $x \wedge y = x$. Hence

$$\pi'(a) = \pi'(a) \wedge \pi'(d) = (\pi'(a) \wedge \pi'(d)) \vee \pi'(c) = \pi'((a \wedge d) \vee c),$$

and similarly $\pi'(a') = \pi'((a' \wedge d) \vee c)$. So $\pi'(a) = \pi'(a')$, since $(a \wedge d) \vee c = (a' \wedge d) \vee c$. □

Remark 6.2. In the light of this proposition, more can be said about the structure of primitive quotients of a left-handed skew distributive lattice S. We may put a partial order on quotients of S by saying a quotient $q : S \to Q$ is *below* another quotient $q' : S \to Q'$ if the map q factors through q'. Suppose $p : S \to P$ is any primitive quotient of S. Then $h := \alpha \circ p : S \to \mathbf{2}$ is a minimal quotient of S below the primitive quotient P, and $S/{\sim_h}$ is a maximal primitive quotient of S which is above P. The partially ordered set of primitive quotients of S is thus partitioned, and each primitive quotient lies between a unique maximal and minimal primitive quotient of S. The minimal primitive quotients of S are exactly the elements of the base space X, and the non-zero elements of the maximal primitive quotients will be the elements of the étalé space S_*, see below.

Remark 6.3. An alternative way to define the equivalence relation $\sim_h$ on S is the following. Let us call a subset F of S a *preprime filter over* h if it satisfies the following properties:

(i) If $a \in F$, $b \in S$ and $a \leq b$, then $b \in F$;
(ii) If $a, b \in F$ then $a \wedge b \in F$;
(iii) If $a \in F$, $b \in S$ and $h(b) = 0$, then $a \vee b \in F$;
(iv) If $a \in F$, then $h(a) = 1$;
(v) If $b \in S$ and $h(b) = 1$, then there is $a \in F$ such that $[a]_{\mathcal{D}} = [b]_{\mathcal{D}}$.

We call a preprime filter over h a *prime filter over* h if it is minimal among the preprime filters over h. One may then show that the non-zero equivalence classes in $S/{\sim_h}$ (viewed as subsets of S) are exactly the prime filters over h. Therefore, the equivalence relation $\sim_h$ can also be described as the equivalence relation inducing the partition whose classes are the prime filters over h, and $h^{-1}(0)$.

6.3. **The étalé space.** We are now ready to define the étalé space $S_\star$. The stalk over $h \in X$ will be the non-zero $\mathcal{D}$-class of $S/{\sim_h}$, or, equivalently, the set of prime filters over h, as defined in Remark 6.3. Put more formally, the underlying set of the étalé space $S_\star$ is

$$S_\star := \bigsqcup_{h \in X} (S/{\sim_h})^1 = \{(h, [a]_{\sim_h}) \mid h \in X,\ h(a) = 1\}.$$

The function $q : S_\star \to X$ is given by $q((h, [a]_{\sim_h}) := h$. For any $a \in S$, we define a function $s_a : \widehat{a} \to S_\star$ by $s_a(h) := (h, [a]_{\sim_h})$. We now define the topology on $S_\star$ by taking the sets $\mathrm{im}(s_a)$ as a subbasis for the open sets, where a ranges over S.

Lemma 6.4. *Each function $s_a : \widehat{a} \to S_\star$ is continuous and $q : S_\star \to X$ is an étale map.*

Proof. Let $a, b \in S$ be arbitrary. We need to show that the set $s_a^{-1}(\mathrm{im}(s_b))$ is open in X. Notice that

$$s_a^{-1}(\mathrm{im}(s_b)) = \{h \in X \mid a \sim_h b\} \cap \widehat{a}.$$

Suppose $a \sim_h b$ and $h \in \widehat{a}$. Then also $h(b) = 1$. Pick $c, d \in S$ such that $h(c) = 0$, $h(d) = 1$ and $(a \wedge d) \vee c = (b \wedge d) \vee c$. Let $U_h := (\widehat{c})^c \cap \widehat{d} \cap \widehat{a} \cap \widehat{b}$. Then, for any $h' \in U_h$, we have $h'(a) = 1 = h'(b)$, $h'(c) = 0$ and $h'(d) = 1$, so that $a \sim_{h'} b$. So $h \in U_h \subseteq s_a^{-1}(\mathrm{im}(s_b))$.

