Generalised Kripke semantics for the Lambek–Grishin calculus

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1 Introduction

Kripke semantics plays a fundamental role for modal logics where it often provides modular semantics allowing a clear treatment of various logics related by the addition of axioms or connectives. Kripke semantics is also the "natural semantics" for classical modal logic in the sense that it is given by the representation theory for modal algebras. The algebraic structure corresponding to a traditional Kripke model is based on the *powerset* of a set of worlds. Powerset algebras are very special Boolean algebras, but they are general enough that every Boolean algebra can be embedded in one.

Kripke semantics have also been applied in the setting of other logics, including intuitionistic and substructural logics. However, Heyting algebras that can be embedded in powerset Boolean algebras are already themselves Boolean algebras and thus the model class must be broadened. This is done for intuitionistic logic by considering sets of worlds enriched with an order. The corresponding algebraic structures are *up-set lattices*, which form a rich enough class of Heyting algebras to allow the embedding of arbitrary Heyting algebras. In the setting of substructural logics, there may be no lattice operations or very weak ones that are not necessarily distributive. Based on the representation theory of such posets and lattices, the papers [DGP05, Geh06] suggest a natural generalisation of Kripke semantics to an analogous kind of semantics (generalised Kripke semantics) for a broader setting including that of substructural logics. This generalisation allows us to build models on a more general algebraic structure, viz. on a kind of lattice generalising powersets and up-set lattices.

The generalised Kripke semantics of [Geh06] provides the possibility of a modular treatment of various substructural logics obtained by axiomatic extension or enrichment of the connective type. This treatment is based on canonicity and correspondence as in the classical modal logic setting. A modular treatment for the basic hierarchy of substructural logics based just on the implication-fusion fragment was given in the paper [DGP05] where the generalised Kripke semantics was first introduced. This semantics and the issuing method of canonicity and correspondence has also been applied to analyse negations in the substructural setting in the paper [Alm09]. The paper [CGv11] considers generalised Kripke semantics for full linear logic and derives in an algorithmic and modular fashion complete semantics based on a class of models generalising but closely related to the phase space models of Girard [Gir95]. Since generalised Kripke semantics are the ones corresponding to the topological representation theory for lattices, they are closely related to the semantics in the book by J. Michael Dunn and Gary Hardegree [DH01] and in [BD09, AD93]. However, the approach we take is a closer parallel to the classical canonicity and correspondence treatment as in [BdV02].

In this paper we will consider a kind of substructural logic (the so-called Lambek-Grishin calculus) that was first mentioned in [Gri83] and discussed in more detail in [Moo09]. This calculus extends the basic implication-fusion fragment (the non-associative Lambek calculus) in the following way: together with the fusion residuated family of connectives, there is a second residuated family. In addition to this, some interaction axioms between these two families of connectives are introduced, which are identified as pertinent to linguistic phenomena in [Moo09]. Traditional Kripke semantics for the Lambek-Grishin calculus was discussed in [Che07] and [KM10]. However, as soon as additional axioms or additional connectives are present, one must start over to obtain semantics for such richer logics. For example, in order to obtain semantics for the Lambek-Grishin calculus with interaction axioms in the presence of lattice operations, semantics based on phase spaces are announced in [Bas10]. This is hard to compare to the semantics given in [KM10].

Here, we consider semantics based on the generalised Kripke frames naturally associated with the algebraic semantics of the logics in question via their representation theory. The two residuated families of the Lambek-Grishin calculus have order dual properties, so the symmetric set-up required by the general lattice setting lends itself particularly well to this. In addition, the interaction axioms provide an interesting case study for exploring the expressive power of these semantics compared to the traditional ones.

The main advantage of this approach is that each axiom and each connective modularly slots in as an additional first order property or additional relational component. This allows a clear comparison of various logics and a fully modular family of completeness results. Accordingly, in addition to the basic Lambek-Grishin operations and axioms, we consider all four families of interaction axioms as well as associativity, commutativity, weakening and contraction, and additional connectives such as lattice operations and linear logic-type negation. The presence of a linear logic-type negation tying together the two families of operations in the Lambek-Grishin calculus yields that all the possible interaction axioms are equivalent. In particular, the interaction axioms considered in [Moo09] are in this setting equivalent to associativity of both the product and the plus connective. A quick overview of our results may be found in Section 6.

Our approach to completeness for these various logics is parallel to the approach taken in [DGP05]. Thus, we prove canonicity and Sahlqvist-style correspondence, thereby providing complete relational semantics. Several of the equivalent versions of the interaction axioms in each group may be seen to be Sahlqvist in an appropriate generalised sense, and thus our derivations produce first-order conditions on the corresponding frames. Remarkably, each such frame condition turns out to be equivalent to a version of the initial axiom itself that is restricted to just the completely join resp. meet irreducible elements of the complex algebra.

The paper is structured as follows: in Section 2 we briefly repeat the theory of generalised Kripke models. In Section 3 we apply this to capture the basis of the Lambek-Grishin calculus. In Section 4 we define canonical extensions of Lambek-Grishin algebras and sketch the approach to completeness results via these canonical extensions. This is applied in Section 5, where we explore the influence of interaction axioms and give our canonicity and correspondence results for the extended Lambek-Grishin calculus. Finally, in Section 6 we give our results on complete relational semantics for the Lambek-Grishin calculus extended with different additional axioms that can be handled in a modular way, and we consider the influence of enriching the connective type with a linear logic-type negation.

2 Generalised Kripke models

2.1 Generalising the set of worlds

In traditional Kripke semantics, a frame is built on a non-empty set of worlds W which does not, generally speaking, have any particular structure. That is, an interpretant of a logical formula in a frame may be any subset of the set of worlds. However, in Kripke-style semantics for intuitionistic logic, frames are equipped with an order and interpretants must be hereditary sets. The idea of the generalised Kripke frames is similar, but is tailored to the substructural setting. What we need in this setting is that an interpretant is not only described by the worlds at which it holds, but also by the 'information quanta' contained in it (so-called **co-worlds**), moreover, either of these completely determine the interpretant.

A **polarity** is a triple $F = (X, Y, \leqslant)$, where X and Y are non-empty sets and $\leqslant \subseteq X \times Y$ is any binary relation from X to Y. [1] Here X is a **set of worlds**, Y a **set of co-worlds**, and a co-world, or information quantum, $y \in Y$ is a **part of the world** $x \in X$ provided $x \leqslant y$ holds.

Such a polarity yields a Galois connection:

$$\begin{array}{rcrcl} (\_)^{\leqslant} & : & \mathcal{P}(X) & \to & \mathcal{P}(Y) \\ & & A & \mapsto & \{y \in Y \mid \forall x \in X \ (x \in A \Rightarrow x \leqslant y)\} \end{array}$$

$$\begin{array}{rcrcl} (\_)^{\geqslant} & : & \mathcal{P}(Y) & \to & \mathcal{P}(X) \\ & & B & \mapsto & \{x \in X \mid \forall y \in Y \ (y \in B \Rightarrow y \geqslant x)\}. \end{array}$$

That is, for all $A \subseteq X$ and $B \subseteq Y$ we have

$$B \subseteq A^{\leqslant} \quad \iff \quad A \subseteq B^{\geqslant}.$$

For more details on the basic theory of polarities and Galois connections, see any standard text on lattice theory such as [DP02]. Interpretants in generalised Kripke semantics are taken from among the Galois-stable subsets of X:

$$\mathcal{G}(F) = \{A \subseteq X \mid A = (A^{\leqslant})^{\geqslant}\}$$

[1] Somewhat non-standardly we use the notation $\leqslant$ for an arbitrary relation from a set X to a set Y. Note that such a relation mostly has no chance of being reflexive or transitive as X need not be the same set as Y. However, when binary relations are used to make lattices of Galois closed sets, $\mathcal{G}(F)$, one has $(\{x\}^{\leqslant})^{\geqslant} \subseteq (\{y\})^{\geqslant}$ if and only if x and y are related in the polarity. Thus the relation of the polarity ends up being the order of $\mathcal{G}(F)$ restricted to $X \times Y$. This is the reason that it is an aid to the intuition to call the relation $\leqslant$ even though it is no order in its own right.

Define $\uparrow x = \{y \in Y \mid x \leqslant y\}$ and $\downarrow y = \{x \in X \mid x \leqslant y\}$. A polarity F is called an **S-frame** (a **separating frame**) provided that

$$\forall x_1, x_2 \in X \quad \Big(\uparrow x_1 = \uparrow x_2 \;\Rightarrow\; x_1 = x_2\Big) \quad \text{and} \quad \forall y_1, y_2 \in Y \quad \Big(\downarrow y_1 = \downarrow y_2 \;\Rightarrow\; y_1 = y_2\Big).$$

Informally, these conditions mean that $\uparrow x$ ($\downarrow y$, respectively) completely describes the world x (the co-world y, resp.). More formally, it means that the mappings $X \to \mathcal{G}(F)$ defined as $x \mapsto (\uparrow x)^{\geqslant}$ and $Y \to \mathcal{G}(F)$ defined as $y \mapsto \downarrow y$ are injective.[2] Accordingly, we will often write x instead of $(\uparrow x)^{\geqslant}$, and we will use $\leqslant$ instead of $\subseteq$ for the order in $\mathcal{G}(F)$. Thus for $A \in \mathcal{G}(F)$, $x \in X$, and $y \in Y$ we write $x \leqslant A$ instead of $(\uparrow x)^{\geqslant} \subseteq A$, and $A \leqslant y$ instead of $A \subseteq \downarrow y$. Note that for $x \in X$ and $y \in Y$ we have $x \leqslant y$, where $\leqslant$ is the relation in the frame F, if and only if $(\uparrow x)^{\geqslant} \subseteq \downarrow y$, so this causes no confusion. Finally, we write $<$ for the strict order on $\mathcal{G}(F)$. In particular, for $x, x' \in X$, we have $x' < x$ if and only if $(\uparrow x')^{\geqslant} \subset (\uparrow x)^{\geqslant}$ if and only if $\uparrow x \subset \uparrow x'$ and similarly for elements of Y.

