DISTRIBUTIVE ENVELOPES AND TOPOLOGICAL DUALITY FOR LATTICES VIA CANONICAL EXTENSIONS

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Abstract. We establish a topological duality for bounded lattices. The two main features of our duality are that it generalizes Stone duality for bounded distributive lattices, and that the morphisms on either side are not the standard ones. A positive consequence of the choice of morphisms is that those on the topological side are functional.

In the course of the paper, we obtain the following results: (1) canonical extensions of bounded lattices are the algebraic versions of the existing dualities for bounded lattices by Urquhart and Hartung, (2) there is a universal construction which associates to an arbitrary lattice two distributive lattice envelopes with an adjoint pair between them, and (3) we give a topological duality for bounded lattices which generalizes Priestley duality and which shows precisely which maps between bounded lattices admit functional duals. For the result in (1), we rely on previous work of Gehrke, Jónsson and Harding. For the universal construction in (2), we modify a construction of the injective hull of a semilattice by Bruns and Lakser, adjusting their concept of 'admissibility' to the finitary case. For (3), we use Priestley duality for distributive lattices and our universal characterization of the distributive envelopes.

1. Introduction

Topological duality for Boolean algebras [22] and distributive lattices [23] is a useful tool for studying relational semantics for propositional logics. Canonical extensions [16, 17, 11, 10] provide a way of looking at these semantics algebraically. In the absence of a satisfactory topological duality, canonical extensions have been used [3] to treat relational semantics for substructural logics. The relationship between canonical extensions and topological dualities in the distributive case suggests that canonical extensions should be taken into account when looking for a topological duality for arbitrary bounded[1] lattices. The main aim of this paper is to investigate this line of research.

Let us outline our approach to duality for arbitrary lattices, by analogy with the more special case of distributive lattices. The starting point of the representation theory of distributive lattices is the following classical theorem of Birkhoff.

The authors would like to thank Andrew Craig and Hilary Priestley for their input in the initial stages of the project leading to this paper. The first-named author would like to thank ANR 2010 BLAN 0202 02 FREC for partial support during the later stages of this paper. The PhD research project of the second-named author has been made possible by NWO grant 617.023.815 of the Netherlands Organization for Scientific Research (NWO).

[1]From here on, we will drop the adjective 'bounded', adopting the convention that all lattices considered in this paper are bounded.

Theorem 1.1 (Birkhoff)**.** *Any finite distributive lattice D is isomorphic to the lattice of downsets of the partially ordered set of join-irreducible elements of D.*

Proof. The isomorphism between D and the lattice of downsets of the partially ordered set $(J(D), \leq_D)$ of join-irreducible elements is given by sending $a \in D$ to the set

$$\widehat{a} := {\downarrow} a \cap J(D) = \{p \in D \mid p \text{ join-irreducible}, p \leq a\}. \qquad \square$$

Thus, for finite distributive lattices, the poset $J(D)$ of join-irreducible elements determines the lattice D. Examining the proof sketched here in some more detail, it is not hard to see that the same proof goes through for all *perfect* distributive lattices, that is, for completely distributive complete lattices in which the completely join-irreducible elements are $\bigvee$-dense [21].[2]

Corollary 1.2 (Raney [21])**.** *Any perfect distributive lattice C is isomorphic to the lattice of downsets of the partially ordered set of completely join-irreducible elements of C.*

Topological dualities for distributive lattices (Stone [23], Priestley [20]) generalize Birkhoff's result to distributive lattices which are no longer assumed to be finite. The points of the dual space $X(D)$ of a distributive lattice D are now defined to be the *prime filters* of D. Any element $a \in D$ yields a subset $\widehat{a} := \{F \in X(D) \mid a \in F\}$ of $X(D)$. Taking the collection $\{\widehat{a} : a \in D\}$ as a basis for the open sets of a topology on $X(D)$ yields the *Stone dual space* of D.
We will now explain how Birkhoff's representation of perfect distributive lattices relates to the Stone dual space. Consider the situation depicted in Figure 1.

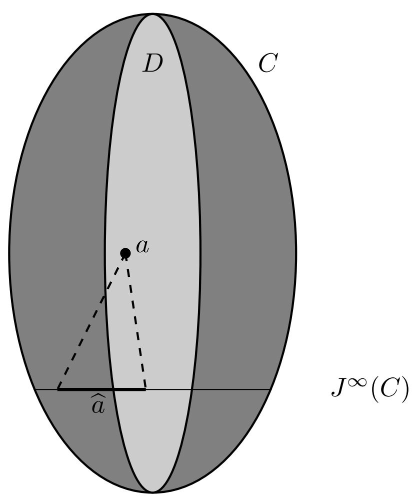


FIGURE 1. A topological space from an embedding $D \hookrightarrow C$

Here, $e : D \hookrightarrow C$ is an embedding of a distributive lattice D into some perfect distributive lattice C. By Corollary 1.2, since C is perfect, the lattice C is determined entirely by the poset $J^\infty(C)$ of completely join-irreducible elements of C.

[2]These lattices go by many different names. They are also called doubly algebraic, bi-algebraic, or completely prime-algebraic distributive lattices.

Any $a \in D$ drops a 'shadow' on $J^\infty(C)$, namely the set $\widehat{a} := {\downarrow} e(a) \cap J^\infty(C)$. Thus, *$D$ can be represented as a collection of subsets of $J^\infty(C)$.* One can now use this collection of subsets $\{\widehat{a} : a \in D\}$ to define a topology on $J^\infty(C)$.

We emphasize that 'topologizing' $J^\infty(C)$ in this manner is possible for *any* embedding $e : D \hookrightarrow C$. However, for an arbitrary choice of embedding $e : D \hookrightarrow C$ into a perfect lattice, the topological space thus obtained will not in general allow one to recover D. For the *canonical* extension $e : D \hookrightarrow D^\delta$ of a distributive lattice D, on the other hand, the following theorem holds.

Theorem 1.3 (Gehrke, Jónsson [11]). *Let D be a distributive lattice. The canonical extension of D is the unique embedding of D into a perfect lattice for which the associated topological space of completely join-irreducibles is exactly the Stone dual space of D.*

The definition of canonical extension, that we will give in Section 2 for arbitrary lattices, is purely algebraic and does not refer to the Stone dual itself. In this sense, canonical extensions of distributive lattices provide an algebraic description of Stone duality.

In the case of an arbitrary lattice L, an analogue of Stone duality is not available, but canonical extensions are [10]. However, Birkhoff's representation of finite and perfect distributive lattices to the non-distributive setting has been generalized [6, 7]: a *perfect lattice* C is a complete lattice in which $J^\infty(C)$ is $\bigvee$-dense and $M^\infty(C)$, the set of completely meet-irreducibles, is $\bigwedge$-dense. The structure $(J^\infty(C), M^\infty(C), \leq)$ completely describes the perfect lattice C (see Section 2 for details). Therefore, the canonical extension L^δ, which is a perfect lattice, may be used to *define* a dual space for a lattice L, in a way completely analogous to the distributive case. This is the approach that we will take in this paper.

We give a brief outline of the rest of this paper. In Section 2, we will recall the basics of canonical extensions and show how they can be used to put existing topological dualities, both for distributive and for arbitrary lattices, in a common framework. This will enable us to provide some examples which show that the topological dual of an arbitrary lattice may lack properties that the topological dual of a distributive lattice always has, such as sobriety and coherence. In the light of these 'bad' properties, we consider an alternative to topology in the form of quasi-uniform spaces in Section 3. The bicompletions of the quasi-uniform spaces introduced there are the Stone dual spaces of certain distributive lattices, that we call distributive envelopes, a name that is justified by the theory we develop in Section 4. The results in that section will enable us, in Section 5, to define a class of 'admissible' morphisms between lattices, which are precisely the maps that admit liftings to the distributive envelopes. Still in Section 5, we then obtain a topological duality for the category of lattices with admissible morphisms which generalizes Stone duality for the category of distributive lattices with homomorphisms.

2. Existing dualities from canonical extensions

As outlined in the introduction, we aim to use canonical extensions to obtain topological representations of arbitrary lattices, which generalize Stone duality. In this section, we will first derive Stone's dual topological space from the canonical extension of a distributive lattice, and then also Urquhart's and Hartung's dual objects from the canonical extension of an arbitrary lattice. We will finish the section by showing that the topological dual frame of a lattice obtained in this way may have undesirable properties.

We first briefly recall the basics of canonical extensions for arbitrary lattices.

Definition 2.1. Let L be a lattice. A *canonical extension* of L is an embedding $e : L \hookrightarrow C$ of L into a complete lattice C satisfying

(i) (dense) For all $u \in C$, we have

$$\bigvee \left\{ \bigwedge e[S] \mid S \subseteq L, \bigwedge e[S] \leq u \right\} = u = \bigwedge \left\{ \bigvee e[T] \mid T \subseteq L, u \leq \bigvee e[T] \right\}.$$

(ii) (compact) For all $S, T \subseteq L$, if $\bigwedge e[S] \leq \bigvee e[T]$ in C, then there are finite $S' \subseteq S$ and $T' \subseteq T$ such that $\bigwedge S' \leq \bigvee T'$ in L.

We denote by $F(L)$ and $I(L)$ the sets of *filters* and *ideals* of a lattice L, respectively. If $e : L \hookrightarrow C$ is an embedding of L into a complete lattice C, we let $F_e(C) := \{\bigwedge e[S] : S \subseteq L\}$ and $I_e(C) := \{\bigvee e[T] : T \subseteq L\}$ be the meet- and join-closure of L in C. In the case where C is a canonical extension of L, we call elements of $F_e(C)$ *filter elements*, and elements of $I_e(C)$ *ideal elements*. With this terminology, the 'denseness' condition in the definition of canonical extension can be rephrased as: *the filter elements are $\bigvee$-dense and the ideal elements are $\bigwedge$-dense.* The following proposition justifies this terminology: filter and ideal elements of a canonical extension correspond to filters and ideals of the lattice L.

Proposition 2.2. *Let $e : L \hookrightarrow C$ be a canonical extension of L. Define*

$$\phi_F : F(L) \to F_e(C) \text{ by } F \mapsto \bigwedge e[F],$$

$$\phi_I : I(L) \to I_e(C) \text{ by } I \mapsto \bigvee e[I].$$

Then the following properties hold.