To prove that q is an étale map, let $e = (h, [a]_{\sim_h}) \in S_\star$. Then $q|_{\mathrm{im}(s_a)} : \mathrm{im}(s_a) \to \widehat{a}$ has s_a as its continuous inverse. □

7. Proof of the main theorem

In this section, we will prove that the contravariant functor $(-)^\star : \mathsf{Sh}(\mathsf{LPS}) \to \mathsf{SDL}$ is essentially surjective, full and faithful. By a basic result from category theory, it then follows that $(-)^\star$ is part of a dual equivalence of categories, proving Theorem 3.6.

The proof that $(-)^\star$ is full and faithful is reasonably straightforward.

Proposition 7.1. *The contravariant functor $(-)^\star$ is full and faithful.*

Proof. Let E and F be sheaves over local Priestley spaces X and Y, respectively. We show that the assignment which sends a morphism $(f, \lambda) : (X, E) \to (Y, F)$ to $(f, \lambda)^\star : (Y, F)^\star \to (X, E)^\star$ is a bijection between the sets $\mathrm{Hom}_{\mathsf{Sh}(\mathsf{LPS})}((X, E), (Y, F))$ and $\mathrm{Hom}_{\mathsf{SDL}}((Y, F)^\star, (X, E)^\star)$.

If $(f, \lambda)^\star = (g, \mu)^\star$ then in particular $\mathcal{L}(f) = \overline{(f, \lambda)^\star} = \overline{(g, \mu)^\star} = \mathcal{L}(g)$, using Lemma 4.2. Therefore, by classical Priestley duality, $f = g$. Moreover, if U is a basic open set in Y, then $\lambda_U = \mu_U$, using the definition of $(f, \lambda)^\star = (g, \mu)^\star$. Since a natural transformation between sheaves is entirely determined by its action on basic opens (cf. Lemma A.1), it follows that $\lambda = \mu$. So $(f, \lambda) = (g, \mu)$, proving that $(-)^\star$ is faithful.

If $h : (Y, F)^\star \to (X, E)^\star$ is a homomorphism of skew lattices, then $\overline{h}$ is a proper homomorphism, so by classical Priestley duality, there is a unique $f : X \to Y$ such that $\overline{h} = \mathcal{L}(f) = f^{-1}$. For U a basic open, define $\lambda_U : F(U) \to E(f^{-1}(U))$ by sending $s \in F(U)$ to $h(s)$, which is indeed an element of $E(\overline{h}(s)) = E(f^{-1}(U))$. Now, if U is an arbitrary open and $s \in F(U)$, we can write U as a union of basic open sets $(U_i)_{i\in I}$. Then also $f^{-1}(U)$ is the union of the basic open sets $(f^{-1}(U_i))_{i\in I}$. It follows from the fact that h is a homomorphism that $(h(s)|_{f^{-1}(U_i)})_{i\in I}$ is a compatible family, so there is a unique patch in $E(f^{-1}(U))$, which we define to be $\lambda_U(s)$. We leave it to the reader to check that λ is a natural transformation and that $(f, \lambda)^\star = h$. □

The proof that $(-)^\star$ is essentially surjective is more involved, and will take up the rest of this section.

Let S be a left-handed skew distributive lattice. By the construction from Section 6, we have a sheaf $S_\star$ over the local Priestley space $X = \mathsf{SDL}(S, \mathbf{2})$. Then $(S_\star)^\star$ is the skew lattice of local sections of $S_\star$ with compact open downward closed domains. We will show in the following three propositions that the map ϕ, which sends $a \in S$ to $s_a \in (S_\star)^\star$ (cf. Lemma 6.4), is an isomorphism of skew lattices.