An S-frame F is called **reduced** (or an **RS-frame**) if the following two properties hold:

$$\forall x \; \exists y \; \Big(x \nleqslant y \wedge \forall x' \; (x' < x \Rightarrow x' \leqslant y)\Big) \quad \text{and} \quad \forall y \; \exists x \; \Big(x \nleqslant y \wedge \forall y' \; (y < y' \Rightarrow x \leqslant y')\Big).$$

These two conditions basically say that all worlds are completely join-irreducible (i.e., it is not possible that $x = \bigvee\{x' \mid \uparrow x \subset \uparrow x'\}$) and all co-worlds are completely meet-irreducible. This notion of an RS-frame gives a generalisation of the usual notions of Kripke structure where an unstructured set of worlds W corresponds to the RS-frame $(W, W, \neq)$ and an intuitionistic Kripke frame $(W, \leq)$ corresponds to the RS-frame $(W, W, \ngeq)$.

Finally, in our process of generalising Kripke models, we introduce a notion of a valuation function. Given an RS-frame F and a set of variables Var, a **valuation** is a mapping $V\colon Var \to \mathcal{G}(F)$.

2.2 Models for substructural logics

The paper [Geh06] provides complete semantics for the implication-fusion fragment of various substructural logics. The basic logic under consideration is the non-associative Lambek calculus. When we speak of a logic here, we mean a set of sequents $A \vdash B$ with A and B formulae in some connective type. For the Lambek calculus, the set of formulae $\mathcal{F}$ is built from a set of variables Var with a product connective (which will be denoted as $\otimes$ here) and its two residuals denoted as $/$ and $\backslash$. The rules of the system are as follows ($A, B, C \in \mathcal{F}$):

- an axiom scheme: $A \vdash A$
- a transitivity or cut rule: if $A \vdash B$ and $B \vdash C$ then $A \vdash C$
- residuation rules: $A \vdash C/B \;\Leftrightarrow\; A \otimes B \vdash C \;\Leftrightarrow\; B \vdash A \backslash C$.

The Lambek calculus is then the collection of all sequents $A \vdash B$ derivable using these axioms and rules.

In [DGP05] it was shown that we may capture this logic semantically by adding a ternary relation $R_{\otimes} \subseteq X \times X \times Y$ modelling the behaviour of the connectives to the basic RS-frames

[2] As a consequence, X and Y may be understood as subsets of $\mathcal{G}(F)$.

introduced above. Since interpretants are Galois-stable sets, the following compatibility conditions on $R_\otimes$ are imposed:

$$\begin{array}{llll} \forall x_1, x_2 \in X & (R_\otimes[x_1, x_2, \_]^{\geqslant})^{\leqslant} & = & R_\otimes[x_1, x_2, \_] \\ \forall x_1 \in X, y \in Y & (R_\otimes[x_1, \_, y]^{\leqslant})^{\geqslant} & = & R_\otimes[x_1, \_, y] \\ \forall x_2 \in X, y \in Y & (R_\otimes[\_, x_2, y]^{\leqslant})^{\geqslant} & = & R_\otimes[\_, x_2, y] \end{array} \qquad (1)$$

Here the $R[\ ]$ notation denotes a relational image. For example,

$$R_\otimes[x_1, x_2, \_] = \{y \in Y \mid R_\otimes(x_1, x_2, y)\}.$$

We call $(F, R_\otimes)$ a **Lambek frame**, provided $F = (X, Y, \leqslant)$ is an RS-frame and $R_\otimes$ is a compatible relation. A **model** is a triple $\mathcal{M} = (F, R_\otimes, V)$ where $(F, R_\otimes)$ is a Lambek frame and $V\colon Var \to \mathcal{G}(F)$ is a valuation of variables. Given such a model, relations $\Vdash\ \subseteq X \times \mathcal{F}$ and $\succ\ \subseteq Y \times \mathcal{F}$ are defined inductively:

- for $p \in Var$: $x \Vdash p \;\Leftrightarrow\; x \leqslant V(p)$ and $y \succ p \;\Leftrightarrow\; y \geqslant V(p)$
- if $x \Vdash A$, $x \Vdash B$, $y \succ A$, $y \succ B$ are defined for all $x \in X$ and $y \in Y$, the relations for complex formulae are defined as follows:

$$\begin{array}{lcl} y \succ A \otimes B & \Leftrightarrow & \forall x_1, x_2 \in X\ \Big((x_1 \Vdash A \ \wedge\ x_2 \Vdash B) \Rightarrow R_\otimes(x_1, x_2, y)\Big) \\ x \Vdash A \otimes B & \Leftrightarrow & \forall y \in Y\ (y \succ A \otimes B \Rightarrow x \leqslant y) \\[1ex] x \Vdash A \backslash B & \Leftrightarrow & \forall x' \in X,\ \forall y \in Y\ \Big((x' \Vdash A \ \wedge\ y \succ B) \Rightarrow R_\otimes(x', x, y)\Big) \\ y \succ A \backslash B & \Leftrightarrow & \forall x \in X\ (x \Vdash A \backslash B \Rightarrow x \leqslant y) \\[1ex] x \Vdash B \,/\, A & \Leftrightarrow & \forall x' \in X,\ \forall y \in Y\ \Big((x' \Vdash A \ \wedge\ y \succ B) \Rightarrow R_\otimes(x, x', y)\Big) \\ y \succ B \,/\, A & \Leftrightarrow & \forall x \in X\ (x \Vdash B \,/\, A \Rightarrow x \leqslant y) \end{array} \qquad (2)$$

We note here that these forcing conditions may seem strange to readers used to Kripke frames corresponding to complex algebras that are based on distributive lattices or Boolean algebras. Because of the compatibility relations that $R_\otimes$ satisfies, it is interdefinable with the relation $R_\otimes^\downarrow \subseteq X^3$ given by

$$R_\otimes^\downarrow(x_1, x_2, x_3) \quad\Longleftrightarrow\quad x_3 \in R_\otimes[x_1, x_2, \_]^{\geqslant}.$$

As was shown in Remark 4.41 in [Geh06], *if* the underlying RS-frame is distributive, then, relative to this relation, we have

$$x_3 \Vdash A \otimes B \quad\Leftrightarrow\quad \exists x_1, x_2 \in X\ \Big(x_1 \Vdash A \wedge x_2 \Vdash B \wedge R_\otimes^\downarrow(x_1, x_2, x_3)\Big),$$

which is the usual forcing condition in the distributive setting. However, this is not true in general for RS-frames, though one can use the relation $R_\otimes^\downarrow$ as basic instead of $R_\otimes$ if one wishes. For more details, please see [Geh06].

Given a formula A, we define the interpretation of A in a model $\mathcal{M} = (F, R_\otimes, V)$ to be

$$[A]_\mathcal{M} = \{x \in X \mid x \Vdash A\}.$$

It is not difficult to check that $[A]_{\mathcal{M}}$ always is a Galois closed subset of X. Now a sequent $A \vdash B$ is said to hold in $\mathcal{M}$ provided $[A]_{\mathcal{M}} \leq [B]_{\mathcal{M}}$ and $A \vdash B$ is said to be valid over a class of Lambek frames if and only if it holds in every model based on a frame from the class. The following theorem (proven in [Geh06]) states soundness and completeness of the non-associative Lambek calculus with respect to the class of models based on Lambek frames:

Theorem: For all formulae A and B the sequent $A \vdash B$ is derivable in the non-associative Lambek calculus if and only if it is valid over the class of all Lambek frames.

Or, equivalently, the non-associative Lambek calculus is complete with respect to the class of Lambek frames.

While this theorem is proved both in [Geh06] and in [DGP05], we sketch the proof of soundness here to give the flavour of these semantics. The point of the semantics as given in (2) is that the elements of X may be viewed as a join-dense subset of $\mathcal{G}(F)$ (via $x \mapsto (\downarrow x)^{\geqslant} = (x^{\leqslant})^{\geqslant}$), the elements of Y may be viewed as a meet-dense subset of $\mathcal{G}(F)$ (via $y \mapsto \downarrow y = y^{\leqslant}$), and the conditions $x \Vdash A$ and $y \succ A$ correspond to the statements $x \leqslant \tilde{V}(A)$ and $\tilde{V}(A) \leqslant y$ where $\tilde{V}$ is the homomorphic extension of V with respect to the operations $\otimes, \backslash, /$ as described by the clauses in (2). That is, the operations $\otimes, \backslash, /$ on $\mathcal{G}(F)$ are defined for $S, T \in \mathcal{G}(F)$ by

$$\begin{array}{ll} S \otimes T & = \bigwedge \{y \in Y \mid \forall x_1, x_2 \in X \ \ [(x_1 \leqslant S \ \wedge \ x_2 \leqslant T) \Rightarrow R_{\otimes}(x_1, x_2, y)]\} \\ S \backslash T & = \bigvee \{x \in X \mid \forall x' \in X, \ \forall y \in Y \ \ [(x' \leqslant S \ \wedge \ y \geqslant T) \Rightarrow R_{\otimes}(x', x, y)]\} \\ T / S & = \bigvee \{x \in X \mid \forall x' \in X, \ \forall y \in Y \ \ [(x' \leqslant S \ \wedge \ y \geqslant T) \Rightarrow R_{\otimes}(x, x', y)]\}. \end{array} \tag{3}$$

Thus, to prove soundness, it suffices to show that the operations thus defined form a residuated family of operations on $\mathcal{G}(F)$. To this end, let $R, S, T \in \mathcal{G}(F)$, we prove that

$$R \otimes S \leqslant T \quad \Longleftrightarrow \quad S \leqslant R \backslash T.$$