(i) *ϕ_F and ϕ_I are bijections, which moreover satisfy $F \supseteq G \iff \phi_F(F) \leq \phi_F(G)$ and $I \subseteq J \iff \phi_I(I) \leq \phi_I(J)$;*

(ii) *For $F \in F(L)$ and $I \in I(L)$, we have $\phi_F(F) \leq \phi_I(I)$ iff $F \cap I \neq \emptyset$;*

(iii) *For $a \in L$, we have $\phi_F(\uparrow a) = e(a) = \phi_I(\downarrow a)$.*

Proof. For item (i), see [10], Lemma 3.3. The second and third items follow from the compactness property of the canonical extension. □

The following two theorems now say that every lattice has a unique canonical extension.

Theorem 2.3. *Let L be a lattice. There exists a canonical extension $e : L \hookrightarrow C$.*

Proof ([10]*, Proposition 2.6*). Consider the Galois connection $u : \mathcal{P}(F(L)) \leftrightarrows \mathcal{P}(I(L)) : l$ given by

$$u(A) := \{I \in I(L) \mid \forall F \in A : F \cap I \neq \emptyset\} \qquad (A \subseteq F(L)),$$

$$l(B) := \{F \in F(L) \mid \forall I \in B : F \cap I \neq \emptyset\} \qquad (B \subseteq I(L)).$$

Since (u, l) is a Galois connection, lu is a closure operator on $\mathcal{P}(F(L))$. Let C be the complete lattice of stable sets, i.e., $C := \{A \subseteq F(L) \mid lu(A) = A\}$. Define the map $e : L \to C$ by sending $a \in L$ to $e(a) := \{F \in F(L) \mid a \in F\}$. One may check that e is a well-defined dense and compact embedding. □

Theorem 2.4. *Let L be a lattice. Suppose $e : L \hookrightarrow C$ and $e' : L \hookrightarrow C'$ are canonical extensions of L. Then there is a complete lattice isomorphism $\phi : C \to C'$ such that $\phi \circ e = e'$.*

Proof. See [10], Proposition 2.7. □

Because of the existence and uniqueness of the canonical extension of a lattice, we may henceforth speak of *the* canonical extension of a lattice L, and we will denote it by L^δ. We will usually omit reference to the embedding e, and regard L as a sublattice of L^δ.
We remarked above that the denseness condition says that the filter elements are $\bigvee$-dense in L^δ. In fact, one may show, using the axiom of choice, that the set of completely join-irreducible elements is already $\bigvee$-dense in L^δ.

Proposition 2.5 (Canonical extensions are perfect lattices)**.** *Let L be a lattice, L^δ its canonical extension. Then the set of completely join-irreducible elements $J^\infty(L^\delta)$ is $\bigvee$-dense in L^δ, and the set of completely meet-irreducible elements $M^\infty(L^\delta)$ is $\bigwedge$-dense in L^δ.*

Proof. See [10], Lemma 3.4. □

We now expand on the claim that canonical extensions of distributive lattices 'capture' Stone duality. We saw in Proposition 2.2 that filters and ideals of a lattice L correspond to elements of the canonical extension L^δ. In the case of distributive lattices, *prime* filters can also be retrieved as particular elements of the canonical extension, namely, as the completely join-irreducible elements.

Proposition 2.6. *Let D be a distributive lattice. The bijection ϕ_F from Proposition 2.2 restricts to a bijection between the set of prime filters and the set $J^\infty(D^\delta)$. Also, ϕ_I restricts to a bijection between the set of prime ideals and the set $M^\infty(D^\delta)$.*

Proof. Note that if $x \in J^\infty(D^\delta)$, then by denseness, $x = \bigvee\{\bigwedge e[S] \mid S \subseteq L, \bigwedge e[S] \leq x\}$, so $x = \bigwedge e[S]$ for some $S \subseteq L$, and indeed $x \in F_e(L)$. If F is a prime filter, then $\bigwedge e[F]$ is a completely join-irreducible element of D^δ, as can be seen easily using compactness and the assumption that F is a prime filter. Conversely, if $x \in J^\infty(D^\delta)$, then one easily checks that $F := {\uparrow}x \cap D$ is a prime filter of D, using distributivity of D^δ. The statement about prime ideals is order dual. □

Corollary 2.7. *Let D be a distributive lattice. In D^δ, the completely join-irreducible elements are completely join-prime.*

Proof. By the proposition, a completely join-irreducible element of D^δ is equal to $\bigwedge e[F]$ for a prime filter F. Using compactness, one may show that meets of prime filters are completely join-prime. □

In fact, the bijection from Proposition 2.6 extends to a homeomorphism, establishing that the Stone dual space of a distributive lattice can be retrieved from the canonical extension, thus proving part of Theorem 1.3.

Proposition 2.8. *Let D be a distributive lattice. Let X_1 be the space of prime filters of D, whose topology is given by taking $\{\{F \in X_1 : a \in F\} : a \in D\}$ as a basis for the open sets. Let X_2 be the space $J^\infty(D^\delta)$, whose topology is given by taking $\{{\downarrow}a \cap J^\infty(D^\delta) : a \in D\}$ as a basis for the open sets. Then ϕ_F is a homeomorphism from X_1 to X_2.*

Proof. By Proposition 2.6, ϕ_F is a bijection. One may easily check that it carries basic opens to basic opens, so that it is open and continuous. □

A slight variant of this proposition shows that the *Priestley dual space* of D can also be retrieved from the canonical extension: the Priestley topology on X_1 is given by taking $\{\{F \in X_1 : a \in F\}, \{F \in X_1 : a \notin F\} : a \in D\}$ as a subbasis for the open sets, and correspondingly the topology on X_2 is given by taking $\{{\downarrow}a \cap J^\infty(D^\delta), ({\downarrow}a)^c \cap J^\infty(D^\delta) : a \in D\}$ as a subbasis for the open sets. The *Priestley order* of reverse inclusion of prime filters on X_1 corresponds to the restriction of the order of the canonical extension to X_2.
Given a distributive lattice D, we will denote the Stone dual space of D by D_*. One may use the presentation of the dual space as X_2 to prove Stone's duality result using canonical extensions. We will not do this completely, as it is not our main aim here, but we illustrate the idea by giving a quick proof of Stone's representation theorem for distributive lattices.

Proposition 2.9. *Let D be a distributive lattice. Then D is isomorphic to the lattice of compact open sets of D_*.*

Proof. We regard the set D_* as $J^\infty(D^\delta)$. For any $a \in D$, recall that $\widehat{a} := \{x \in D_* \mid x \leq a\}$ is the basic open set corresponding to a. We prove that $\widehat{a}$ is compact. If $\{\widehat{t}\}_{t\in T}$ is a cover of $\widehat{a}$ by basic opens, then, since $J^\infty(D^\delta)$ is $\bigvee$-dense in D^δ, we get $a \leq \bigvee T$. By the compactness property of D^δ, there is a finite subset $T' \subseteq T$ such that $a \leq \bigvee T'$. It follows that $\{\widehat{t}\}_{t\in T'}$ covers $\widehat{a}$. Since $J^\infty(D^\delta)$ is $\bigvee$-dense in D^δ, it now follows that the map $a \mapsto \widehat{a}$ is an order embedding from D into the lattice of compact open sets of D_*. We now show it is surjective. Let K be a compact open subset of D_*. Then $K = \bigcup\{\widehat{a} : \widehat{a} \subseteq K\}$ since K is open, and therefore $K = \bigcup_{i=1}^n \widehat{a_i}$ for some $a_1, \ldots, a_n \in D$ since K is compact. By Corollary 2.7, join-irreducibles of D^δ are join-prime, so it follows that $K = \widehat{\bigvee_{i=1}^n a_i}$. □

We have explained above how *the canonical extension of a distributive lattice captures the dual space*. We will now show that this idea can be generalized to arbitrary lattices.
Following the successful program of Stone duality for distributive lattices via canonical extensions, a first attempt at defining a topological dual for an arbitrary lattice L would be to take $J^\infty(L^\delta)$ with topology generated by taking the sets $\widehat{a}$, $a \in L$, as a basis for the open sets, where $\widehat{a} := {\downarrow}a \cap J^\infty(L^\delta) = \{x \in J^\infty(L^\delta) : x \leq a\}$. However, this first attempt does not work as nicely for arbitrary lattices as it did for distributive lattices. As a case in point, the following example shows that one can not hope to retrieve the lattice L from only the poset $J^\infty(L^\delta)$, even for finite lattices (where the topology does not carry any additional information, as it is given by the order).

Example 2.10. Recall that the canonical extension of any finite lattice is equal to the lattice itself: if L is finite, then L is complete, and $L \hookrightarrow L$ is a compact and dense embedding. In particular, the poset $J^\infty(L^\delta)$ is simply the poset of join-irreducibles of L.
Let L and L' be the lattices depicted in Figure 2. Note that both posets $J^\infty(L^\delta)$ and $J^\infty((L')^\delta)$ of join-irreducibles of $L = L^\delta$ and $L' = (L')^\delta$ are the three element antichain: the 'new' element in L' is not join-irreducible. The topologies are discrete, so the spaces $J^\infty(L^\delta)$ and $J^\infty((L')^\delta)$ are homeomorphic.

FIGURE 2. Two non-isomorphic lattices L and L', whose posets J^∞ are isomorphic.

Observe that the posets of *meet*-irreducibles of L and L' are not isomorphic. Thus, in a representation of lattices, both J^∞ and M^∞ will need to play a role.

The above example indicates that *two-sorted* dual structures are more natural when representing arbitrary lattices. This observation has led to the development of *formal concept analysis* [6].

Definition 2.11. A *polarity* (*context*)[3] is a tuple $P = \langle X, Y, R\rangle$, where X and Y are sets and $R \subseteq X \times Y$ is a relation. Any polarity naturally induces a Galois connection $u : \mathcal{P}(X) \leftrightarrows \mathcal{P}(Y) : l$, where

$$u(A) := \{y \in Y \mid \forall a \in A : aRy\} \qquad (A \subseteq X),$$
$$l(B) := \{x \in X \mid \forall b \in B : xRb\} \qquad (B \subseteq Y).$$

Since (u, l) is a Galois connection, the map $A \mapsto lu(A)$ is a closure operator on the poset $\mathcal{P}(X)$. A set $A \in \mathcal{P}(X)$ is called a *stable set* (*closed concept*) if $lu(A) = A$. The collection of stable sets of a polarity P, ordered by inclusion, forms a complete lattice, that we will denote by P^+.