Proposition 7.2. *The function $\phi : S \to (S_\star)^\star$ is a homomorphism of skew lattices.*

Proof. It is clear that ϕ preserves 0. Let $a, b \in S$. We need to show that $s_{a\vee b} = s_a \vee s_b$ and $s_{a\wedge b} = s_a \wedge s_b$. Note that in these equations, the operations $\vee$ and $\wedge$ on the right hand side are the operations defined in (3) and (4) of Section 4, whereas the operations $\vee$ and $\wedge$ on the left hand side are the operations of the given left-handed skew distributive lattice S.

Note that the domain of $s_{a\vee b}$ is $\widehat{a \vee b} = \widehat{a} \cup \widehat{b}$, which is also the domain of $s_a \vee s_b$. We now claim that $s_{a\vee b}(x) = s_b(x)$ for $x \in \widehat{b}$ and $s_{a\vee b}(x) = s_a(x)$ for $x \in \widehat{a} \setminus \widehat{b}$, agreeing with the definition of $s_a \vee s_b$.

- Let $x \in \widehat{b}$. For $d := b$ and $c := 0$, it is easy to show that $((a \vee b) \wedge d) \vee c = (b \wedge d) \vee c$, so $[a \vee b]_{\sim_x} = [b]_{\sim_x}$, by definition of $\sim_x$.
- Let $x \in \widehat{a} \setminus \widehat{b}$. For $d := a$ and $c := b$, we then have $((a\vee b)\wedge d)\vee c = (a\wedge d)\vee c$, so that $[a \vee b]_{\sim_x} = [a]_{\sim_x}$.

Similarly, the domain of $s_{a\wedge b}$ is equal to the domain of $s_a \wedge s_b$, and if x is an element of this domain, then we have $((a \wedge b) \wedge d) \vee c = (a \wedge d) \vee c$, for $d := a \wedge b$ and $c := 0$, proving that $[a \wedge b]_{\sim_x} = [a]_{\sim_x}$. □

To establish surjectivity of ϕ, we will need the following lemma.

Lemma 7.3. *For each $n \in \mathbb{N}$, the following holds.*

If $s : U \to S_\star$ is a section on a compact open decreasing subset U of X, and if $a_1, \dots, a_n$, $c_1, \dots, c_n$, $d_1, \dots, d_n$ are elements of S such that

(i) for each $i \in \{1, \dots, n\}$, $\widehat{c_i} \subseteq \widehat{d_i}$;
(ii) $U = \bigcup_{i=1}^n (\widehat{d_i} \cap \widehat{c_i}^c)$;
(iii) for each $i \in \{1, \dots, n\}$, $\widehat{d_i} \cap \widehat{c_i}^c \subseteq \widehat{a_i}$, and $s|_{\widehat{d_i}\cap\widehat{c_i}^c} = s_{a_i}|_{\widehat{d_i}\cap\widehat{c_i}^c}$,

then there exists an $a \in S$ such that $s = s_a$.

Proof. By a natural induction on $n \in \mathbb{N}$. For $n = 0$, it follows from assumption (ii) that $U = \emptyset$, so s is the empty function, and the (unique) element of S such that $s = s_a$ is $a = 0$.

Now let $n \geq 1$, and assume the statement is true for $n-1$. Suppose that $s : U \to S_\star$, $a_1, \ldots, a_n$, $c_1, \ldots, c_n$, and $d_1, \ldots, d_n$ satisfy the assumptions (i)–(iii).