We start by proving that this is true in the special case where $R = x \in X$, $S = x' \in X$, and $T = y \in Y$. First note that by the third compatibility condition on $R_{\otimes}$, we have that $R_{\otimes}(x, x', y)$ and $x_1 \leq x$ implies $R_{\otimes}(x_1, x', y)$. This is because $R_{\otimes}(x, x', y)$ means $x \in R_{\otimes}[\_, x', y]$ and $x_1 \leq x$ means $x_1 \in (\downarrow x)^{\geqslant}$ and thus $x_1 \in (\downarrow x)^{\geqslant} \subseteq (R_{\otimes}[\_, x', y]^{\leqslant})^{\geqslant}$. But by the third compatibity condition on $R_{\otimes}$, we have $(R_{\otimes}[\_, x', y]^{\leqslant})^{\geqslant} = R_{\otimes}[\_, x', y]$ so that $x_1 \in R_{\otimes}[\_, x', y]$. That is, $R_{\otimes}(x_1, x', y)$. The corresponding result for the second coordinate is proved similarly from the second compatibility condition. And finally, by the first compatibility condition on $R_{\otimes}$, we have $R_{\otimes}(x, x', y)$ and $y \leqslant y_1$ implies $R_{\otimes}(x, x', y_1)$. From the definition of $\otimes$ as given in (3) and these monotonicity properties of $R_{\otimes}$, we obtain

$$\begin{aligned} x \otimes x' \leqslant y \iff & \forall x_1, x_2 \in X \ \Big((x_1 \leqslant x \ \wedge \ x_2 \leqslant x') \Rightarrow R_{\otimes}(x_1, x_2, y)\Big) \\ \iff & R_{\otimes}(x, x', y). \end{aligned}$$

Similarly, one may prove using the definition of $\backslash$ that

$$x' \leqslant x \backslash y \iff R_{\otimes}(x, x', y).$$

Thus indeed

$$x \otimes x' \leqslant y \quad \Longleftrightarrow \quad x' \leqslant x \backslash y.$$

Now finally, for arbitrary $R, S, T \in \mathcal{G}(F)$ we have

$$
\begin{aligned}
& R \otimes S \leqslant T \\
\Longleftrightarrow\; & \bigvee \{x \otimes x' \mid X \ni x \leqslant R \,\wedge\, X \ni x' \leqslant S\} \leqslant \bigwedge \{y \in Y \mid T \leqslant y\} \\
\Longleftrightarrow\; & \forall x, x' \in X \;\forall y \in Y \left[(x \leqslant R \,\wedge\, x' \leqslant S \,\wedge\, T \leqslant y) \Rightarrow x \otimes x' \leqslant y\right] \\
\Longleftrightarrow\; & \forall x, x' \in X \;\forall y \in Y \left[(x \leqslant R \,\wedge\, x' \leqslant S \,\wedge\, T \leqslant y) \Rightarrow x' \leqslant x \backslash y\right] \\
\Longleftrightarrow\; & \bigvee \{x' \in X \mid x' \leqslant S\} \leqslant \bigwedge \{x \backslash y \mid X \ni x \leqslant R \,\wedge\, T \leqslant y \in Y\} \\
\Longleftrightarrow\; & S \leqslant R \backslash T.
\end{aligned}
$$

3 Generalised Kripke semantics for $LG_\emptyset$

In this section we illustrate the theory described above on another kind of substructural logic, viz., the Lambek-Grishin calculus. We start with its minimal version $LG_\emptyset$, in which no interaction axioms are added. The formulae of $LG_\emptyset$ are built from variables taken from the set Var with six connectives: product $\otimes$ and its upper residuals $/$ and $\backslash$, and plus $\oplus$ and its lower residuals $\obackslash$ and $\oslash$. The rules are as follows:

- an axiom scheme: $A \vdash A$
- a transitivity rule: if $A \vdash B$ and $B \vdash C$ then $A \vdash C$
- residuation rules for the product family: $A \vdash C/B \;\Leftrightarrow\; A \otimes B \vdash C \;\Leftrightarrow\; B \vdash A \backslash C$
- residuation rules for the plus family: $B \obackslash C \vdash A \;\Leftrightarrow\; C \vdash B \oplus A \;\Leftrightarrow\; C \oslash A \vdash B$

In order to model the behaviour of the plus family of connectives, we add to the frames for the Lambek calculus another ternary relation $R_\oplus \subseteq X \times Y \times Y$ satisfying the appropriate compatibility conditions:

$$
\begin{array}{lll}
\forall y_1, y_2 \in Y & (R_\oplus[\_, y_1, y_2]^{\leqslant})^{\geqslant} & = \;R_\oplus[\_, y_1, y_2] \\
\forall y_1 \in Y, x \in X & (R_\oplus[x, y_1, \_]^{\geqslant})^{\leqslant} & = \;R_\oplus[x, y_1, \_] \\
\forall y_2 \in Y, x \in X & (R_\oplus[x, \_, y_2]^{\geqslant})^{\leqslant} & = \;R_\oplus[x, \_, y_2]
\end{array}
\tag{4}
$$

Let us call $(F, R_\otimes, R_\oplus)$ a **Lambek-Grishin frame**, provided $(F, R_\otimes)$ is a Lambek frame and $R_\oplus$ is compatible in the sense given above. The truth conditions for the new connectives are defined as follows:

$$
\begin{array}{lcl}
x \Vdash A \oplus B & \Leftrightarrow & \forall y_1, y_2 \in Y \left((y_1 \succ A \;\wedge\; y_2 \succ B) \Rightarrow R_\oplus(x, y_1, y_2)\right) \\
y \succ A \oplus B & \Leftrightarrow & \forall x \in X \;(x \Vdash A \oplus B \Rightarrow x \leqslant y) \\
& & \\
y \succ A \obackslash B & \Leftrightarrow & \forall y' \in Y,\; \forall x \in X \left((y' \succ A \;\wedge\; x \Vdash B) \Rightarrow R_\oplus(x, y', y)\right) \\
x \Vdash A \obackslash B & \Leftrightarrow & \forall y \in Y \;(y \succ A \obackslash B \Rightarrow x \leqslant y) \\
& & \\
y \succ B \oslash A & \Leftrightarrow & \forall y' \in Y,\; \forall x \in X \left((y' \succ A \;\wedge\; x \Vdash B) \Rightarrow R_\oplus(x, y, y')\right) \\
x \Vdash B \oslash A & \Leftrightarrow & \forall y \in Y \;(y \succ B \oslash A \Rightarrow x \leqslant y)
\end{array}
\tag{5}
$$

Again, $[A]_{\mathcal{M}}$ is defined accordingly and a sequent $A \vdash B$ holds in a Lambek-Grishin frame provided it holds in all models based on that frame. We have the following soundness and completeness theorem for the basic Lambek-Grishin calculus $LG_\emptyset$ with respect to the class of frames described above:

Theorem: For all formulae A and B of the Lambek-Grishin calculus, the sequent $A \vdash B$ is derivable in $LG_\emptyset$ if and only if it is valid over the class of all Lambek-Grishin frames $(X, Y, \leqslant, R_\otimes, R_\oplus)$.

This theorem is a direct consequence of the completeness theorem for the non-associative Lambek calculus given in the previous section, since each of the residuated families separately gives rise to an isomorphic or an order dual non-associative Lambek system, and no interaction between the two is stipulated. While this basic completeness theorem for $LG_\emptyset$ is a direct consequence of the work in [Geh06], it requires some work to develop the modular correspondence theory for the interaction axioms. As in Kripke semantics for modal logics, two components are needed: canonicity and correspondence. Our results for both of these components are described in Section 5 after the necessary preliminaries have been introduced.

4 Introduction to canonical extensions

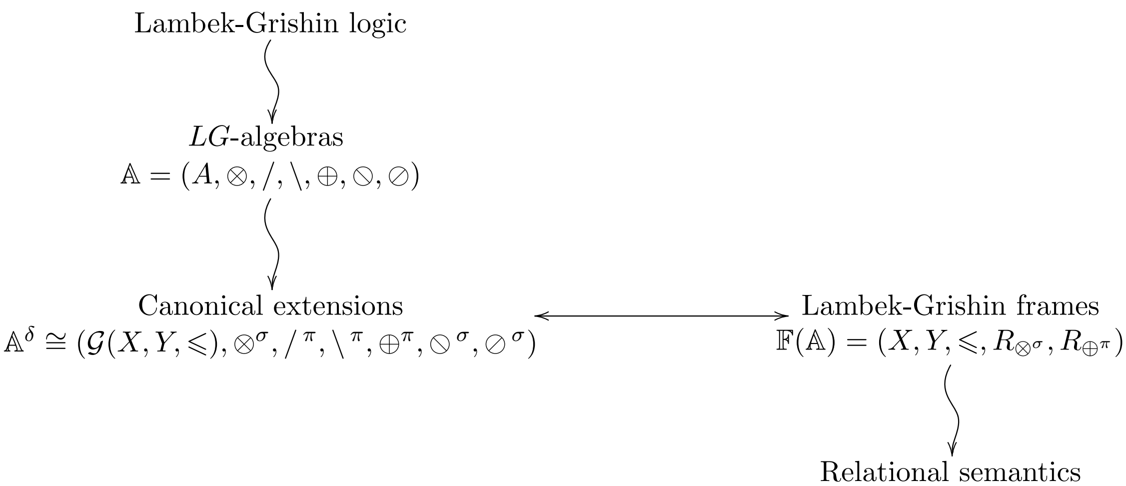
In this section we will first sketch the approach to completeness results via canonical extensions that is behind the results given above, and which will be applied to the Lambek-Grishin calculus with interaction axioms in Section 5. Next, we will give some preliminaries needed on canonical extensions of posets and of additional operations on posets in order, finally, to define canonical extensions of what we will call Lambek-Grishin algebras.

4.1 Completeness via canonicity

Our proof method is different from the one most frequently used in logic. Typically, given a propositional logic L, a so-called canonical frame model is constructed using some notion of maximal consistent theories or related notions. This is also possible for the notion of frame used here and this approach to the completeness of the Lambek calculus is given in section 4 of [Geh06]. By contrast, the completeness of the Lambek calculus and of several of its extensions is proved in the paper [DGP05] using a combination of algebraic canonicity results and correspondence results. We will use this latter approach here as it is particularly well-suited for dealing with additional laws such as the interaction axioms. For those familiar with standard canonicity and correspondence proofs in modal logic, the most important divergence from the standard procedure is that our correspondence results are just for frames and their complex algebras, and we do not use correspondence results when proving canonicity: we treat correspondence and canonicity separately. For canonicity, our proof method takes advantage of developments within the algebraic theory of canonical extensions of posets expanded with further operations and is a direct generalisation of the papers [DGP05] and [GNV05].