Note that with this definition, we may rephrase the proof of the existence of the canonical extension (Theorem 2.3 above) as follows. For an arbitrary lattice L, its canonical extension L^δ can be constructed as the complete lattice $(P_L)^+$ from the polarity $P_L = \langle X_L, Y_L, R_L\rangle$, where $X_L = F(L)$, $Y_L = I(L)$, and $F\, R_L\, I$ iff $F \cap I \neq \emptyset$.
For finite lattices, one arrives at the following generalization of Birkhoff's representation theorem (Theorem 1.1).

Theorem 2.12. *Let L be a finite lattice. Let $L_+ = \langle J(L), M(L), \leq\rangle$, where $J(L)$ is the set of join-irreducibles, $M(L)$ is the set of meet-irreducibles, and $\leq$ is the restriction of the order of L to $J(L) \times M(L)$. Then L is isomorphic to $(L_+)^+$.*

Proof. There is a function $L \to (L_+)^+$ which sends an element $a \in L$ to the stable set $ul({\downarrow}a \cap J(L)) \in (L_+)^+$. One may readily check that this is indeed a lattice isomorphism. □

Remark 2.13. To see that this theorem indeed generalizes Theorem 1.1, note that the join-irreducible elements of a finite distributive lattice D are the join-prime elements, and the meet-irreducible elements are the meet-prime elements. Moreover, there is a bijection between the sets of join- and meet-primes which sends a join-prime element $x \in J(D)$ to $\kappa(x)$, defined as the largest element of D

[3]A note on terminology. In this paper, we largely follow the terminology of "Generalized Kripke Frames" [7]. In this definition, to accomodate the reader who is more familiar with the terminology of formal concept analysis [6], we give the equivalent terms between parentheses.

which is not above x. One may show that $\kappa(x)$ is a meet-prime and κ defines a bijection between $J(D)$ and $M(D)$, and $x \leq a$ iff $a \nleq \kappa(x)$. So the polarity D_+ is isomorphic to the polarity $\langle J(D), J(D), \ngeq \rangle$. The stable subsets of $J(D)$ are exactly the downsets.

As with Birkhoff's theorem, the following is a straightforward corollary of the proof of Theorem 2.12.

Corollary 2.14. *Let C be a perfect lattice. Let $C_+ = \langle J^\infty(C), M^\infty(C), \leq \rangle$, where $J^\infty(C)$ is the set of completely join-irreducibles, $M^\infty(C)$ is the set of completely meet-irreducibles, and $\leq$ is the restriction of the order of C to $J^\infty(C) \times M^\infty(C)$. Then C is isomorphic to $(C_+)^+$.*

We will now outline how one may arrive at dual structures for an arbitrary lattice from its canonical extension. This will be a generalization of the approach to Stone duality from canonical extensions outlined above. We use the canonical extension L^δ of an arbitrary lattice L to describe the topological dual structures defined by Urquhart [24] and Hartung [15].
The following generalization of Proposition 2.6 is crucial. Following Urquhart [24], we call $(F, I) \in F(L) \times I(L)$ a *maximal pair* if F is maximal among all filters disjoint from I and I is maximal among all ideals disjoint from F.

Proposition 2.15. *Let L be a lattice. The bijection ϕ_F from Proposition 2.2 restricts to a bijection between the set $J^\infty(L^\delta)$ and the set of filters F which are in a maximal pair (F, I), for some ideal I. Also, ϕ_I restricts to a bijection between the set $M^\infty(L^\delta)$ and the set of ideals I which are in a maximal pair (F, I), for some filter F.*

Proof. See [10], Lemma 3.4. □

In a distributive lattice D, a filter F is in some maximal pair (F, I) iff the filter F is prime. Moreover, if F is a prime filter, then $I = F^c$ is an ideal, so it is the *unique* ideal such that (F, I) is a maximal pair.
We briefly recall how Hartung [15] defined the dual structure of a lattice, and then show that a more transparent definition of the same object can be given using the canonical extension of L. For an arbitrary lattice L, Hartung defines its dual *topological polarity* $L_* := \langle X_L, Y_L, R_L \rangle$ as follows. The points of X_L are the filters which are in some maximal pair, the points of Y_L are the ideals which are in some maximal pair, and R_L is the relation defined by $F \, R_L \, I$ iff $F \cap I \neq \emptyset$. The topology on X_L is defined by taking the sets $\{F \in X_L \mid a \in F\}$, for $a \in L$, as a subbasis for the closed sets, and the topology on Y_L is defined by taking the sets $\{I \in Y_L \mid b \in I\}$, for $b \in L$, as a subbasis for the closed sets.
From the point of view of the canonical extension, the topological polarity dual to a lattice L can be equivalently defined as follows (cf. Figure 3). For $a \in L$ we define $\widehat{a} := {\downarrow} a \cap J^\infty(L^\delta)$ and $\check{a} := {\uparrow} a \cap M^\infty(L^\delta)$. Let τ_c^J be the topology on $J^\infty(L^\delta)$ given by taking $\{\widehat{a} : a \in L\}$ as a subbasis for the closed sets. Let τ_c^M be the topology on $M^\infty(L^\delta)$ given by taking $\{\check{a} : a \in L\}$ as a subbasis for the closed sets. It may come as a surprise that the sets $\widehat{a}$ are taken to be *closed*, whereas these sets are the compact *open* sets in the usual topology on the Stone spectrum of a distributive lattice. However, we will see in Example 2.20, among other things, that the topology on $J^\infty(L^\delta)$ obtained by taking the sets $\widehat{a}$ to be open may fail to be compact when L is a non-distributive lattice.

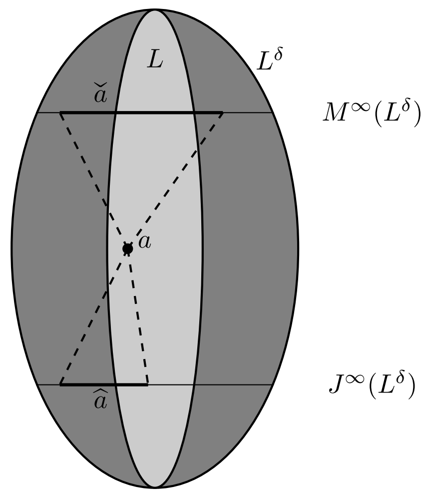


FIGURE 3. Topological spaces from the embedding of a lattice L into its canonical extension L^δ.

The following proposition is a precise statement of Remark 2.10 in [10].

Proposition 2.16. *Let L be a lattice. Define the topological polarity*

$$L_* := \langle (J^\infty(L^\delta), \tau_c^J), (M^\infty(L^\delta), \tau_c^M), \leq \rangle.$$

Then L_* is isomorphic to Hartung's topological polarity $\langle X_L, Y_L, R_L \rangle$.

Proof. The bijections ϕ_F and ϕ_I from Proposition 2.2 restrict to homeomorphisms between X_L and $J^\infty(L^\delta)$, and between Y_L and $M^\infty(L^\delta)$, respectively, and $\phi_F(F) \leq \phi_F(I)$ iff $F \, R_L \, I$. □

In the same Remark 2.10 of [10], the following fact is left to the reader as a "non-trivial exercise". This fact now follows immediately from the above results.

Proposition 2.17. *The map $L \to (L_*)^+$ defined by $a \mapsto \widehat{a}$ is the canonical extension of L.*

Proof. Combine Proposition 2.5, Corollary 2.14 and Proposition 2.16. □

To show that the structure L_* is sufficient to recover the original lattice L, one needs the following remarkable fact, which is the non-distributive analogue of Proposition 2.9.

Proposition 2.18. *The lattice L is isomorphic to the sublattice C_L of $(L_*)^+$, which consists of those $A \in \mathcal{P}(J^\infty(L^\delta))$ such that both A and $u(A)$ are closed in the topologies τ_c^J and τ_c^M, respectively.*

Proof (Sketch). By Proposition 2.17 and the definitions of the topologies τ_c^J and τ_c^M, it is clear that L order-embeds into the sublattice C_L via $a \mapsto \widehat{a}$. To show that this embedding is surjective, one needs the non-trivial argument which is given in the proof of Proposition 2.1.7 of Hartung [14] and uses the axiom of choice in the form of Rado's Selection Theorem [13]. □

Remark 2.19. Before Hartung, Urquhart [24] had already defined the dual structure of a lattice to be a doubly ordered topological space $\langle Z, \tau, \leq_1, \leq_2 \rangle$ whose points are maximal pairs (F, I). We briefly outline how this structure can be obtained from the canonical extension.
Let P be the subset of $J^\infty(L^\delta) \times M^\infty(L^\delta)$ consisting of pairs (x, y) such that $x \not\leq_{L^\delta} y$, i.e., P is the set-theoretic complement of the relation R_L of L_*. Then P inherits the subspace topology from the product topology $\tau_c^J \times \tau_c^M$ on $J^\infty(L^\delta) \times M^\infty(L^\delta)$. We define an order $\preceq$ on P by $(x, y) \preceq (x', y')$ iff $x \geq_{L^\delta} x'$ and $y \leq_{L^\delta} y'$; in other words, $\preceq$ is the restriction of the product of the dual order and the usual order of L^δ. Urquhart's space $\langle Z, \tau \rangle$ then corresponds to the subspace of $\preceq$-maximal points of P, and the orders $\leq_1$ and $\leq_2$ correspond to the projections of the order $\preceq$ onto the first and second coordinate, respectively.

We have shown how Hartung's and Urquhart's constructions of the dual structure of a lattice both arise naturally from the embedding of a lattice L into its canonical extension L^δ. This perspective enables us to prove that the spaces which occur in Hartung's duality may be rather badly behaved. We will show that the dual structures of an arbitrary lattice may lack the nice properties that dual spaces of distributive lattices always have. In particular, the topology on the dual space may not be sober, and the compact open sets will not in general be closed under finite intersections.

Example 2.20 (A lattice whose dual topology is not sober)**.** Let L be a countable antichain with top and bottom, as depicted in Figure 4.

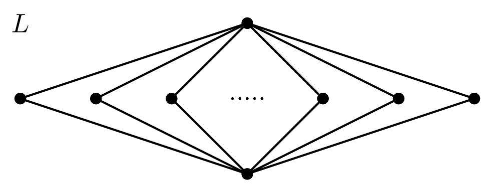


FIGURE 4. The lattice L, a countable antichain with top and bottom.