Let $j \in \{1, \ldots, n\}$ be arbitrary. We are going to apply the induction hypothesis to the function $s|_{\widehat{c_j}} : \widehat{c_j} \to S_\star$. For $i \in \{1, \ldots, n\}$ with $i \neq j$, define $c_{i,j} := c_i \wedge c_j$ and $d_{i,j} := d_i \wedge c_j$. Note that

$$\widehat{d_{i,j}} \cap \widehat{c_{i,j}}^c = (\widehat{d_i} \cap \widehat{c_i}^c) \cap \widehat{c_j},$$

so that $\widehat{c_j} = U \cap \widehat{c_j} = \bigcup_{i \neq j} (\widehat{d_{i,j}} \cap \widehat{c_{i,j}}^c)$, and assumptions (i) and (iii) also clearly hold for the elements $a_i, c_{i,j}, d_{i,j}$, where i ranges over $\{1, \ldots, n\} \setminus \{j\}$. By the induction hypothesis, there exists an $f_j \in S$ such that $s|_{\widehat{c_j}} = s_{f_j}$.

Since j was arbitrary, we get that for each j, there exists an $f_j \in S$ such that $s|_{\widehat{c_j}} = s_{f_j}$. Now consider

$$a := \bigvee_{j=1}^{n} ((a_j \wedge d_j) \vee f_j).$$

We claim that $s = s_a$. Note first that

$$\mathrm{dom}(s_a) = \widehat{a} = \bigcup_{j=1}^{n} ((\widehat{a_j} \cap \widehat{d_j}) \cup \widehat{f_j}) = \bigcup_{j=1}^{n} \widehat{d_j} = U.$$

Now let $x \in U$ be arbitrary, and let j be the largest number in $\{1, \ldots, n\}$ such that $x \in \widehat{d_j}$. Then, using Proposition 7.2 and the definition of $\vee$ in $(S_\star)^\star$, we see that

$$s_a(x) = \begin{cases} s_{f_j}(x) & \text{if } x \in \widehat{c_j}, \\ s_{a_j}(x) & \text{if } x \notin \widehat{c_j}, \end{cases}$$

If $x \in \widehat{c_j}$, then $s_{f_j}(x) = s(x)$ by the choice of f_j, and if $x \notin \widehat{c_j}$, then $x \in \widehat{d_j} \cap \widehat{c_j}^c$, so $s_{a_j}(x) = s(x)$ by assumption (iii). □

The above lemma exactly enables us to prove surjectivity of ϕ: it is now an application of compactness, as follows.

Proposition 7.4. *The function $\phi : S \to (S_\star)^\star$ is surjective, and in particular it is a morphism of* SDL.

Proof. Let $s \in (S_\star)^\star$, so s is a continuous section over a compact open decreasing set U. For each $x \in U$, we have $s(x) \in (S_\star)_x = (S/{\sim_x})^1$, so we can pick $a_x \in S$ such that $s(x) = [a_x]_{\sim_x}$, and define

$$T_x := \|s = s_{a_x}\| = \{y \in U \mid s(y) = [a_x]_{\sim_y}\} = s^{-1}(\mathrm{im}(s_{a_x})) \cap U.$$

Now T_x is open in X, because s is continuous, $\mathrm{im}(s_{a_x})$ is open in $S_\star$, and U is open in X. Since $x \in T_x$, there exist $c_x, d_x \in S$ such that $x \in \widehat{d_x} \cap \widehat{c_x}^c \subseteq T_x$. We now have

$$U \subseteq \bigcup_{x \in U} (\widehat{d_x} \cap \widehat{c_x}^c) \subseteq \bigcup_{x \in U} T_x \subseteq U,$$

so equality holds throughout. Since U is compact, there exist indices $x_1, \ldots, x_n \in U$ such that $U = \bigcup_{i=1}^{n} (\widehat{d_{x_i}} \cap \widehat{c_{x_i}}^c)$. We will write c_i and d_i for c_{x_i} and d_{x_i}, respectively. Note that, for each $i \in \{1, \ldots, n\}$, we have $\widehat{d_i} \cap \widehat{c_i}^c \subseteq T_{x_i}$, so $s|_{\widehat{d_i} \cap \widehat{c_i}^c} = s_{a_{x_i}}|_{\widehat{d_i} \cap \widehat{c_i}^c}$. By Lemma 7.3, we get $a \in S$ such that $s = s_a$, proving that ϕ is surjective. For the in particular part, note that surjective homomorphisms are always proper. □

By a similar method, one may prove that ϕ is injective. Again, a lemma which is proved by natural induction is crucial.