Starting from the Lindenbaum algebra of the logic, the algebraic theory provides a certain completion, called the canonical extension, which is unique with respect to a few simple abstract properties. It turns out that this extension is precisely the algebra of Galois closed sets of the canonical frame as defined in section 2 of [Geh06]. Thus we get a simple abstract manner of working with the canonical frame. This makes it easy to treat additional operations and their interaction axioms. In particular, from $\mathbb{A} = (A, \otimes, /, \backslash, \oplus, \oslash, \oslash)$ we get $\mathbb{A}^\delta = (A^\delta, \otimes^\sigma, /^\pi, \backslash^\pi, \oplus^\pi, \oslash^\sigma, \oslash^\sigma)$ and $\mathbb{A}^\delta$ is the algebra of Galois closed sets for some frame which we denote by $(X, Y, \leqslant, R_{\otimes^\sigma}, R_{\oplus^\pi}) =: \mathbb{F}(\mathbb{A})$.

The central role of the canonical extension in the process of finding relational semantics is illustrated by the following diagramme.



This process works for the Lindenbaum algebra $\mathbb{A}$ of the $LG_\emptyset$-logic in the sense that the canonical frame $\mathbb{F}(\mathbb{A})$ with the interpretation given by the embedding map is a model of precisely those sequents that are deducible in $LG_\emptyset$, as the following equivalences demonstrate.

$$\begin{array}{ccl} & A \vdash B & \text{holds in } LG_\emptyset \\ \Leftrightarrow & A \leqslant B & \text{holds in } \mathbb{A} \\ \overset{*}{\Leftrightarrow} & [A]_{\mathbb{A}^\delta} \leqslant [B]_{\mathbb{A}^\delta} & \text{holds in } \mathbb{A}^\delta \\ \Leftrightarrow & \mathbb{F}(\mathbb{A}) \Vdash A \vdash B, & \end{array}$$

where the subscript $\mathbb{A}^\delta$ means that the formula is interpreted in $\mathbb{A}^\delta$.

We call $\mathbb{A} = (A, \otimes, /, \backslash, \oplus, \oslash, \oslash)$ an ***LG*-algebra** provided A is a poset, and the operations of $\mathbb{A}$ satisfy the rules of $LG_\emptyset$, i.e. $/$ and $\backslash$ are upper residuals of $\otimes$, while $\oslash$ and $\oslash$ are lower residuals of $\oplus$.

The process of getting a canonical extension and from a canonical extension a Lambek-Grishin frame works for any LG-algebra. When dealing with a class of algebras that satisfy additional interaction axioms (e.g. one of Grishin's groups, see next section), our aim is to find out which first-order condition is imposed by these axioms on the class of Lambek-Grishin frames.

In order to prove the completeness of an axiomatic extension of $LG_\emptyset$, two components are needed:

1. We have to show that if we start with $\mathbb{A}$, the Lindenbaum algebra for some extension of $LG_\emptyset$, then for each additional axiom, the equivalence indicated by $*$ still works. This

equivalence is called the **canonicity** of the additional axiom. This canonicity implies completeness with respect to the class of frames satisfying the axioms. However this algebraic part of the work does not necessarily give us a good (preferably first-order) class of frames with respect to which the logic is complete.

2. The second part of our work then, typically referred to as **correspondence**, consists in showing that the axioms holding in a frame are equivalent to appropriate first-order properties of the frames.

4.2 LG-algebras and their canonical extensions

In defining the canonical extension of an LG-algebra $\mathbb{A}$, we will start with abstractly defining the canonical extension of a poset and we will apply this construction to the poset reduct P of $\mathbb{A}$. This poset extension can then be expanded by an additional operation for each additional operation of the original $\mathbb{A}$, in such a way that $\mathbb{A}^\delta$ will be an LG-algebra.

In order to be able to give the abstract definition of the canonical extension of a poset, we first make the following definitions:

Definition 4.1. Let P be a poset.

1. A **filter** of P is a non-empty subset F of P satisfying:
   (a) F is an up-set, that is, if $x \in F$, $y \in P$, and $x \leqslant y$, then $y \in F$;
   (b) F is down-directed, that is, if $x, y \in F$, then there exists $z \in F$ with $z \leqslant x$ and $z \leqslant y$.

   An ideal of P is defined dually. That is, I is an **ideal** of P provided I is a non-empty up-directed down-set of P.

2. An **extension** of P is an order embedding $e : P \to Q$, i.e., for every $x, y \in P$, $x \leqslant y$ if and only if $e(x) \leqslant e(y)$. For ease of notation we will suppress e and call Q an extension of P and assume that P is a subposet of Q.

3. Given an extension Q of P, an element of Q is called a **filter element** provided it is the infimum in Q of some filter F of P. We denote the set of all filter elements of Q by $F(Q)$. Dually, an element of Q is called an **ideal element** provided it is the supremum in Q of some ideal I of P. We denote the set of all ideal elements of Q by $I(Q)$. Note that filter, respectively ideal, elements are also known under the names closed, respectively open, elements.

4. An extension Q of P is said to be **dense** provided each element of Q is both the supremum of all the filter elements below it and the infimum of all the ideal elements above it.

5. An extension Q of P is said to be **compact** provided $\bigwedge F \leq \bigvee I$ in Q implies $F \cap I \neq \emptyset$ for F a filter of P and I an ideal of P.

6. A **completion** of a poset P is an extension Q of P which also happens to be a complete lattice.

We are now ready to give the abstract definition of the canonical extension of a poset.

Definition 4.2. Let P be a poset. A **canonical extension** of P is a dense and compact completion of P.

In [DGP05], it is shown that every poset P has a canonical extension that is unique up to an isomorphism fixing P. We will denote this extension by P^δ. In earlier papers P^σ was used instead of P^δ, but we will use the new notation as introduced in [GP08].

Using the axiom of choice one can prove that the canonical extension P^δ of an arbitrary poset P is a perfect lattice, that is, it is a complete lattice such that for all $x \in P^\delta$:

$$x = \bigvee\{j \in \mathcal{J}^\infty(P^\delta) \mid j \leqslant x\} \text{ and } x = \bigwedge\{m \in \mathcal{M}^\infty(P^\delta) \mid x \leqslant m\},$$

where $\mathcal{J}^\infty(P^\delta)$, respectively $\mathcal{M}^\infty(P^\delta)$, denotes the set of completely join, respectively meet, irreducible elements of P^δ. This enables us to identify each canonical extension with a frame: because P^δ is a perfect lattice, it can be shown that $P^\delta \cong \mathcal{G}(\mathcal{J}^\infty(P^\delta), \mathcal{M}^\infty(P^\delta), \leqslant)$ and that the frame $(\mathcal{J}^\infty(P^\delta), \mathcal{M}^\infty(P^\delta), \leqslant)$ is an RS-frame.

In order to extend an LG-algebra $\mathbb{A}$, we still need to extend our residuated families to $\mathbb{A}^\delta$. First, we look at posets that come with an additional order-preserving unary map f. There are two different ways of extending f to the canonical extension, namely:

Definition 4.3. Let P and Q be posets, and $f : P \to Q$ an order preserving map. Define maps $f^\sigma, f^\pi : P^\delta \to Q^\delta$ by setting:

$$\begin{aligned} f^\sigma(u) &= \bigvee\Big\{\bigwedge\{f(p) \mid x \leqslant p \in P\} \mid u \geqslant x \in F(P^\delta)\Big\}, \\ f^\pi(u) &= \bigwedge\Big\{\bigvee\{f(p) \mid y \geqslant p \in P\} \mid u \leqslant y \in I(P^\delta)\Big\}. \end{aligned}$$

It is clear that for $u \in P^\delta$, $x \in F(P^\delta)$, and $y \in I(P^\delta)$, this gives

$$\begin{aligned} f^\sigma(u) &= \bigvee\{f^\sigma(x) \mid u \geqslant x \in F(P^\delta)\}, \\ f^\sigma(x) &= \bigwedge\{f(p) \mid x \leqslant p \in P\}, \\ f^\pi(u) &= \bigwedge\{f^\pi(y) \mid u \leqslant y \in I(P^\delta)\}, \\ f^\pi(y) &= \bigvee\{f(p) \mid y \geqslant p \in P\}. \end{aligned}$$

Furthermore, it can be shown that f^σ and f^π are order-preserving extensions of f, that send filter elements to filter elements and ideal elements to ideal elements.

Canonical extension commutes with taking the order dual version of a poset and with taking a finite product of posets, that is $(P^{\text{dual}})^\delta = (P^\delta)^{\text{dual}}$ and $(P^n)^\delta = (P^\delta)^n$. This enables us to use the previous definition to find the σ- and π-extension of more complicated maps. If f is order-reversing, it can be seen as an order-preserving map from the order dual version of its domain to its codomain, to which the definition can be applied. Similarly, in order to deal with operations of arity greater than 1, that are in each coordinate either order-preserving or order-reversing, we regard these as maps of which the domain is a finite product of P's and order dual versions of P.

When extending a particular operation, one has to choose between the σ- and the π-extension, depending on the properties (e.g. join-preserving, meet-preserving) of the particular operation.

In the paper [DGP05] it is shown in general how to extend residuated maps. Applying this to our operations $\otimes, /, \backslash, \oplus, \obslash, \oslash$ gives that if we extend an LG-algebra $(A, \otimes, /, \backslash, \oplus, \obslash, \oslash)$ as $(A^{\delta}, \otimes^{\sigma}, /^{\pi}, \backslash^{\pi}, \oplus^{\pi}, \obslash^{\sigma}, \oslash^{\sigma})$, then the basic Lambek-Grishin rules will hold again on this canonical extension. That is the canonicity result for $LG_{\emptyset}$.

5 Completeness for the interaction axioms

In this section we extend $LG_{\emptyset}$ with one of the families of interderivable interaction axioms. We show that these axioms are canonical, and we show how the axioms give rise to corresponding first-order conditions that tell us how to adapt the class of Kripke frames.

5.1 Interaction axioms extending $LG_{\emptyset}$

Interactions involving only connectives from one family are well-known from the Lambek calculus. For example, adding to the non-associative Lambek calculus an axiom $A \otimes (B \otimes C) \dashv\vdash (A \otimes B) \otimes C$ yields the associative Lambek calculus.