One may easily show that $L \hookrightarrow L$ is a canonical extension, so $L = L^\delta$. The set $J^\infty(L)$ is the countable antichain (as is the set $M^\infty(L)$). The topology τ_c^J is the cofinite topology on a countable set, which is not sober: the entire space is itself a closed irreducible subset which is not the closure of a point.
Also note that if one instead would define a topology on $J^\infty(L)$ by taking the sets $\widehat{a}$, for $a \in L$, to be open, instead of closed, then one obtains the discrete topology on $J^\infty(L)$, which is in particular not compact.

In the light of the above example, one may wonder whether the *sobrification* of the space $(J^\infty(L), \tau_c^J)$ may have better properties, and in particular whether it will be spectral. However, the following example shows that it can not be, since the frame of opens of the space $(J^\infty(L), \tau_c^J)$ in this example is not *arithmetic*: intersections of compact opens do not need to be compact open.

Example 2.21 (A lattice whose dual topology is not spectral)**.** Consider the lattice K depicted in Figure 5. In this figure, the elements of the original lattice K are

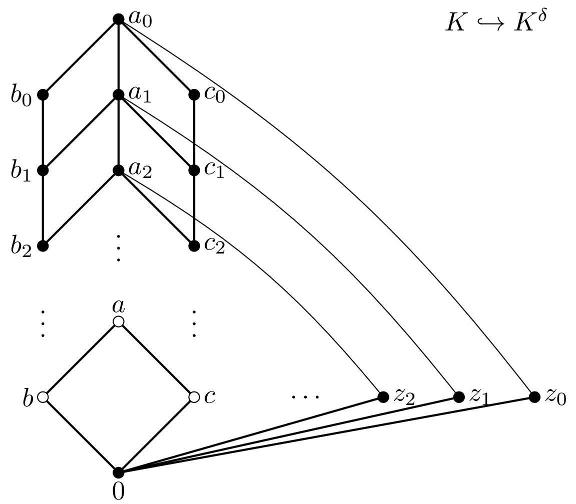


FIGURE 5. The lattice K, for which $(J^\infty(K^\delta), \tau_c)$ is not spectral.

drawn as filled dots, and the three additional elements a, b and c of the canonical extension K^δ are drawn as white dots.

The set $J^\infty(K^\delta)$ is $\{b_i, c_i, z_i \mid i \geq 0\} \cup \{b, c\}$. We take $\{\widehat{a} : a \in K\}$ as a subbasis for the closed sets, so $\{(\widehat{a})^c : a \in K\}$ is a subbasis for the open sets.

In particular, $(\widehat{b_0})^c$ and $(\widehat{c_0})^c$ are compact open sets. However, their intersection is not compact: $\{(\widehat{a_n})^c\}_{n=0}^\infty$ is an open cover of $(\widehat{b_0})^c \cap (\widehat{c_0})^c = \{z_i : i \geq 0\}$ with no finite subcover.

The above examples indicate that the spaces obtained from Hartung's duality can be badly behaved. In particular, they do not fit into the framework of the duality between sober spaces and spatial frames. The individual spaces which occur in Hartung's duality may fail to be the Stone duals of any distributive lattice.

In the next section, we will show how one may put a *quasi-uniform structure* rather than a topology on the sets J^∞ and M^∞ to associate a quasi-uniform space to a lattice. This will naturally lead to the definition of the distributive envelope of a lattice.

3. Quasi-uniform spaces associated with a lattice

So far we have obtained a pair of representations $L \hookrightarrow \mathcal{P}(X_L), a \mapsto \widehat{a}$ and $L \hookrightarrow \mathcal{P}(Y_L)$, $a \mapsto \breve{a}$ where $X_L = J^\infty(L^\delta)$ and $Y_L = M^\infty(L^\delta)$ (see figure 3). In the case of a distributive lattice, the topological space obtained by taking the sets representing the elements of L as a basis is a particularly nice sober space. For a lattice in general this is not so, as we have seen at the end of Section 2 above.

In this section we consider what happens if we take instead the *Pervin quasi-uniform space* generated by the representations of L in $\mathcal{P}(X_L)$ and $\mathcal{P}(Y_L)$, respectively. As seen in [9] this links up with duality theory for bounded distributive lattices. It will allow us to give a spatial meaning to certain distributive lattices associated with a lattice L.

Uniform spaces generalize metric spaces beyond the setting of spaces with a countable basis. Uniform spaces, as well as metric spaces, encode more information than topologies and this approach will allow us to encode important information about a lattice. In a metric space, for each $\varepsilon > 0$, we get, at each point of the space, an ε-ball; this is modeled in a uniform space, X, by a subset $U \subseteq X \times X$ that should be thought of as the set of all (x, y) so that y is in the ε-ball centered at x. Quasi-uniform spaces is a non-symmetric generalization of uniform spaces. We give the basic notions needed here and for further details on the theory of uniform spaces and quasi-uniform spaces we refer to [1] and [4], respectively.

A *quasi-uniform space* is a pair $(X,\mathcal{U})$, where X is a set, and $\mathcal{U}$ is a collection of subsets of $X \times X$ having the following properties:

(i) $\mathcal{U}$ is a filter of subsets of $X \times X$ contained in the up-set of the diagonal $\Delta = \{(x, x) \mid x \in X\}$;
(ii) for each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$;

The collection $\mathcal{U}$ is called a quasi-uniformity and its elements are called *entourages*. The condition (ii) corresponds to the triangle inequality. A quasi-uniform space is said to be a *uniform space* provided $\mathcal{U}$ is symmetric in the sense that the converse, U^{-1}, of each entourage U is again an entourage of the space.

A function $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ between quasi-uniform spaces is *uniformly continuous* provided $(f \times f)^{-1}(V) \in \mathcal{U}$ for each $V \in \mathcal{V}$. Sometimes we will write $f : X \to Y$ is $(\mathcal{U},\mathcal{V})$-uniformly continuous instead to express this fact. A quasi-uniform space $(X,\mathcal{U})$ always gives rise to a topological space. This is the space X with the *induced topology*, which is given by letting $V \subseteq X$ be open provided that, for each $x \in V$, there is $U \in \mathcal{U}$ such that $U(x) = \{y \in X \mid (x, y) \in U\} \subseteq V$. Just as for metric spaces, in general, several different quasi-uniformities on X give rise to the same topology. We will assume that all spaces are *Komolgorov*, that is, the induced topology is T_0. A quasi-uniform space is Kolmogorov if, and only if, for any two distinct points $x, y \in X$, there is an entourage U with $(x, y) \notin U$ or $(y, x) \notin U$. This requirement is equivalent to the intersection of all the entourages in $\mathcal{U}$ being a partial order rather than just a quasi-order on X.

Given a set X, we denote, for each subset $A \subseteq X$, by U_A the subset

$$(A^c \times X) \cup (X \times A) = \{(x, y) \mid x \in A \implies y \in A\}$$

of $X \times X$. Given a topology τ on X, the filter $\mathcal{U}_\tau$ in the power set of $X \times X$ generated by the sets U_A for $A \in \tau$ is a quasi-uniformity on X. The quasi-uniform spaces $(X,\mathcal{U}_\tau)$ were first introduced by Pervin [19] and are now known in the literature as Pervin spaces. Generalizing this idea, given any subcollection $\mathcal{K} \subseteq \mathcal{P}(X)$, we define $(X,\mathcal{U}_\mathcal{K})$ to be the quasi-uniform space whose quasi-uniformity is the filter generated by the entourages U_A for $A \in \mathcal{K}$. Here we will call this larger class of quasi-uniform spaces *Pervin spaces*.

The first crucial point is that, for any collection $\mathcal{K} \subseteq \mathcal{P}(X)$, the bounded distributive sublattice $D(\mathcal{K})$ of $\mathcal{P}(X)$ generated by $\mathcal{K}$ may be recovered from $(X,\mathcal{U}_\mathcal{K})$ even though this cannot be done in general from the associated topology. The generated sublattice is recovered as the blocks of the quasi-uniform space: The *blocks* of a space $(X,\mathcal{U})$ are the subsets $A \subseteq X$ such that U_A is an entourage of the space, or equivalently, those for which the characteristic function $\chi_A : X \to 2$ is uniformly continuous with respect to the Sierpiński quasi-uniformity on 2, which is the one containing just 2^2 and $\{(0, 0), (1, 1), (1, 0)\}$.

Further, it is not hard to see that if $\mathcal{D} \subseteq \mathcal{P}(Y)$ and $\mathcal{E} \subseteq \mathcal{P}(X)$ are bounded sublattices of the respective power sets, then a map $f : (X, \mathcal{U}_{\mathcal{E}}) \to (Y, \mathcal{U}_{\mathcal{D}})$ is uniformly continuous if and only if f^{-1} induces a lattice homomorphism from $\mathcal{D}$ to $\mathcal{E}$ by restriction. Thus, the category of sublattices of power sets with morphisms that are commuting diagrams

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\;h\;} & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{P}(Y) & \xrightarrow{\;\phi\;} & \mathcal{P}(X) \end{array}$$

where ϕ is a complete lattice homomorphism, is dually isomorphic to the category of Pervin spaces with uniformly continuous maps.

To be able to state the main result from [9] that we want to apply here, we need to define *bicompletions* of quasi-uniform spaces. For more details see [4, Chapter 3]. Bicompleteness generalizes the notion of completeness for uniform spaces, which is well-understood (see, e.g., [1, Chapter II.3]): a uniform space $(X, \mathcal{U})$ is *complete* if every Cauchy filter converges. Here, we recall that a proper filter $\mathcal{F}$ of $\mathcal{P}(X)$ is *Cauchy* if every entourage $U \in \mathcal{U}$ contains a set of the form $F \times F$ for some $F \in \mathcal{F}$, and the filter $\mathcal{F}$ is said to *converge* to $x \in X$ if $U(x) \in \mathcal{F}$ for all $U \in \mathcal{U}$. In a uniformity that comes from a metric space, Cauchy filters correspond exactly to Cauchy sequences.

Now let $(X, \mathcal{U})$ be a quasi-uniform space. The converse, $\mathcal{U}^{-1}$, of the quasi-uniformity $\mathcal{U}$, consisting of the converses U^{-1} of the entourages $U \in \mathcal{U}$, is again a quasi-uniformity on X. Further, the *symmetrisation*, $\mathcal{U}^s$, which is the filter of $\mathcal{P}(X \times X)$ generated by the union of $\mathcal{U}$ and $\mathcal{U}^{-1}$, is a uniformity on X. A quasi-uniform space $(X, \mathcal{U})$ is called *bicomplete* if and only if $(X, \mathcal{U}^s)$ is a complete uniform space. It has been shown by Fletcher and Lindgren [4, Chapter 3.3] that the full subcategory of bicomplete quasi-uniform spaces forms a reflective subcategory of the category of quasi-uniform spaces with uniformly continuous maps. Thus, for each quasi-uniform space $(X, \mathcal{U})$, there is a bicomplete quasi-uniform space $(\widetilde{X}, \widetilde{\mathcal{U}})$ and a uniformly continuous map $\eta_X : (X, \mathcal{U}) \to (\widetilde{X}, \widetilde{\mathcal{U}})$ with a universal property.