Lemma 7.5. *For each $n \in \mathbb{N}$, the following holds.*

If a, b, $c_1, \ldots, c_n$ and $d_1, \ldots d_n$ are elements of S such that:

(i) *for each* $i \in \{1, \dots, n\}$, $\widehat{c_i} \subseteq \widehat{d_i} \subseteq \widehat{a}$;
(ii) $\widehat{a} = \bigcup_{i=1}^n (\widehat{d_i} \cap \widehat{c_i}^c) = \widehat{b}$;
(iii) *for each* $i \in \{1, \dots, n\}$, $(a \wedge d_i) \vee c_i = (b \wedge d_i) \vee c_i$,

then $a = b$.

Proof. For $n = 0$, we get that $\widehat{a} = \emptyset = \widehat{b}$, so $a = 0 = b$.

Let $n \geq 1$, and suppose the statement is proved for $n-1$. Let $c_1, \dots, c_n, d_1, \dots d_n$ be elements of S satisfying the assumptions. Then in particular $\widehat{a} = \bigcup_{i=1}^n \widehat{d_i} = \widehat{\bigvee_{i=1}^n d_i}$, so that $[a]_{\mathcal{D}} = [\bigvee_{i=1}^n d_i]_{\mathcal{D}}$. Therefore,

$$a = a \wedge \left(\bigvee_{i=1}^n d_i \right) = \bigvee_{i=1}^n (a \wedge d_i).$$

Similarly, since $\widehat{b} = \bigcup_{i=1}^n \widehat{d_i}$, we get that $b = \bigvee_{i=1}^n (b \wedge d_i)$.

Let $j \in \{1, \dots, n\}$ be arbitrary. For $i \neq j$, define $a_j := a \wedge c_j$, $b_j := b \wedge c_j$, $d_{i,j} := d_i \wedge c_j$, and $c_{i,j} := c_i \wedge c_j$. Note that $\widehat{a_j} = \widehat{b_j}$, and also that for each $i \neq j$, we have $\widehat{c_{i,j}} \subseteq \widehat{d_{i,j}} \subseteq \widehat{a_j}$. Moreover:

$$\begin{aligned}
(a_j \wedge d_{i,j}) \vee c_{i,j} &= (a \wedge c_j \wedge d_i \wedge c_j) \vee (c_i \wedge c_j) && \text{(definitions of } a_j,\ d_{i,j} \text{ and } c_{i,j}))\\
&= (a \wedge d_i \wedge c_j) \vee (c_i \wedge c_j) && \text{(left normality)}\\
&= ((a \wedge d_i) \vee c_i) \wedge c_j && \text{(distributivity)}\\
&= ((b \wedge d_i) \vee c_i) \wedge c_j && \text{(assumption)}\\
&= (b \wedge d_i \wedge c_j) \vee (c_i \wedge c_j) && \text{(as above)}\\
&= (b_j \wedge d_{i,j}) \vee c_{i,j}.
\end{aligned}$$

By the induction hypothesis, we thus conclude that $a \wedge c_j = a_j = b_j = b \wedge c_j$. Now, to show $a \wedge d_j = b \wedge d_j$, we calculate:

$$\begin{aligned}
a \wedge d_j &= a \wedge (d_j \vee c_j) && (\widehat{c_j} \subseteq \widehat{d_j})\\
&= (a \wedge d_j) \vee (a \wedge c_j) && \text{(distributivity)}\\
&= (a \wedge d_j) \vee (b \wedge c_j) && (a \wedge c_j = b \wedge c_j)\\
&= (a \wedge d_j) \vee c_j \vee (b \wedge c_j) && (\widehat{c_j} \subseteq \widehat{b})\\
&= (b \wedge d_j) \vee c_j \vee (b \wedge c_j) && \text{(assumption)}\\
&= (b \wedge d_j) \vee (b \wedge c_j) && (\widehat{c_j} \subseteq \widehat{b})\\
&= b \wedge d_j. && \text{(as above)}
\end{aligned}$$