In the setting of $LG_{\emptyset}$ it is most interesting to consider mixed axioms, i.e. those involving connectives from both residuated families. In [Gri83] Grishin studied and classified all of these. In particular, he showed that because of the rules of the basic calculus $LG_{\emptyset}$, the additional axioms come in groups of six that are mutually interderivable on the basis of $LG_{\emptyset}$. The paper [Moo09] identified the axioms in Grishin's fourth group as most relevant for linguistic applications. However, from the logical perspective, Grishin's first group of axioms seems equally interesting. In Section 6 we give our results for all of these four groups. In this section we will consider Grishin's fourth group in detail. For completeness, we give all four groups here:

$$\begin{array}{llll} (a) & (B \oplus C) \otimes A \vdash B \oplus (C \otimes A) & (d) & (A \obslash B) \backslash C \vdash B \backslash (A \oplus C) \\ (b) & A \oplus (C / B) \vdash (A \oplus C) / B & (e) & (C \otimes B) \oslash A \vdash C \oslash (A / B) \\ (c) & A \obslash (C \otimes B) \vdash (A \obslash C) \otimes B & (f) & A \otimes (C \backslash B) \vdash (A \oslash C) \oplus B \end{array} \qquad \text{(I)}$$

$$\begin{array}{llll} (a) & (B \backslash C) \otimes A \vdash B \backslash (C \otimes A) & (d) & (A \otimes B) \backslash C \vdash B \backslash (A \backslash C) \\ (b) & A \backslash (C / B) \vdash (A \backslash C) / B & (e) & (A / B) / C \vdash A / (C \otimes B) \\ (c) & A \otimes (C \otimes B) \vdash (A \otimes C) \otimes B & (f) & A \otimes (C \backslash B) \vdash (C / A) \backslash B \end{array} \qquad \text{(II)}$$

$$\begin{array}{llll} (a) & A \obslash (C \oplus B) \vdash (A \obslash C) \oplus B & (d) & B \oslash (C \oplus A) \vdash (B \oslash A) \oslash C \\ (b) & (A \oplus C) \oplus B \vdash A \oplus (C \oplus B) & (e) & A \obslash (B \obslash C) \vdash (B \oplus A) \obslash C \\ (c) & (A \obslash C) \oslash B \vdash A \obslash (C \oslash B) & (f) & (C \oslash B) \obslash A \vdash B \oplus (C \obslash A) \end{array} \qquad \text{(III)}$$

$$\begin{array}{llll} (a) & (B \backslash C) \oslash A \vdash B \backslash (C \oslash A) & (d) & (A \backslash C) \obslash B \vdash C \obslash (A \otimes B) \\ (b) & B \backslash (C \oplus A) \vdash (B \backslash C) \oplus A & (e) & (A \oplus B) / C \vdash A / (C \oslash B) \\ (c) & A \otimes (C \oslash B) \vdash (A \otimes C) \oslash B & (f) & A \oslash (B \obslash C) \vdash (C / A) \backslash B \end{array} \qquad \text{(IV)}$$

5.2 Canonicity of the interaction axioms

Our first step in proving the completeness is showing that if an interaction axiom holds in an LG-algebra, then it will hold in its canonical extension. In order to do this, the operations $\otimes, /, \backslash$ and $\oplus, \oslash, \oslash$ are extended to the canonical extension, as explained in Section 4. This will give again an LG-algebra. To prove canonicity of the interaction axioms of group (IV), it suffices to prove that one of them lifts to the canonical extension. The validity of one axiom together with the interderivability of the axioms will yield canonicity of the other ones.

We will now prove the canonicity of the interaction axiom (b) given in (IV). This means that we will show that for an LG-algebra $\mathbb{A}$ in which for all $a, b, c \in A$ it holds that

$$b \backslash (c \oplus a) \leqslant (b \backslash c) \oplus a,$$

this axiom will also hold for the extended operations in the canonical extension $\mathbb{A}^\delta$. First, we show that the axiom holds on certain combinations of filter and ideal elements of A^δ, and we will use this to show that it holds on all elements of A^δ. Thus, we want to show that for $x_1 \in F(A^\delta)$ and $y_2,\ y_3 \in I(A^\delta)$ it holds that

$$x_1 \backslash^\pi (y_2 \oplus^\pi y_3) \leqslant (x_1 \backslash^\pi y_2) \oplus^\pi y_3.$$

For $y_2,\ y_3 \in I(A^\delta)$

$$y := y_2 \oplus^\pi y_3 = \bigvee \Big\{ a_2 \oplus a_3 \mid a_i \leqslant y_i, a_i \in A \Big\}. \tag{6}$$

Note that $y \in I(A^\delta)$. Therefore, for $x_1 \in F(A^\delta)$ we have:

$$x_1 \backslash^\pi y = \bigvee \Big\{ a_1 \backslash a \mid a_1 \geqslant x_1, a \leqslant y, a_1, a \in A \Big\}. \tag{7}$$

Combining (6) and (7) gives

$$x_1 \backslash^\pi (y_2 \oplus^\pi y_3) = \bigvee \Big\{ a_1 \backslash a \mid a_1 \geqslant x_1, a \leqslant \bigvee \{a_2 \oplus a_3 \mid a_i \leqslant y_i, a_i \in A\}, a_1, a \in A \Big\}. \tag{8}$$

We show that the set $S := \{a_2 \oplus a_3 \mid a_i \leqslant y_i, a_i \in A\}$ is a directed set in A^δ. For $a_2 \oplus a_3,\ a_2' \oplus a_3' \in S$, we have that $(a_2 \vee a_2') \oplus (a_3 \vee a_3') \in S$, and since $\oplus$ is order preserving in both coordinates,

$$(a_2 \vee a_2') \oplus (a_3 \vee a_3') \geqslant a_2 \oplus a_3 \quad \text{and} \quad (a_2 \vee a_2') \oplus (a_3 \vee a_3') \geqslant a_2' \oplus a_3',$$

thus S is directed.

We now look at the condition for the a's in (8): $a \leqslant \bigvee S$. Since $a \in A$, it is a filter element. Using the compactness of the canonical extension, $a \leqslant \bigvee S$ implies that there is $a_2 \oplus a_3 \in S$ such that $a \leqslant a_2 \oplus a_3$. This means that there are $a_2, a_3 \in A$ with $a_2 \leqslant y_2$, $a_3 \leqslant y_3$ and $a \leqslant a_2 \oplus a_3$.

The map $\backslash$ is the upper residual of $\otimes$ and thus it is order preserving in its second coordinate. So for all a satisfying the condition in (8) there exists $a_2 \oplus a_3 \in S$ such that $a_1 \backslash a \leqslant a_1 \backslash (a_2 \oplus a_3)$, implying

$$\bigvee \Big\{ a_1 \backslash a \mid x_1 \leqslant a_1 \in A \ni a \leqslant \bigvee S \Big\} \leqslant \bigvee \Big\{ a_1 \backslash (a_2 \oplus a_3) \mid x_1 \leqslant a_1 \in A \ni a_2 \leqslant y_2, A \ni a_3 \leqslant y_3 \Big\}. \tag{9}$$

Thus, from (8) and (9):

$$x_1 \setminus^\pi (y_2 \oplus^\pi y_3) \leqslant \bigvee \Big\{ a_1 \setminus (a_2 \oplus a_3) \mid a_1 \geqslant x_1, a_2 \leqslant y_2, a_3 \leqslant y_3, a_i \in A \Big\}.$$

Now we can use the assumption on the elements of A, which yields

$$x_1 \setminus^\pi (y_2 \oplus^\pi y_3) \leqslant \bigvee \Big\{ (a_1 \setminus a_2) \oplus a_3 \mid a_1 \geqslant x_1, a_2 \leqslant y_2, a_3 \leqslant y_3, a_i \in A \Big\}.$$

Note that $\oplus$ is order preserving in its first coordinate and that

$$x_1 \setminus^\pi y_2 = \bigvee \Big\{ a_1 \setminus a_2 \mid a_1 \geqslant x_1, a_2 \leqslant y_2, a_1, a_2 \in A \Big\},$$

so $a_1 \setminus a_2 \leqslant x_1 \setminus^\pi y_2$ for all a_1, a_2 in this join. Therefore

$$\begin{aligned} x_1 \setminus^\pi (y_2 \oplus^\pi y_3) &\leqslant \bigvee \Big\{ (a_1 \setminus a_2) \oplus a_3 \mid a_1 \geqslant x_1, a_2 \leqslant y_2, a_3 \leqslant y_3, a_i \in A \Big\} \\ &\leqslant \bigvee \Big\{ (x_1 \setminus^\pi y_2) \oplus^\pi a_3 \mid a_3 \leqslant y_3, a_3 \in A \Big\}. \end{aligned}$$

And similarly,

$$x_1 \setminus^\pi (y_2 \oplus^\pi y_3) \leqslant (x_1 \setminus^\pi y_2) \oplus^\pi y_3.$$

Thus, we have proved that axiom (b) from (IV) holds for all combinations of an $x_1 \in F(A^\delta)$ in the first spot and $y_2,\ y_3 \in I(A^\delta)$ in the second and third spot. Using this result for filter and ideal elements, we will prove that the inequality on the elements of A implies the inequality for all elements of the canonical extension A^δ. Let $u_1, u_2, u_3 \in A^\delta$:

$$u_1 \setminus^\pi (u_2 \oplus^\pi u_3) = u_1 \setminus^\pi \Big(\bigwedge \Big\{ y_2 \oplus^\pi y_3 \mid u_2 \leqslant y_2, u_3 \leqslant y_3, y_i \in I(A^\delta) \Big\} \Big)$$

Since $\setminus^\pi$ is meet preserving in the second coordinate, and by definition of $\setminus^\pi$, since $y_2 \oplus^\pi y_3 \in I(A^\delta)$, we obtain:

$$\begin{aligned} u_1 \setminus^\pi (u_2 \oplus^\pi u_3) &= \bigwedge \Big\{ u_1 \setminus^\pi (y_2 \oplus^\pi y_3) \mid u_2 \leqslant y_2, u_3 \leqslant y_3, y_i \in I(A^\delta) \Big\} \\ &= \bigwedge \Big\{ x_1 \setminus^\pi (y_2 \oplus^\pi y_3) \mid u_1 \geqslant x_1, x_1 \in F(A^\delta), u_2 \leqslant y_2, u_3 \leqslant y_3, y_i \in I(A^\delta) \Big\}. \end{aligned}$$