Now we are ready to state the main result of Section 1 of [9]: The set representation of a distributive lattice D given by Stone/Priestley duality is obtainable from *any* bounded lattice embedding $e : D \hookrightarrow \mathcal{P}(X)$ of D into a power set by taking the *bicompletion* of the corresponding quasi-uniform Pervin space $(X, \mathcal{U}_{Im(e)})$. To be more precise, we have:

Theorem 3.1. [9, Theorem 1.6] *Let D be a bounded distributive lattice, and let $e : D \hookrightarrow \mathcal{P}(X)$ be any bounded lattice embedding of D in a power set lattice and denote by $\mathcal{D}$ the image of the embedding e. Let $\widetilde{X}$ be the bicompletion of the Pervin space $(X, \mathcal{U}_{\mathcal{D}})$. Then $\widetilde{X}$ with the induced topology is the Stone dual space of D.*

Alternatively, one can think of the quasi-uniform space $(\widetilde{X}, \widetilde{\mathcal{U}_{\mathcal{D}}})$ as an ordered uniform space, as follows. Equip the uniform space $(\widetilde{X}, \widetilde{\mathcal{U}^s_{\mathcal{D}}})$ with the order $\leq$ defined by $\bigcap_{a \in \mathcal{D}} U_{\widehat{a}}$. Then $(\widetilde{X}, \widetilde{\mathcal{U}^s_{\mathcal{D}}}, \leq)$ is a uniform version of the *Priestley dual space* of D.

We now want to apply this theorem to the setting of this paper. Let L be a bounded lattice, L^δ, the canonical extension of L, and $X_L = J^\infty(L^\delta)$ and $Y_L = M^\infty(L^\delta)$. Then L induces quasi-uniform space structures $(X_L, \mathcal{U}_{\widehat{L}})$ and $(Y_L, \mathcal{U}_{\check{L}})$ on X_L and

Y_L, respectively. Here $\mathcal{U}_{\widehat{L}}$ is the Pervin quasi-uniformity generated by the image $\widehat{L} = \{\widehat{a} \mid a \in L\}$ and $\mathcal{U}_{\check{L}}$ is the Pervin quasi-uniformity generated by the image $\check{L} = \{\check{a} \mid a \in L\}$. By Theorem 3.1, the bicompletions of these Pervin spaces are Stone spaces and the corresponding bounded distributive lattices are the sublattices of $\mathcal{P}(X_L)$ and $\mathcal{P}(Y_L)$ generated by $\widehat{L}$ and $\check{L}$, respectively.

Definition 3.2. Let L be a bounded lattice, $X_L = J^\infty(L^\delta)$, and $Y_L = M^\infty(L^\delta)$. Then we denote by $D^\wedge(L)$ the sublattice of $\mathcal{P}(X_L)$ generated by $\widehat{L} = \{\widehat{a} \mid a \in L\}$ and by $D^\vee(L)$ the sublattice of $\mathcal{P}(Y_L)$ generated by $\check{L}$. We call $D^\wedge(L)$ the *distributive* $\wedge$*-envelope* and $D^\vee(L)$ the *distributive* $\vee$*-envelope* of D. We will see in the next section that these names are appropriate.

The following theorem is then a corollary of Theorem 3.1.

Theorem 3.3. *Let L be a bounded lattice, then the bicompletion of the associated quasi-uniform Pervin space, $(X_L, \mathcal{U}_{\widehat{L}})$, is the dual space of the distributive $\wedge$-envelope, $D^\wedge(L)$, of L. Order dually, the bicompletion of the associated quasi-uniform Pervin space, $(Y_L, \mathcal{U}_{\check{L}})$, is the dual space of the distributive $\vee$-envelope, $D^\vee(L)$, of L.*

Example 3.4. For any finite lattice L, the distributive envelope $D^\wedge(L)$ is the lattice of downsets of the poset $J(L)$, with the order inherited from L. Thus, in the finite case, the quasi-uniform space X_L is already bicomplete, and hence equal to its own bicompletion. The same of course holds for $D^\vee(L)$ and Y_L. In the finite case, we therefore stay faithful to Hartung's duality.
For the lattice L discussed in Example 2.20 above, the distributive envelope $D^\wedge(L)$ is (isomorphic to) the lattice consisting of all finite subsets of the countable antichain, and a top element. Thus, in the bicompletion of X_L, we find one new point, corresponding to the prime filter consisting of only the top element.
For the lattice K discussed in Example 2.21, the distributive envelope $D^\wedge(K)$ is a much bigger lattice than K, and the bicompletion of X_L will contain many new points. In particular, the bicompletion will not just be the sobrification of X_L.

4. Distributive envelopes

In this section we will study the two *distributive envelopes* $D^\wedge(L)$ and $D^\vee(L)$ of L, which appeared in the previous section as the distributive lattices dual to the bicompletions of the quasi-uniform spaces $(X_L, \mathcal{U}_{\widehat{L}})$ and $(Y_L, \mathcal{U}_{\check{L}})$, respectively. We will give algebraic constructions of the distributive envelopes, and characterize the envelopes by universal properties. Some of the results in this section can be seen as finitary versions of the results on injective hulls of semilattices of Bruns and Lakser [5]. We will relate our results to theirs in Remark 4.16. However, the reader who is not familiar with [5] should be able to read this section independently.
We give an algebraic construction of the distributive $\wedge$-envelope $D^\wedge(L)$ of L. The construction of the distributive $\vee$-envelope $D^\vee(L)$ of L is order dual, cf. Remark 4.17 at the end of this section. The following definition is central, being the finitary version of the definition of *admissible* given in [5].

Definition 4.1. A finite subset $M \subseteq L$ is *join-admissible* if its join distributes over all meets with elements from L, i.e., if, for all $a \in L$,

$$a \wedge \bigvee M = \bigvee_{m \in M} (a \wedge m) \tag{1}$$

The join-admissible subsets of L are those subsets whose join 'is already distributive' in L. From the perspective of the canonical extension L^δ, a set is join-admissible iff the join-irreducibles behave like join-primes with respect to the join of that set. This is made precise in the lemma below.

Lemma 4.2. *Let L be a lattice and $M \subseteq L$ a finite subset. Then the following are equivalent.*

(i) *M is join-admissible;*
(ii) *For any $x \in J^\infty(L^\delta)$, if $x \leq \bigvee M$, then $x \leq m$ for some $m \in M$.*

Proof. (i) $\Rightarrow$ (ii). Suppose M is join-admissible, and let $x \in J^\infty(L^\delta)$ such that $x \leq \bigvee M$. Define $x' := \bigvee_{m \in M}(x \wedge m)$. It is obvious that $x' \leq x$. We show that $x \leq x'$. Let y be an ideal element of L^δ such that $x' \leq y$. Then, for each $m \in M$, we have $x \wedge m \leq y$, so by compactness, there exists $a_m \in L$ such that $x \leq a_m$ and $a_m \wedge m \leq y$. Let $a := \bigwedge_{m \in M} a_m$. Since M is join-admissible, we get

$$x \leq a \wedge \bigvee M = \bigvee_{m \in M} (a \wedge m) \leq \bigvee_{m \in M} (a_m \wedge m) \leq y.$$

Since y was an arbitrary ideal element above x', by denseness we conclude that $x \leq x'$. So $x = x' = \bigvee_{m \in M}(x \wedge m)$. Since x is join-irreducible, we get $x = x \wedge m$ for some $m \in M$, so $x \leq m$.

(ii) $\Rightarrow$ (i). Let $a \in L$ be arbitrary. Because the other inequality is obvious, we only need to show that $a \wedge \bigvee M \leq \bigvee_{m \in M}(a \wedge m)$ holds in L. We show the inequality holds in L^δ and use that $L \hookrightarrow L^\delta$ is an embedding. Let $x \in J^\infty(L^\delta)$ such that $x \leq a \wedge \bigvee M$. By (ii), pick $m \in M$ such that $x \leq m$. Then $x \leq a \wedge m \leq \bigvee_{m \in M}(a \wedge m)$. Since $x \in J^\infty(L^\delta)$ was arbitrary, by Proposition 2.5 we conclude $a \wedge \bigvee M \leq \bigvee_{m \in M}(a \wedge m)$. □

The above lemma will be our main tool in studying admissible sets. It is a typical example of the usefulness of canonical extensions: one can formulate an algebraic property (join-admissibility) in a spatial manner (using the 'points', i.e., completely join-irreducibles, of the canonical extension).

Note that the same proof goes through without the restriction that M is finite, if one extends the definition of join-admissibility to include infinite sets. We will not expand on this point here, because we will only need the result for finite sets.

To construct $D^\wedge(L)$, we want to 'add joins' to L. This can of course be done with ideals. In the case of $D^\wedge(L)$ the required ideals will be closed under admissible joins. To formalize this idea, we define a-ideals.

Definition 4.3. A subset $A \subseteq L$ is called an *a-ideal* if (i) A is a downset, i.e., if $a \in A$ and $b \leq a$ then $b \in A$, and (ii) A is closed under admissible joins, i.e., if $M \subseteq A$ is join-admissible, then $\bigvee M \in A$.

Remark 4.4. Note that any (lattice) ideal of a lattice L is in particular an a-ideal. Moreover, any intersection of a-ideals is again an a-ideal. In particular, the poset $\mathsf{aIdl}(L)$ of all a-ideals of L is a closure system. Therefore, for any subset T of L, there exists a smallest a-ideal containing T. We will denote this a-ideal by $\langle T \rangle_{ai}$ and call it *the a-ideal generated by* T. As usual, we say that an a-ideal A is *finitely generated* if there is a finite set T such that $A = \langle T \rangle_{ai}$. In a distributive lattice D, all joins are admissible, and a-ideals coincide with lattice ideals.