Now $a = \bigvee_{j=1}^n (a \wedge d_j) = \bigvee_{j=1}^n (b \wedge d_j) = b$, as required. □

Proposition 7.6. *The function* $\phi : S \to (S_\star)^\star$ *is injective.*

Proof. Let $a, b \in S$, and suppose that $s_a = s_b$. Then in particular $\widehat{a} = \text{dom}(s_a) = \text{dom}(s_b) = \widehat{b}$. For each $x \in \widehat{a}$, we have $[a]_{\sim_x} = s_a(x) = s_b(x) = [b]_{\sim_x}$, so by definition of $\sim_x$, we may pick $c_x, d_x \in S$ such that $(a \wedge d_x) \vee c_x = (b \wedge d_x) \vee c_x$, and $x \in \widehat{d_x} \cap \widehat{c_x}^c$. We thus get that the collection $(\widehat{d_x} \cap \widehat{c_x}^c)_{x \in \widehat{a}}$ is an open cover of $\widehat{a}$. Since $\widehat{a}$ is compact, we can pick a finite subcover, indexed by $x_1, \dots, x_n \in \widehat{a}$. We will write c_i and d_i for c_{x_i} and d_{x_i}, respectively. Without loss of generality, we may assume that $\widehat{c_i} \subseteq \widehat{d_i} \subseteq \widehat{a}$ for each i, by replacing c_i by $c_i \wedge d_i \wedge a$ and d_i by $d_i \wedge a$, and checking that the new c_i and d_i still satisfy the same properties. Now it follows from Lemma 7.5 that $a = b$. □

We have thus established that $\phi : S \to (S_\star)^\star$ is an isomorphism in SDL, so:

Proposition 7.7. *The contravariant functor* $(-)^\star : \mathsf{Sh}(\mathsf{LPS}) \to \mathsf{SDL}$ *is essentially surjective.*

It now follows from Propositions 7.1 and 7.7 that $(-)^\star$ is part of a dual equivalence. This concludes the proof of our main theorem, Theorem 3.6.

Appendix A. Sheaves and étalé spaces

Preliminaries on sheaves and étalé spaces can be found in any textbook on sheaf theory, e.g. in [3, 16]. We will recall the basics and notation that we will use here.

A.1. **Sheaves.** Let X be a topological space. We denote by $\Omega(X)$ the poset of open subsets of X, ordered by inclusion. In particular, $\Omega(X)$ is a category. A *presheaf* on X is a contravariant functor E from $\Omega(X)$ to the category Set of non-empty sets.[4] If the presheaf E is clear from the context, and $U, V \in \Omega(X)$ with $U \subseteq V$, then we write $(-)|_U : E(V) \to E(U)$ for the morphism $E(U \subseteq V)$, and call it the *restriction map from* V *to* U.

If $(U_i)_{i\in I}$ is a cover of an open set U, then we say a family of elements $(s_i)_{i\in I}$, where $s_i \in E(U_i)$ for each $i \in I$, is *compatible* if for all $i, j \in I$, $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$. A presheaf E on X is called a *sheaf* if for any such compatible family there exists a unique $s \in E(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$. We will also denote this unique element s by $\bigvee_{i\in I} s_i$, and call it the *patch* of the family $(s_i)_{i\in I}$.

If E is a sheaf on a topological space X and $f : X \to Y$ is a continuous map, we define the functor f_*E on $\Omega(Y)$ on objects by $(f_*E)(V) := E(f^{-1}(V))$, and we call f_*E the *direct image sheaf* of E under f. It is a well known fact in sheaf theory that f_*E is indeed a sheaf [16, Ch. II, §1].

In this paper, a *morphism* from a sheaf E on X to a sheaf F on Y is a pair (f, λ), where $f : X \to Y$ is a morphism of the base spaces, and $\lambda : F \Rightarrow f_*E$ is a natural transformation. In the proof of Proposition 7.1, we use the following lemma.