Using the inequality for filter and ideal elements, the definition of $\oplus^\pi$, and then the fact that it is meet preserving in its first coordinate, we get:

$$\begin{aligned} u_1 \setminus^\pi (u_2 \oplus^\pi u_3) &\leqslant \bigwedge \Big\{ (x_1 \setminus^\pi y_2) \oplus^\pi y_3 \mid u_1 \geqslant x_1, x_1 \in F(A^\delta), u_2 \leqslant y_2, u_3 \leqslant y_3, y_i \in I(A^\delta) \Big\} \\ &= \bigwedge \Big\{ (x_1 \setminus^\pi y_2) \oplus^\pi u_3 \mid u_1 \geqslant x_1, x_1 \in F(A^\delta), u_2 \leqslant y_2, y_2 \in I(A^\delta) \Big\} \\ &= \Big(\bigwedge \Big\{ x_1 \setminus^\pi y_2 \mid u_1 \geqslant x_1, x_1 \in F(A^\delta), u_2 \leqslant y_2, y_2 \in I(A^\delta) \Big\} \Big) \oplus^\pi u_3 \\ &= (u_1 \setminus^\pi u_2) \oplus^\pi u_3, \end{aligned}$$

where the last step is obtained by definition of $\setminus^\pi$. Thus we have proved that for all $u_1, u_2, u_3 \in A^\delta$ holds that

$$u_1 \setminus^\pi (u_2 \oplus^\pi u_3) \leqslant (u_1 \setminus^\pi u_2) \oplus^\pi u_3.$$

5.3 Correspondence for interaction axioms

In the paper [DGP05] completeness theorems for the implication-fusion fragment of various substructural logics are proven algebraically. For each axiom that is added to the pure implication-fusion fragment, a first-order frame condition is derived by Sahlqvist correspondence methods and it is proven that these conditions are canonical (i.e. hold in the canonical frame for each substructural logic considered). We stress here that this approach is parallel to the one for modal logics and that this was the first modular approach to these completeness results.

For example, the frame condition imposed by postulating associativity for the product connective looks as follows [DGP05]:

$$\begin{aligned}&\forall x_1, x_2, x_3 \in X\ \forall y \in Y\\ &\quad\Big[\forall x \in X\ \Big(R^{\downarrow}_{\otimes}(x_2, x_3, x) \Rightarrow R_{\otimes}(x_1, x, y)\Big) \ \Leftrightarrow\ \forall x \in X\ \Big(R^{\downarrow}_{\otimes}(x_1, x_2, x) \Rightarrow R_{\otimes}(x, x_3, y)\Big)\Big].\end{aligned}$$

Here $R^{\downarrow}_{\otimes}$ is a ternary relation in $X \times X \times X$ which is defined as follows:

$$(x_1, x_2, x) \in R^{\downarrow}_{\otimes} \ \Leftrightarrow\ \forall y \in Y\ \Big((x_1, x_2, y) \in R_{\otimes} \Rightarrow x \leqslant y\Big)$$

Informally, $R_{\otimes}(x_1, x_2, y)$ reflects the fact that the information y is contained in the 'fusion of the worlds' x_1 and x_2, whereas $R^{\downarrow}_{\otimes}(x_1, x_2, x)$ means that the 'fusion of the worlds' x_1 and x_2 makes x possible. We note that the relation $R^{\downarrow}_{\otimes}$ is the one more commonly used in Kripke semantics for substructural logics.

5.3.1 Frame conditions for interaction axioms

Let us illustrate how adding to the basic Lambek-Grishin calculus only one axiom of interaction between the $\otimes$ and $\oplus$ families influences the relations $R_{\otimes}$ and $R_{\oplus}$. As an example, we chose the family of axioms given above in (IV), and work on the form given as the axiom (b): $B \backslash (C \oplus A) \vdash (B \backslash C) \oplus A$. The corresponding frame condition looks as follows:

$$\begin{aligned}&\forall x, x', x_2 \in X\ \forall y, y' \in Y\\ &\quad\Big[\Big(\forall x_1 \in X\ \Big[R^{\downarrow}_{\otimes}(x', x, x_1) \Rightarrow\ R_{\oplus}(x_1, y', y)\Big] \text{ and } R^{\downarrow}_{\oslash}(x, y, x_2)\Big) \Rightarrow\ R_{\otimes}(x', x_2, y')\Big],\end{aligned} \tag{10}$$

where $R^{\downarrow}_{\oslash}$ is defined in a similar way as $R^{\downarrow}_{\otimes}$.

In order to be able to give a proof of this correspondence result, we will work in the 'complex algebra' for these non-distributive structures. That is, we work in $\mathcal{G}(X, Y, \leqslant)$. Now the x's are completely join irreducible elements of $\mathcal{G}(X, Y, \leqslant)$ (atoms in the Boolean setting), the y's are completely meet irreducible elements of $\mathcal{G}(X, Y, \leqslant)$ (co-atoms), and A, B, C are general elements of $\mathcal{G}(X, Y, \leqslant)$. Also $\otimes, /, \backslash$ and $\oplus, \otimes, \oslash$ are operations on $\mathcal{G}(X, Y, \leqslant)$ specified by (2) and (5), where $A \vdash B$ means $A \leqslant B$, $x \Vdash A$ means $x \leqslant A$ and $y \succ A$ means $y \geqslant A$.

The content of the completeness theorem for the basic Lambek-Grishin calculus is, in algebraic terms, that on $\mathcal{G}(X, Y, \leqslant)$ the connectives $\otimes, /, \backslash$ and $\oplus, \otimes, \oslash$ satisfy the rules of $LG_{\emptyset}$.

Note that each canonical extension $\mathbb{A}^\delta$ of an LG-algebra $\mathbb{A}$ is in fact such a structure, since a canonical extension is a perfect lattice (i.e. a corresponding RS-frame can be found) and the extended operations coincide.

We will prove that axiom (b) holds in an algebra $\mathcal{G}(X, Y, \leqslant)$ if and only if the first-order condition (10) holds for the underlying frame $(X, Y, \leqslant)$.

Take the axiom: $\forall A, B, C \ \Big(B \setminus (C \oplus A) \leqslant (B \setminus C) \oplus A\Big)$. Since the completely join irreducibles, $x \in X$, join-generate $\mathcal{G}(X, Y, \leqslant)$, the axiom holds in $\mathcal{G}(X, Y, \leqslant)$ if and only if

$$\forall x \ \forall A, B, C \ \Big(x \leqslant B \setminus (C \oplus A) \ \Rightarrow \ x \leqslant (B \setminus C) \oplus A\Big). \tag{11}$$

By residuation, $x \leqslant B \setminus (C \oplus A)$ is equivalent to $B \leqslant (C \oplus A) \,/\, x$, such that (11) holds if and only if

$$\forall x \ \forall A, B, C \ \Big(B \leqslant (C \oplus A) \,/\, x \ \Rightarrow \ x \leqslant (B \setminus C) \oplus A\Big). \tag{12}$$

Instantiating $B = (C \oplus A) \,/\, x$, this implies

$$\forall x \ \forall A, C \ \Big(x \leqslant \Big([(C \oplus A) \,/\, x] \setminus C\Big) \oplus A\Big). \tag{13}$$

On the other hand, we can get from (13) to (12) since for every B, we have for every x, A, C that

$$\begin{aligned} B \leqslant (C \oplus A) \,/\, x \quad &\Rightarrow \quad B \setminus C \geqslant [(C \oplus A) \,/\, x] \setminus C \\ &\Rightarrow \quad (B \setminus C) \oplus A \geqslant \Big([(C \oplus A) \,/\, x] \setminus C\Big) \oplus A. \end{aligned}$$

Combining this with (13) yields the desired conclusion in (12): $x \leqslant (B \setminus C) \oplus A$, thus the axiom is equivalent to (13).

By residuation, the following equivalence holds:

$$x \leqslant \Big([(C \oplus A) \,/\, x] \setminus C\Big) \oplus A \ \Leftrightarrow \ \Big([(C \oplus A) \,/\, x] \setminus C\Big) \oslash x \leqslant A.$$

And since the $y \in Y$ meet-generate $\mathcal{G}(X, Y, \leqslant)$, (13) is equivalent to

$$\forall x, y \ \forall A, C \ \Big(y \geqslant A \ \Rightarrow \ y \geqslant \Big([(C \oplus A) \,/\, x] \setminus C\Big) \oslash x\Big). \tag{14}$$

Instantiating $A = y$, (14) implies

$$\forall x, y \ \forall C \ \Big(y \geqslant \Big([(C \oplus y) \,/\, x] \setminus C\Big) \oslash x\Big). \tag{15}$$

On the other hand, assuming (15), we have the following for every A and every x, y, C

$$\begin{aligned} y \geqslant A \quad &\Rightarrow \quad C \oplus A \leqslant C \oplus y \\ &\Rightarrow \quad (C \oplus A) \,/\, x \leqslant (C \oplus y) \,/\, x \\ &\Rightarrow \quad [(C \oplus A) \,/\, x] \setminus C \geqslant [(C \oplus y) \,/\, x] \setminus C \\ &\Rightarrow \quad \Big([(C \oplus A) \,/\, x] \setminus C\Big) \oslash x \leqslant \Big([(C \oplus y) \,/\, x] \setminus C\Big) \oslash x. \end{aligned}$$

Using $y \geqslant \Big([(C \oplus y) \,/\, x] \setminus C\Big) \oslash x$ from (15), this gives (14), so that the axiom is equivalent to (15).