Recall that $D^\wedge(L)$ was defined in the previous section as the sublattice of $\mathcal{P}(J^\infty(L^\delta))$ generated by the sets $\widehat{a} := \{x \in J^\infty(L^\delta) : x \leq a\}$, where a ranges over the elements of L. Note that, for any finite subset $T \subseteq L$, we have

$$\bigcap_{a \in T} \widehat{a} = \widehat{\bigwedge T}.$$

Hence, an arbitrary element of $D^\wedge(L)$ is a finite union of sets of the form $\widehat{a}$, where $a \in L$. We aim to establish an isomorphism between the poset of finitely generated a-ideals and the poset $D^\wedge(L)$. To do so, the following criterium will be useful.

Lemma 4.5. *Let L be a lattice, $T \subseteq L$ a finite subset and $b \in L$. The following are equivalent:*

(i) *$b \in \langle T \rangle_{ai}$;*
(ii) *$\widehat{b} \subseteq \bigcup_{a \in T} \widehat{a}$;*
(iii) *There exists a finite join-admissible $M \subseteq {\downarrow} T$ such that $b = \bigvee M$.*

Proof. (i) $\Rightarrow$ (ii). Note that $A := \{b \in L : \widehat{b} \subseteq \bigcup_{a \in T} \widehat{a}\}$ is an a-ideal which contains T: it is clearly a downset, and it is closed under admissible joins, using Lemma 4.2. Hence, $b \in \langle T \rangle_{ai} \subseteq A$, as required.
(ii) $\Rightarrow$ (iii). Let $M := \{b \wedge a \mid a \in T\}$. We claim that $b = \bigvee M$ and M is join-admissible. Note that $\bigvee M \leq b$, so $\widehat{\bigvee M} \subseteq \widehat{b}$. Using (iii), we also get:

$$\widehat{b} = \widehat{b} \cap \bigcup_{a \in T} \widehat{a} = \bigcup_{a \in T} (\widehat{b} \cap \widehat{a}) = \bigcup_{a \in T} \widehat{b \wedge a} = \bigcup_{m \in M} \widehat{m} \subseteq \widehat{\bigvee M} \subseteq \widehat{b}.$$

Therefore, equality holds throughout, and in particular $b = \bigvee M$ and $\bigcup_{m \in M} \widehat{m} = \widehat{\bigvee M}$, so that M is join-admissible by Lemma 4.2.
(iii) $\Rightarrow$ (i). Since $\langle T \rangle_{ai}$ is a downset containing T, $\langle T \rangle_{ai}$ contains M, and therefore, being closed under admissible joins, it contains $b = \bigvee M$. $\square$

We can now prove that the finitely generated a-ideals form a lattice isomorphic to $D^\wedge(L)$.

Proposition 4.6. *Let ϕ be the function which sends a finitely generated a-ideal $A = \langle T \rangle_{ai}$ to the set $\bigcup_{a \in T} \widehat{a} \in D^\wedge(L)$. Then ϕ is a well-defined order isomorphism between the poset of finitely generated a-ideals and $D^\wedge(L)$.*

Proof. Let T and U be finite subsets of L. Note that if $\langle T \rangle_{ai} = A = \langle U \rangle_{ai}$, then in particular $b \in \langle T \rangle_{ai}$ for each $b \in U$, so $\widehat{b} \subseteq \bigcup_{a \in T} \widehat{a}$ by Lemma 4.5. Hence, $\bigcup_{b \in U} \widehat{b} \subseteq \bigcup_{a \in T} \widehat{a}$. The proof of the other inclusion is symmetric, so indeed $\bigcup_{a \in T} \widehat{a} = \bigcup_{b \in U} \widehat{b}$, and ϕ is well defined. This argument also shows that ϕ is order preserving. Finally, if $\phi(\langle U \rangle_{ai}) = \bigcup_{b \in U} \widehat{b} \subseteq \bigcup_{a \in T} \widehat{a} = \phi(\langle T \rangle_{ai})$, then for each $b \in U$ we have $\widehat{b} \subseteq \bigcup_{a \in T} \widehat{a}$, so by Lemma 4.5 we get $b \in \langle T \rangle_{ai}$. Since this holds for each $b \in U$, we get $\langle U \rangle_{ai} \subseteq \langle T \rangle_{ai}$, so ϕ is order reflecting. It is immediate from the definition of $D^\wedge(L)$ as finite unions of sets of the form $\widehat{a}$ that ϕ is surjective. $\square$

Note that this proposition implies in particular that the finitely generated a-ideals form a distributive lattice which is isomorphic to $D^\wedge(L)$. However, we have not said anything so far about the operations $\wedge$ and $\vee$ between finitely generated a-ideals. Clearly, the join of two finitely generated a-ideals is generated by the union of the sets of generators. Intersection is also well-behaved, as we will prove now.

Proposition 4.7. *Let L be an arbitrary lattice, and let T and U be finite subsets of L. Then*
$$\langle T\rangle_{ai} \cap \langle U\rangle_{ai} = \langle t \wedge u \mid t \in T, u \in U\rangle_{ai}.$$
In particular, the intersection of two finitely generated a-ideals is again finitely generated.

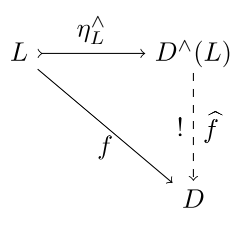
Proof. By Lemma 4.5, we have that
$$\langle T\rangle_{ai} \cap \langle U\rangle_{ai} = \{b \in L \mid \widehat{b} \subseteq \bigcup_{t\in T} \widehat{t} \text{ and } \widehat{b} \subseteq \bigcup_{u\in U} \widehat{u}\} = \{b \in L \mid \widehat{b} \subseteq \left(\bigcup_{t\in T} \widehat{t}\right) \cap \left(\bigcup_{u\in U} \widehat{u}\right)\}.$$
Note that
$$\left(\bigcup_{t\in T} \widehat{t}\right) \cap \left(\bigcup_{u\in U} \widehat{u}\right) = \bigcup_{t\in T, u\in U} (\widehat{t} \cap \widehat{u}) = \bigcup_{t\in T, u\in U} \widehat{t \wedge u}.$$
So we get that $\langle T\rangle_{ai} \cap \langle U\rangle_{ai} = \{b \in L \mid \widehat{b} \subseteq \bigcup_{t\in T, u\in U} \widehat{t \wedge u}\} = \langle t\wedge u \mid t \in T, u \in U\rangle_{ai}$, again by Lemma 4.5. □

We will now prove that $D^{\wedge}(L)$ and $D^{\vee}(L)$ satisfy universal properties, showing in what sense exactly they are 'distributive envelopes' of L. We will regard $D^{\wedge}(L)$ as the distributive lattice of finitely generated a-ideals, by Proposition 4.6. We denote by $\eta_L^{\wedge} : L \hookrightarrow D^{\wedge}(L)$ the embedding which sends $a \in L$ to $\eta_L^{\wedge}(a) := \langle a\rangle_{ai}$, which is simply the downset of a in L. We say that a function $f : L_1 \to L_2$ between lattices *preserves admissible joins* if, for each finite join-admissible set $M \subseteq L_1$, we have $f(\bigvee M) = \bigvee_{m\in M} f(m)$. We will show that $D^{\wedge}(L)$ has a universal property with respect to the class of maps which preserve finite meets and admissible joins.

Lemma 4.8. *Let L be a lattice. Then $\eta_L^{\wedge} : L \hookrightarrow D^{\wedge}(L)$ preserves finite meets and admissible joins.*

Proof. Clearly, $\eta_L^{\wedge}(a \wedge b) = \eta_L^{\wedge}(a) \cap \eta_L^{\wedge}(b)$ and $\eta_L^{\wedge}(1) = 1_{D^{\wedge}(L)}$. Let M be a finite join-admissible set. Then $\bigvee_{m\in M} \eta_L^{\wedge}(m) = \langle M\rangle_{ai}$. By Lemma 4.5, $b \in \langle M\rangle_{ai}$ iff $\widehat{b} \subseteq \bigcup_{m\in M} \widehat{m}$, and $\bigcup_{m\in M} \widehat{m} = \widehat{\bigvee M}$ by Lemma 4.2, since M is join-admissible. So $b \in \bigvee_{m\in M} \eta_L^{\wedge}(m) = \langle M\rangle_{ai}$ iff $\widehat{b} \subseteq \widehat{\bigvee M}$ iff $b \in \langle \bigvee M\rangle_{ai} = \eta_L^{\wedge}(\bigvee M)$. □

Theorem 4.9. *Let L be a lattice and D a distributive lattice. If $f : L \to D$ preserves finite meets and admissible joins, then there exists a unique homomorphism $\widehat{f} : D^{\wedge}(L) \to D$ such that $\widehat{f} \circ \eta_L^{\wedge} = f$.*



Proof. Let $f : L \to D$ be a function which preserves meets and admissible joins. If $g : D^{\wedge}(L) \to D$ is a homomorphism such that $g \circ \eta_L^{\wedge} = f$, then, for any finite subset $T \subseteq L$, we have
$$g(\langle T\rangle_{ai}) = g\left(\bigvee_{t\in T} \eta_L^{\wedge}(t)\right) = \bigvee_{t\in T} g(\eta_L^{\wedge}(t)) = \bigvee_{t\in T} f(t).$$

So there is at most one homomorphism $g : D^\wedge(L) \to D$ satisfying $g \circ \eta_L^\wedge = f$.
Let $\widehat{f} : D^\wedge(L) \to D$ be the function defined for a finite subset $T \subseteq L$ by

$$\widehat{f}(\langle T \rangle_{ai}) := \bigvee_{t \in T} f(t).$$

We show that $\widehat{f}$ is a well-defined homomorphism. For well-definedness, suppose that $\langle T \rangle_{ai} = \langle U \rangle_{ai}$ for some finite subsets $T, U \subseteq L$. Let $u \in U$ be arbitrary. We then have $u \in \langle T \rangle_{ai}$. By Lemma 4.5, $u = \bigvee M$ for some finite join-admissible $M \subseteq {\downarrow} T$. Using that f preserves admissible joins and order, we get

$$f(u) = f\left(\bigvee M\right) = \bigvee_{m \in M} f(m) \leq \bigvee_{t \in T} f(t).$$

Since $u \in U$ was arbitrary, we have shown that $\bigvee_{u \in U} f(u) \leq \bigvee_{t \in T} f(t)$. The proof of the other inequality is the same. We conclude that $\bigvee_{t \in T} f(t) = \bigvee_{u \in U} f(u)$, so $\widehat{f}$ is well-defined.
It is clear that $\widehat{f} \circ \eta_L^\wedge = f$. In particular, $\widehat{f}$ preserves 0 and 1, since f does. It remains to show that $\widehat{f}$ preserves $\vee$ and $\wedge$. Let $T, U \subseteq L$ be finite subsets. Then $\langle T \rangle_{ai} \vee \langle U \rangle_{ai} = \langle T \cup U \rangle_{ai}$, so

$$\widehat{f}(\langle T \rangle_{ai} \vee \langle U \rangle_{ai}) = \bigvee_{v \in T \cup U} f(v) = \bigvee_{t \in T} f(t) \vee \bigvee_{u \in U} f(u) = \widehat{f}(\langle T \rangle_{ai}) \vee \widehat{f}(\langle U \rangle_{ai}).$$

Using Proposition 4.7 and the assumptions that D is distributive and f is meet-preserving, we have

$$\begin{aligned}\widehat{f}(\langle T \rangle_{ai} \wedge \langle U \rangle_{ai}) = \bigvee_{t \in T, u \in U} f(t \wedge u) &= \bigvee_{t \in T, u \in U} (f(t) \wedge f(u)) \\ &= \bigvee_{t \in T} f(t) \wedge \bigvee_{u \in U} f(u) = \widehat{f}(\langle T \rangle_{ai}) \wedge \widehat{f}(\langle U \rangle_{ai}). \square\end{aligned}$$

It will be useful to know that the extension of an injective map to $D^\wedge(L)$ is injective.