Lemma A.1. *Suppose* (f, λ) *and* (f, λ') *are morphisms from a sheaf* E *on* X *to a sheaf* F *on* Y*, and suppose that* $\mathcal{B}$ *is a basis for the space* Y*. If, for all* $V \in \mathcal{B}$*, we have* $\lambda_V = \lambda'_V$*, then* $\lambda = \lambda'$*.*

A.2. **Étalé spaces.** Let X be a topological space. An *étalé space over* X is a topological space E together with a surjective *local homeomorphism* $p : E \twoheadrightarrow X$, that is, for any $e \in E$, there exists an open neighbourhood V of e such that $p(V)$ is open in X and $p|_V : V \to p(V)$ is a homeomorphism. If U is an open subset of X, a *(local) section over* U is a continuous map $s : U \to E$ such that $p \circ s = \mathrm{id}_U$. We denote by $E(U)$ the set of sections over U. The equivalence classes induced by p are called *stalks* or *fibers*: for $x \in X$, we denote the stalk $p^{-1}(\{x\})$ by E_x.

A.3. **Correspondence between sheaves and étalé spaces.** We now sketch the basic correspondence between sheaves and étalé spaces. See [16, Ch. II, §5] for more details.

If $p : E \twoheadrightarrow X$ is an étalé space, then the assignment $U \mapsto E(U)$, the local sections over U, naturally extends to a sheaf on X: if $U \subseteq V$, then we have the map $E(V) \to E(U)$ which sends a local section s over V to its restriction $s|_U$ over U.

If F is a sheaf on X, then for any $x \in X$ we define the *stalk* F_x to be the colimit of $F(U)$, where U ranges over the open neighbourhoods of x. More explicitly,

$$F_x = \left(\bigsqcup_{x\in U} F(U) \right) / \sim_x,$$

[4]Note that all presheaves and sheaves in this paper are assumed to have global support.

where, for $s \in F(U)$ and $t \in F(V)$, we have $s \sim_x t$ iff there exists an open neighbourhood W of x such that $W \subseteq U \cap V$ and $s|_W = t|_W$. The classes in F_x are called *germs* and denoted by $\mathrm{germ}_x s$. The étalé space associated to F has $\bigsqcup_{x \in X} F_x$ as its underlying set. Any $s \in F(U)$ yields a function $\widehat{s} : U \to \bigsqcup_{x \in X} F_x$ by sending $x \in U$ to $\mathrm{germ}_x s$. The topology on $\bigsqcup_{x \in X} F_x$ is defined by taking the sets $\widehat{s}(U)$ as a basis, where U ranges over $\Omega(X)$ and s ranges over $F(U)$. One may now prove that these assignments are well-defined and mutually inverse up to isomorphism, as in [16, Corollary II.5.3].

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A.B.: University of Ljubljana, Faculty of Mathematics and Physics, Jadranska 19, SI-1001, Ljubljana, Slovenia.
E-mail address: andrej.bauer@andrej.com

K.C.-V.: University of Ljubljana, Faculty of Mathematics and Physics, Jadranska 19, SI-1001, Ljubljana, Slovenia.
E-mail address: karin.cvetko@fmf.uni-lj.si

M.G. (corresponding author): LIAFA, CNRS and Université Paris Diderot – Paris 7, Case 7014, F-75205 Paris Cedex 13, +33 1 5727 9404, France.
E-mail address: mgehrke@liafa.univ-paris-diderot.fr

S.V.G.: LIAFA, Universite Paris Diderot – Paris 7 and IMAPP, Radboud University Nijmegen, P.O. Box 9010 6500 GL Nijmegen, The Netherlands.
E-mail address: samvangool@me.com

G.K.: University of Ljubljana, Faculty of Computer and Information Science, Tržaška cesta 25, SI-1001, Ljubljana, Slovenia.
E-mail address: ganna.kudryavtseva@fri.uni-lj.si