By residuation, $y \geqslant \Big([(C \oplus y) \,/\, x] \setminus C\Big) \oslash x$ is equivalent to $x \oslash y \leqslant [(C \oplus y) \,/\, x] \setminus C$, which is equivalent to $[(C \oplus y) \,/\, x] \otimes [x \oslash y] \leqslant C$. Thus (15) holds if and only if

$$\forall x, y \; \forall C \; \Big([(C \oplus y) \,/\, x] \otimes [x \oslash y] \leqslant C\Big).$$

Equivalently, since the $y' \in Y$ meet generate $\mathcal{G}(X, Y, \leqslant)$,

$$\forall x, y, y' \; \forall C \; \Big(C \leqslant y' \;\Rightarrow\; [(C \oplus y) \,/\, x] \otimes [x \oslash y] \leqslant y'\Big). \tag{16}$$

Instantiating $C = y'$, (16) implies

$$\forall x, y, y' \; \Big([(y' \oplus y) \,/\, x] \otimes [x \oslash y] \leqslant y'\Big). \tag{17}$$

On the other hand, assuming (17), we have that for all x, y, y', C:

$$\begin{aligned} C \leqslant y' \quad &\Rightarrow \quad C \oplus y \leqslant y' \oplus y \\ &\Rightarrow \quad (C \oplus y) \,/\, x \leqslant (y' \oplus y) \,/\, x \\ &\Rightarrow \quad [(C \oplus y) \,/\, x] \otimes [x \oslash y] \leqslant [(y' \oplus y) \,/\, x] \otimes [x \oslash y]. \end{aligned}$$

Using $[(y' \oplus y) \,/\, x] \otimes [x \oslash y] \leqslant y'$ from (17), this gives (16), thus the axiom is equivalent to the first-order condition (17), which is by residuation equivalent to

$$\forall x, y, y' \; \Big((y' \oplus y) \,/\, x \leqslant y' \,/ (x \oslash y)\Big). \tag{18}$$

We reformulate this in order to obtain a first-order condition in terms of relations. Condition (18) holds if and only if

$$\forall x, x', y, y' \; \Big(x' \leqslant (y' \oplus y) \,/\, x \;\Rightarrow\; x' \leqslant y' \,/ (x \oslash y)\Big). \tag{19}$$

By residuation, the antecedent $x' \leqslant (y' \oplus y) \,/\, x$ is equivalent to $x' \otimes x \leqslant y' \oplus y$, which is equivalent to

$$\forall x_1 \Big[x_1 \leqslant x' \otimes x \;\Rightarrow\; x_1 \leqslant y' \oplus y\Big].$$

By definition, this is

$$\forall x_1 \Big[x_1 \leqslant x' \otimes x \;\Rightarrow\; R_{\oplus}(x_1, y', y)\Big],$$

and $x_1 \leqslant x' \otimes x$ is equivalent to $R_{\otimes}^{\downarrow}(x', x, x_1)$, since for all y such that $R_{\otimes}(x', x, y)$, it holds that $x_1 \leqslant x' \otimes x \leqslant y$. In total, (19) is equivalent to

$$\forall x, x', y, y' \; \Big(\forall x_1 \Big[R_{\otimes}^{\downarrow}(x', x, x_1) \Rightarrow\; R_{\oplus}(x_1, y', y)\Big] \Rightarrow\; x' \leqslant y' \,/ (x \oslash y)\Big). \tag{20}$$

The conclusion $x' \leqslant y' \,/ (x \oslash y)$ is equivalent to $x \oslash y \leqslant x' \setminus y'$, which is

$$\forall x_2 \; \Big[x_2 \leqslant x \oslash y \;\Rightarrow\; x_2 \leqslant x' \setminus y'\Big]. \tag{21}$$

The conclusion of (21), $x_2 \leqslant x' \setminus y'$, is equivalent to $x' \otimes x_2 \leqslant y'$, and this can be written as $R_\otimes(x', x_2, y')$.

The antecedent of (21) is equivalent to

$$\forall y_1 \Big[x \oslash y \leqslant y_1 \;\Rightarrow\; x_2 \leqslant y_1\Big],$$

which we will write as $R^\downarrow_\oslash(x, y, x_2)$. Thus, (21) becomes

$$\forall x_2 \Big[R^\downarrow_\oslash(x, y, x_2) \;\Rightarrow\; R_\otimes(x', x_2, y')\Big].$$

Inserting this into (20), gives the following condition in terms of relations:

$$\forall x, x', y, y' \;\Big(\forall x_1 \Big[R^\downarrow_\otimes(x', x, x_1) \Rightarrow\; R_\oplus(x_1, y', y)\Big] \Rightarrow\; \forall x_2 \Big[R^\downarrow_\oslash(x, y, x_2) \Rightarrow\; R_\otimes(x', x_2, y')\Big]\Big),$$

which is

$$\forall x, x', x_2, y, y' \;\Big[\Big(\forall x_1 \Big[R^\downarrow_\otimes(x', x, x_1) \Rightarrow\; R_\oplus(x_1, y', y)\Big] \text{ and } R^\downarrow_\oslash(x, y, x_2)\Big) \Rightarrow\; R_\otimes(x', x_2, y')\Big].$$

5.3.2 Equivalence of restricted interaction axioms

Notice that the first-order condition (18) is precisely the shape of the interaction axiom (e) listed in (IV). What has happened however is that the second-order variables A, B, C in the join-preserving coordinates of the operations have been replaced by variables from X, while the variables in the meet-preserving coordinates have been replaced by variables from Y. It is interesting that there is some kind of calculus behind this that gives the same replacements in axioms (b), (c) and (d). The resulting restricted axioms are mutually interderivable and are particularly compact descriptions but are nevertheless clearly first-order on the frames. As noted below, variants (a) and (f) have ambivalent slots and must be treated differently.

The restricted version of axiom (e) is equivalent to

$$\forall x, x', y, y' \;\Big(x' \leqslant (y' \oplus y) \,/\, x \;\Rightarrow\; x' \leqslant y' \,/ (x \oslash y)\Big),$$

which, by residuation, is equivalent to

$$\forall x, x', y, y' \;\Big(x' \otimes x \leqslant y' \oplus y \;\Rightarrow\; x \oslash y \leqslant x' \setminus y'\Big). \tag{22}$$

Applying residuation with respect to the different operations gives that (22) is equivalent to the restricted versions of axioms (b), (c) and (d):

$$\forall x, x', y, y' \;\Big(x \leqslant x' \setminus (y' \oplus y) \;\Rightarrow\; x \leqslant (x' \setminus y') \oplus y\Big),$$

$$\forall x, x', y, y' \;\Big((x' \otimes x) \oslash y \leqslant y' \;\Rightarrow\; x' \otimes (x \oslash y) \leqslant y'\Big),$$

$$\forall x, x', y, y' \;\Big(y' \obslash (x' \otimes x) \leqslant y \;\Rightarrow\; (x' \setminus y') \obslash x \leqslant y\Big).$$

As for axioms (a) and (f), the replacement of the second-order variables into variables from X and Y cannot be done directly, since there are operations in these axioms that have conflicting demands concerning the join- resp. meet-preservingness for some slots. For axiom (a) we get the following semi-restricted version directly:

$$\forall x', y, C \ \Big((x' \setminus C) \oslash y \leqslant x' \setminus (C \oslash y) \Big).$$

This is equivalent to

$$\forall x, x', y, C \ \Big(x \leqslant x' \setminus C \ \Rightarrow \ x \oslash y \leqslant x' \setminus (C \oslash y) \Big),$$

which is

$$\forall x, x', y, y', C \ \Big((x \leqslant x' \setminus C \text{ and } C \oslash y \leqslant y') \ \Rightarrow \ x \oslash y \leqslant x' \setminus y' \Big).$$

The conditions in the antecedent are by residuation equivalent to $x' \otimes x \leqslant C$ and $C \leqslant y' \oplus y$, thus we have

$$\forall x, x', y, y' \ \Big(x' \otimes x \leqslant y' \oplus y \ \Rightarrow \ x \oslash y \leqslant x' \setminus y' \Big)$$

as a restricted version of axiom (a).

To find the restricted version of axiom (f), we start with

$$\forall x, B, C \ \Big(x \oslash (B \obslash C) \leqslant (C \,/\, x) \setminus B \Big).$$

Since the $x' \in X$ join-generate $\mathcal{G}(X, Y, \leqslant)$ and $\setminus$ is order reversing in its first coordinate, this is

$$\forall x, x', B, C \ \Big(x' \leqslant C \,/\, x \ \Rightarrow \ x \oslash (B \obslash C) \leqslant x' \setminus B \Big).$$

Similarly,

$$\forall x, x', y', B, C \ \Big((x' \leqslant C \,/\, x \text{ and } B \leqslant y') \ \Rightarrow \ x \oslash (B \obslash C) \leqslant x' \setminus y' \Big),$$

and

$$\forall x, x', y, y', B, C \ \Big((x' \leqslant C \,/\, x, B \leqslant y', B \obslash C \leqslant y) \ \Rightarrow \ x \oslash y \leqslant x' \setminus y' \Big).$$

Applying residuation to the conditions in the antecedent yields

$$\forall x, x', y, y' \ \Big(x' \otimes x \leqslant y' \oplus y \ \Rightarrow \ x \oslash y \leqslant x' \setminus y' \Big),$$

thus (f) has the same restricted version as (a).

Note that this restricted version of (a) and (f) is exactly (22), so that the restricted versions of all the axioms listed in (IV) are equivalent. Since the restricted version of axiom IV(c) is equivalent to the frame condition imposed by axiom IV(b) (and thus by our family of axioms), this means any restricted axiom can be used to find this frame condition.

6 Results

In this paper we considered in detail the interaction axioms from Grishin's fourth group. By proving the canonicity of these axioms and by providing a first-order correspondent such that the axioms hold on a canonical extension if and only if the correspondent holds on the Lambek-Grishin frame associated with that canonical extension, we obtained complete relational semantics for $LG_\emptyset$ extended with this group of axioms.

This approach can be used to handle other additional axioms as well, and these results are summarised in the following theorem. The strength of this approach lies in the fact that these extensions work in a modular way — one can extend with any combination of additional axioms that satisfy canonicity, and will find complete relational semantics with respect to the intersection of the corresponding frame classes.

Theorem: For $LG_\emptyset$ extended with any subset of the following axioms, we obtain complete relational semantics with respect to the intersection of the corresponding frame classes.