Proposition 4.10. *Let L be a lattice, D a distributive lattice, and $f : L \to D$ a function which preserves finite meets and admissible joins. If f is injective, then the unique extension $\widehat{f} : D^\wedge(L) \to D$ is injective.*

Proof. Note that f is order reflecting, since f is meet-preserving and injective. Suppose that $\widehat{f}(\langle U \rangle_{ai}) \leq \widehat{f}(\langle T \rangle_{ai})$. We need to show that $\langle U \rangle_{ai} \subseteq \langle T \rangle_{ai}$. Let $u \in U$ be arbitrary. Then $f(u) \leq \widehat{f}(\langle U \rangle_{ai}) \leq \widehat{f}(\langle T \rangle_{ai}) = \bigvee_{t \in T} f(t)$. For any $a \in L$, we then have

$$\begin{aligned}f(a \wedge u) = f(a \wedge u) \wedge \bigvee_{t \in T} f(t) &= \bigvee_{t \in T} (f(a \wedge u) \wedge f(t)) \\ &= \bigvee_{t \in T} f(a \wedge u \wedge t) \leq f\left(\bigvee_{t \in T} (a \wedge u \wedge t)\right).\end{aligned}$$

Since f is order reflecting, we thus get $a \wedge u \leq \bigvee_{t \in T} (a \wedge u \wedge t)$. Since the other inequality is clear, we get

$$a \wedge u = \bigvee_{t \in T} (a \wedge u \wedge t).$$

In particular, putting $a = 1$, we see that $u = \bigvee_{t \in T}(u \wedge t)$, and the above equation then says that $\{u \wedge t \mid t \in T\}$ is join-admissible. So $u \in \langle T \rangle_{ai}$. We conclude that $U \subseteq \langle T \rangle_{ai}$, and therefore $\langle U \rangle_{ai} \subseteq \langle T \rangle_{ai}$. □

From Theorem 4.9, we can deduce that the assignment $L \mapsto D^{\wedge}(L)$ extends to an adjunction between categories. We first define the appropriate categories. We denote by **DL** the category of distributive lattices with homomorphisms. The relevant category of lattices is defined as follows.

Definition 4.11. We say a function $f : L_1 \to L_2$ between lattices is a ($\wedge$,a$\vee$)-*morphism* if f preserves finite meets and admissible joins, and, for any join-admissible set $M \subseteq L_1$, $f[M]$ is join-admissible. We denote by $\mathbf{L}_{\wedge,\mathbf{a}\vee}$ the category of lattices with ($\wedge$,a$\vee$)-morphisms between them. (The reader may verify that $\mathbf{L}_{\wedge,\mathbf{a}\vee}$ is indeed a category.)

We consider this definition in a bit more detail before we proceed, since it is central to what follows. Note that if the lattice L_2 is distributive, then the condition that f sends join-admissible sets to join-admissible sets is vacuously true, since any subset of a distributive lattice is join-admissible. This explains why we did not need to state the condition that f preserves join-admissible sets in the universal property of $D^{\wedge}(L)$ (Theorem 4.9). However, the following example shows that in general the condition 'f sends join-admissible sets to join-admissible sets' can not be omitted from the definition of ($\wedge$,a$\vee$)-morphism.

Example 4.12. *The composition $g \circ f$ of functions $f : L_1 \to L_2$ and $g : L_2 \to L_3$ between lattices which preserve meets and admissible joins need not preserve admissible joins.*
Let L_1 be the diamond distributive lattice, let L_2 be the three-element antichain with 0 and 1 adjoined, and let L_3 be the Boolean algebra with 3 atoms, as in Figure 6 below. Note that $L_3 = D^{\wedge}(L_2)$.

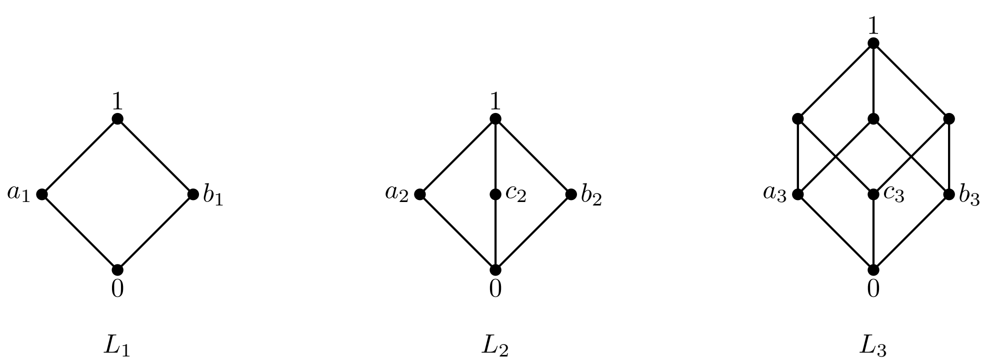


FIGURE 6. The lattices L_1, L_2 and L_3 from Example 4.12.

Let $f : L_1 \to L_2$ be the function defined by $f(0) = 0$, $f(1) = 1$, $f(x_1) = x_2$ for $x \in \{a, b\}$. Let $g : L_2 \to L_3$ be the function defined by $g(0) = 0$, $g(1) = 1$, $g(x_2) = x_3$ for $x \in \{a, b, c\}$. Then f is a lattice homomorphism (moreover, f is injective). Also note that $g = \eta^{\wedge}_{L_2}$, so it is in particular a $(\wedge, a\vee)$-morphism. However, the

composition $g \circ f$ does not preserve (admissible) joins: $gf(a_1 \vee b_1) = gf(1) = 1$, but $gf(a_1) \vee gf(b_1) = a_3 \vee b_3 \neq 1$.
Note that f, despite it being a homomorphism, does not send join-admissible sets to join-admissible sets: the image of $\{a_1, b_1\}$ is $\{a_2, b_2\}$, which is not join-admissible in L_2.

However, the following proposition shows that for *surjective* maps, the condition 'f sends join-admissible sets to join-admissible sets' can be omitted. It was already observed by Urquhart [24] that surjective maps are well-behaved for duality, and accordingly our duality in Section 5 also includes surjective lattice homomorphisms.

Proposition 4.13. *Suppose $f : L_1 \to L_2$ is a surjective function which preserves finite meets and admissible joins. Then f sends join-admissible sets to join-admissible sets (and therefore f is a morphism in $\mathbf{L}_{\wedge,\mathbf{a}\vee}$).*

Proof. Suppose $M \subseteq L_1$ is a join-admissible set. To show that $f[M]$ is join-admissible, first let $a \in L_1$ be arbitrary. Note that it follows from the definition of join-admissibility that $\{a \wedge m \mid m \in M\}$ is also join-admissible in L_1. So, using that f preserves meets and admissible joins, we get

$$f(a) \wedge \bigvee_{m \in M} f(m) = f(a \wedge \bigvee M) = f\left(\bigvee_{m \in M} (a \wedge m)\right) = \bigvee_{m \in M} (f(a) \wedge f(m)).$$

Since f is surjective, any $b \in L_2$ is of the form $b = f(a)$ for some $a \in L_1$. Hence, $f[M]$ is join-admissible. □

Note that, if L_1 and L_2 are distributive, then ($\wedge$,a$\vee$)-morphisms from L_1 to L_2 are exactly bounded lattice homomorphisms. Hence, we have a full inclusion of categories $I^\wedge : \mathbf{DL} \hookrightarrow \mathbf{L}_{\wedge,\mathbf{a}\vee}$. The following is now a consequence of the Theorem 4.9.

Corollary 4.14. *The functor $D^\wedge : \mathbf{L}_{\wedge,\mathbf{a}\vee} \to \mathbf{DL}$, which sends L to $D^\wedge(L)$ and a ($\wedge$,a$\vee$)-morphism $f : L_1 \to L_2$ to the unique homomorphic extension of $\eta^\wedge_{L_2} \circ f : L_1 \to D^\wedge(L_2)$, is left adjoint to $I^\wedge : \mathbf{DL} \hookrightarrow \mathbf{L}_{\wedge,\mathbf{a}\vee}$ and $\eta^\wedge$ is the unit of the adjunction. Moreover, the counit $\epsilon^\wedge : D^\wedge \circ I \to 1_{\mathbf{DL}}$ is an isomorphism.*

The following characterisation of $D^\wedge(L)$ now follows directly from the preceding results.

Corollary 4.15. *Let L be a lattice. If D is a distributive lattice and $f : L \to D$ is a function such that*

(i) *f preserves meets and admissible joins,*
(ii) *f is injective,*
(iii) *$f[L]$ is join-dense in D,*

then D is isomorphic to $D^\wedge(L)$ via the isomorphism $\widehat{f}$.