Additional axioms	Frame class
Grishin's first group	$\forall x, x', x_1, y, y' \left[\left(\forall x_2 \, [R^{\downarrow}_{\oslash}(y, x', x_2) \Rightarrow R_{\otimes}(x_2, x, y')] \text{ and } R^{\downarrow}_{\otimes}(x', x, x_1)\right) \Rightarrow R_{\oplus}(x, y, y')\right]$
Grishin's second group	$\forall x_1, x_2, x_3, y \left(\forall x \, [R^{\downarrow}_{\otimes}(x_1, x_2, x) \Rightarrow R_{\otimes}(x, x_3, y)] \Rightarrow \forall x \, [R^{\downarrow}_{\otimes}(x_2, x_3, x) \Rightarrow R_{\otimes}(x_1, x, y)]\right)$
Grishin's third group	$\forall x, y_1, y_2, y_3 \left(\forall y \, [R^{\downarrow}_{\oplus}(y, y_1, y_2) \Rightarrow R_{\oplus}(x, y, y_3)] \Rightarrow \forall y \, [R^{\downarrow}_{\oplus}(y, y_2, y_3) \Rightarrow R_{\oplus}(x, y_1, y)]\right)$
Grishin's fourth group	$\forall x, x', x_2, y, y' \left[\left(\forall x_1 \, [R^{\downarrow}_{\otimes}(x', x, x_1) \Rightarrow R_{\oplus}(x_1, y', y)] \text{ and } R^{\downarrow}_{\oslash}(x, y, x_2)\right) \Rightarrow R_{\otimes}(x', x_2, y')\right]$
$\wedge, \vee, 0, 1,$ satisfying the bounded lattice axioms	
distributive lattice operations	$(X, Y, \leqslant) \cong (P, P, \not\geqslant)$ for some poset P
$A \otimes (B \otimes C) \dashv\vdash (A \otimes B) \otimes C$	$\forall x_1, x_2, x_3, y \left(\forall x \, [R^{\downarrow}_{\otimes}(x_1, x_2, x) \Rightarrow R_{\otimes}(x, x_3, y)] \Leftrightarrow \forall x \, [R^{\downarrow}_{\otimes}(x_2, x_3, x) \Rightarrow R_{\otimes}(x_1, x, y)]\right)$
$A \oplus (B \oplus C) \dashv\vdash (A \oplus B) \oplus C$	$\forall x, y_1, y_2, y_3 \left(\forall y \, [R^{\downarrow}_{\oplus}(y, y_1, y_2) \Rightarrow R_{\oplus}(x, y, y_3)] \Leftrightarrow \forall y \, [R^{\downarrow}_{\oplus}(y, y_2, y_3) \Rightarrow R_{\oplus}(x, y_1, y)]\right)$
$A \otimes B \dashv\vdash B \otimes A$	$\forall x_1, x_2, y \, [R_{\otimes}(x_1, x_2, y) \Leftrightarrow R_{\otimes}(x_2, x_1, y)]$
$A \oplus B \dashv\vdash B \oplus A$	$\forall x, y_1, y_2 \, [R_{\oplus}(x, y_1, y_2) \Leftrightarrow R_{\oplus}(x, y_2, y_1)]$
$A \otimes B \vdash A$	$\forall x_1, x_2, y \, [x_1 \leqslant y \Rightarrow R_{\otimes}(x_1, x_2, y)]$
$A \vdash A \oplus B$	$\forall x, y_1, y_2 \, [x \leqslant y_1 \Rightarrow R_{\oplus}(x, y_1, y_2)]$
$A \vdash A \otimes A$	$\forall x, y \, [R_{\otimes}(x, x, y) \Rightarrow x \leqslant y]$
$A \oplus A \vdash A$	$\forall x, y \, [R_{\oplus}(x, y, y) \Rightarrow x \leqslant y]$

6.1 Grishin's interaction axioms in the presence of $(\ )^{\perp}$

Next to the axiomatic extensions listed in the table, one can also enrich the connective type. Here we consider adding a linear negation in the fashion of [DL02].Given a poset $(P, \vdash)$ with two binary operations $\otimes$ and $\oplus$, we define a unary operation $(\ )^{\perp}$ that links the two binary ones, such that the following is satisfied for all $A, B, C \in P$:

$$\begin{array}{c} A \vdash B \;\Leftrightarrow\; B^{\perp} \vdash A^{\perp}, \\ (A \otimes B)^{\perp} \dashv\vdash A^{\perp} \oplus B^{\perp}, \\ (A \oplus B)^{\perp} \dashv\vdash A^{\perp} \otimes B^{\perp}, \\ A \vdash C \oplus B^{\perp} \;\Leftrightarrow\; A \otimes B \vdash C \;\Leftrightarrow\; B \vdash A^{\perp} \oplus C, \\ B^{\perp} \otimes C \vdash A \;\Leftrightarrow\; C \vdash B \oplus A \;\Leftrightarrow\; C \otimes A^{\perp} \vdash B. \end{array} \tag{23}$$

We will use the following abbreviations:

$$A \backslash C := A^{\perp} \oplus C,\; C \,/\, B := C \oplus B^{\perp},\; B \oslash C := B^{\perp} \otimes C, \text{ and } C \oslash A := C \otimes A^{\perp}.$$

Clearly, (23) implies that $(\otimes, /, \backslash, \oplus, \oslash, \oslash)$ satisfies the rules of $LG_{\emptyset}$. The link between $\otimes$ and $\oplus$, through $(\ )^{\perp}$, influences the behaviour of Grishin's groups of interaction axioms: whereas the four groups *a priori* have no dependencies, we will see that the presence of $(\ )^{\perp}$ makes all four groups equivalent to each other.

Since the axioms within each group are interderivable, the equivalence of two groups follows from sharing one axiom. Note that, in writing the axioms in terms of $\otimes, \oplus$ and $(\ )^{\perp}$, we can simply leave out the operation $(\ )^{\perp}$ on individual elements. We are allowed to do so since an axiom is a statement about *all* $A, B, C \in P$, and since there will be no occurrences of both an element and its dual in the same axiom.

First group $\Leftrightarrow$ second group

Axiom I(c) is $A \oslash (B \otimes C) \vdash (A \oslash B) \otimes C$. This is $A^{\perp} \otimes (B \otimes C) \vdash (A^{\perp} \otimes B) \otimes C$. As we may leave out the operation $(\ )^{\perp}$ on A, this yields the axiom $A \otimes (B \otimes C) \vdash (A \otimes B) \otimes C$, which is axiom II(c).

Fourth group $\Leftrightarrow$ first group

In a similar way, axiom IV(b), $A \backslash (B \oplus C) \vdash (A \backslash B) \oplus C$, is equivalent to $A \oplus (B \,/\, C) \vdash (A \oplus B) \,/\, C$, which is axiom I(b).

Third group $\Leftrightarrow$ second group

Axiom III(c) is $(A \oslash B) \oslash C \vdash A \oslash (B \oslash C)$, which is $(A^{\perp} \otimes B) \otimes C^{\perp} \vdash A^{\perp} \otimes (B \otimes C^{\perp})$. This is equivalent to $(A \otimes B^{\perp}) \otimes C \vdash A \otimes (B^{\perp} \otimes C)$. Using the first and then the second property of (23), this yields $A^{\perp} \oplus (B \oplus C^{\perp}) \vdash (A^{\perp} \oplus B) \oplus C^{\perp}$, which is equivalent to $A \backslash (B \,/\, C) \vdash (A \backslash B) \,/\, C$, which in turn is axiom II(b).

Thus, in this particular situation, Grishin's groups of interaction axioms are all equivalent. Next, we will show that the first group (and thus each group) holds if and only if the connective $\otimes$ is associative, and similarly if and only if the connective $\oplus$ is associative.

The first group implies associativity of $\otimes$, since axiom I(c) is

$$A \otimes (B \otimes C) \vdash (A \otimes B) \otimes C,$$

and the reverse inequality is obtained using axiom I(e), I(c), and again I(e):

$$(A \otimes B) \otimes C \vdash A \otimes (C \otimes B) \vdash (A \otimes C) \otimes B \vdash A \otimes (B \otimes C).$$

On the other hand, if $\otimes$ is associative, then in particular axiom I(c) holds. From the rules in (23), it is clear that $\otimes$ is associative if and only if $\oplus$ is associative.

The above findings apply to the multiplicative fragment of linear logic: the connectives $\otimes, \invamp$ and $(\ )^{\perp}$ satisfy the equations listed in (23), where $\invamp$ plays the role of $\oplus$. On top of that, $\otimes$ and $\invamp$ are associative operations, thus the results above show that this fragment of linear logic is an example of a logic for which all Grishin's groups of interaction axioms hold.

7 Conclusion

In this paper we presented our work on generalised Kripke semantics for the Lambek-Grishin calculus. The soundness and completeness theorem for the basic calculus $LG_{\emptyset}$ is easily obtainable from the analogous result for the non-associative Lambek calculus. A more interesting piece of work concerns the extensions of $LG_{\emptyset}$ with interaction axioms.

Traditional Kripke semantics for these logics have the drawback that, as soon as additional axioms or additional connectives are present, one may have to start over to obtain semantics for such richer logics. The advantage of our approach via canonical extensions of LG-algebras is that each additional axiom that lifts to the canonical extension gives a first-order condition on the corresponding frame that can be handled in a modular way, whereas additional connectives modularly slot in as additional relational components. We have seen that all groups of axioms presented by Grishin in [Gri83] satisfy canonicity, and we have obtained Sahlqvist-style correspondence results for each of these. The modular set-up allows us to augment this by the correspondence results for associativity, commutativity, weakening and contraction, given in [DGP05]. Thus, we have provided complete relational semantics for various possible extensions of the Lambek-Grishin calculus.

For each of Grishin's groups of additional axioms, we have found that the first-order frame condition imposed by such an axiom is in fact equivalent to a version of the initial axiom itself, in which the second-order variables have been replaced by completely join resp. meet irreducible elements of the complex algebra. The resulting restricted axioms are particularly compact descriptions but are nevertheless clearly first-order on the frames.

In future exploration, the approach taken in this paper can be applied to obtain complete relational semantics for various substructural logics. For example, [CGv11] considers generalized Kripke semantics for full linear logic.

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