Proof. The homomorphism $\widehat{f}$ is injective by Proposition 4.10. It is surjective because $f[L]$ is join-dense in D and $\widehat{f}[D^\wedge(L)] = \{\bigvee f[T] \mid T \subseteq L\}$, by the construction of $\widehat{f}$ in the proof of Theorem 4.9. □

Remark 4.16. We compare our results in this section so far to those of Bruns and Lakser [5]. The equivalence of (i) and (ii) in Lemma 4.5 is very similar to the statement of Lemma 3 in [5]. Our proofs are different from those in [5] in making use of the canonical extension of L; in particular Lemma 4.2 has proven to be useful here. Our Corollary 4.15 is a finitary version of the characterisation in Corollary 2 of [5]. The fact that $D^\wedge$ is an adjoint to a full inclusion can also be seen as a finitary analogue of the result of [5] that their construction provides the injective hull of a meet-semilattice. Note that our construction of $D^\wedge(L)$ could also be applied to the situation where L is only a meet-semilattice, if we modify our definition of join-admissible sets to require that the relevant joins exist in L. The injective hull of L that was constructed in [5] can now be retrieved from our construction by taking the free directedly complete poset (dcpo) over the distributive lattice $D^\wedge(L)$. This is a special case of a general phenomenon, where frame constructions may be seen as a combination of a finitary construction, followed by a dcpo construction [18].

Remark 4.17. We outline the order dual version of the construction given above for later reference. A finite subset $M \subseteq L$ is *meet-admissible* if for all $a \in M$, $a \vee \bigwedge M = \bigwedge_{m \in M}(a \vee m)$. An *a-filter* is an upset which is closed under admissible meets. The poset of finitely generated a-filters of L, ordered by reverse inclusion, is isomorphic to $D^\vee(L)$, by sending the a-filter generated by a finite set T to $\bigcup_{a \in T} \check{a}$. Note that the order on a-filters is taken to be the reverse inclusion order, to ensure that the unit of the adjunction will be order preserving (and not order reversing). The universal property of Theorem 4.9 holds for $D^\vee(L)$, interchanging the words 'join' and 'meet' everywhere. We say $f : L_1 \to L_2$ is a $(\vee,\mathrm{a}\wedge)$-*morphism* if it preserves finite joins, admissible meets, and sends meet-admissible sets to meet-admissible sets. Then $D^\vee$ is a functor from the category $\mathbf{L}_{\vee,\mathbf{a}\wedge}$ to $\mathbf{DL}$ which is left adjoint to the full inclusion $I^\vee : \mathbf{DL} \to \mathbf{L}_{\vee,\mathbf{a}\wedge}$. We denote the unit of the adjunction by $\eta^\vee : 1_{\mathbf{L}_{\vee,\mathbf{a}\wedge}} \to I^\vee \circ D^\vee$. Finally, $D^\vee(L)$ is the (up to isomorphism) unique distributive meet-dense extension of L which preserves finite joins and admissible meets.

We end this section by examining additional structure which links the two distributive envelopes $D^\wedge(L)$ and $D^\vee(L)$, and enables us to retrieve L from the lattices $D^\wedge(L)$ and $D^\vee(L)$. Recall that in Section 2, we described the polarity $(J^\infty(L^\delta), M^\infty(L^\delta), \leq)$ for any lattice L. In Definition 2.11, we described an adjoint pair associated to any polarity. In particular, let $u_L : \mathcal{P}(J^\infty(L^\delta)) \leftrightarrows \mathcal{P}(M^\infty(L^\delta)) : l_L$ be the adjunction coming from the polarity $(J^\infty(L^\delta), M^\infty(L^\delta), \leq)$. Here, the map u_L sends a set $V \subseteq J^\infty(L^\delta)$ to the set $u_L(V) := \{y \in M^\infty(L^\delta) \mid \forall x \in V : x \leq_{L^\delta} y\}$. In particular, if $V = \widehat{a}$ for some $a \in L$, then $u_L(V) = u_L(\widehat{a}) = \check{a}$. Recall that the distributive lattice $D^\wedge(L)$ can be identified with its image in $\mathcal{P}(J^\infty(L^\delta))$ and $D^\vee(L)$ with its image in $\mathcal{P}(M^\infty(L^\delta))$. We then have:

Proposition 4.18. *For any lattice L, the maps u_L and l_L restrict to an adjoint pair $u_L : D^\wedge(L) \leftrightarrows D^\vee(L) : l_L$. The lattice of stable elements under this adjunction is isomorphic to L.*

Proof. Note that $D^\wedge(L)$ consists of finite unions of sets of the form $\widehat{a}$. If $T \subseteq L$, then we have

$$u_L\left(\bigcup_{a \in T} \widehat{a}\right) = \bigcap_{a \in T} \check{a} = \check{t},$$

where $t := \bigvee T$. From this, it follows that $u_L(D^\wedge(L)) \subseteq D^\vee(L)$, and the analogous statement for l_L is proved similarly. The lattice of stable elements under this adjunction is both isomorphic to the image of u_L in $D^\vee(L)$ and the image of l_L in $D^\wedge(L)$. Both of these lattices are clearly isomorphic to L. □

Note that in the presentation of $D^{\wedge}(L)$ and $D^{\vee}(L)$ as finitely generated a-ideals and a-filters, the maps u and l act as follows. Given an a-ideal I which is generated by a finite set $T \subseteq L$, $u_L(I)$ is the principal a-filter generated by $\bigvee T$. Conversely, given an a-filter F which is generated by a finite set $S \subseteq L$, $l_L(F)$ is the principal a-ideal generated by $\bigwedge S$.

5. Topological duality for arbitrary lattices

In this section, we show how the results of this paper can be applied to obtain a new duality for a category of lattices. This category of lattices will be the intersection of the categories $\mathbf{L}_{\wedge,\mathbf{a}\vee}$ and $\mathbf{L}_{\vee,\mathbf{a}\wedge}$ defined in the previous section: its objects are all bounded lattices, its morphisms are admissible homomorphisms (Definition 5.1). We combine the facts from the previous section with the existing Stone/Priestley duality for distributive lattices to obtain a duality for this category. Since the $\mathbf{L}_{\wedge,\mathbf{a}\vee}$-morphisms are *exactly* the morphisms which can be lifted to homomorphisms between the $D^{\wedge}$-envelopes, these morphisms also correspond exactly to the lattice morphisms which have functional duals between the X-sets of the dual frames; the same remark applies to $\mathbf{L}_{\vee,\mathbf{a}\wedge}$-morphisms and the Y-sets of the dual frames. The duals of admissible morphisms will be pairs of functions; one function being the dual of the '$\mathbf{L}_{\wedge,\mathbf{a}\vee}$-part' of the admissible morphism, the other being the dual of the '$\mathbf{L}_{\vee,\mathbf{a}\wedge}$-part' of the morphism.
The functors $D^{\wedge}$ and $D^{\vee}$ that we defined in Section 4 are left adjoint to inclusions of the category **DL** as a full subcategory in the categories $\mathbf{L}_{\wedge,\mathbf{a}\vee}$ and $\mathbf{L}_{\vee,\mathbf{a}\wedge}$, respectively. In the light of Proposition 4.18, we can combine $D^{\wedge}$ and $D^{\vee}$ to obtain a single functor D into a category of *adjoint pairs between distributive lattices*. On *objects*, this functor D sends a lattice L to the adjoint pair $u_L : D^{\wedge}(L) \leftrightarrows D^{\vee}(L) : l_L$ (see Proposition 4.18 above). For the *morphisms* in the domain category of D, we take the intersection of the set of morphisms in $\mathbf{L}_{\wedge,\mathbf{a}\vee}$ and the set of morphisms in $\mathbf{L}_{\vee,\mathbf{a}\wedge}$. This intersection is defined directly in the following definition.

Definition 5.1. A function $f : L \to M$ between lattices is an *admissible homomorphism* if it is a lattice homomorphism which sends join-admissible subsets of L to join-admissible subsets of M and meet-admissible subsets of L to meet-admissible subsets of M. We denote by $\mathbf{L_a}$ the category of lattices with admissible homomorphisms.

Remark 5.2. Note that, indeed, f is an admissible homomorphism if and only if it is a morphism both in $\mathbf{L}_{\wedge,\mathbf{a}\vee}$ and in $\mathbf{L}_{\vee,\mathbf{a}\wedge}$. Although the definition may look rather restrictive, any homomorphism whose codomain is a distributive lattice is automatically admissible. Also, any surjective homomorphism between arbitrary lattices is admissible, by Proposition 4.13. This may be the underlying reason for the fact that both surjective homomorphisms and morphisms whose codomain is distributive have proven to be 'easier' cases in the existing literature on lattice duality (see, e.g., [24, 14]). Of course, not all homomorphisms are admissible, cf. Example 4.12.

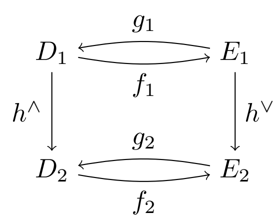
We will now first define an auxiliary category of 'doubly dense adjoint pairs between distributive lattices' (**daDL**) which has the following two features:

(i) The category $\mathbf{L_a}$ can be embedded into **daDL** as a full subcategory (Proposition 5.4);
(ii) There is a natural Stone-type duality for the category **daDL** (Theorem 5.13).

We will then give a dual characterization of the 'special' objects in **daDL** which are in the image of the embedding of $\mathbf{L_a}$ from (i), calling these dual objects *tight* (cf. Definition 5.18). The restriction of the natural Stone-type duality (ii) will then yield our final result: a topological duality for lattices with admissible homomorphisms (Theorem 5.19).

Definition 5.3. We denote by **aDL** the category with:

- *objects:* tuples (D, E, f, g), where D and E are distributive lattices and $f : D \leftrightarrows E : g$ is a pair of maps such that f is lower adjoint to g;
- *morphisms:* an **aDL**-morphism from (D_1, E_1, f_1, g_1) to (D_2, E_2, f_2, g_2) is a pair of homomorphisms $h^\wedge : D_1 \to D_2$ and $h^\vee : E_1 \to E_2$ such that $h^\vee f_1 = f_2 h^\wedge$ and $h^\wedge g_1 = g_2 h^\vee$, i.e., both squares in the following diagram commute:



We call an adjoint pair (D, E, f, g) *doubly dense* if both $g[E]$ is join-dense in D and $f[D]$ is meet-dense in E. We denote by **daDL** the full subcategory of **aDL** whose objects are doubly dense adjoint pairs.

Proposition 5.4. *The category* $\mathbf{L_a}$ *is equivalent to a full subcategory of* **daDL**.

Proof. Let $\mathcal{D} : \mathbf{L_a} \to \mathbf{daDL}$ be the functor defined by sending:

- a lattice L to $\mathcal{D}(L) := (D^\wedge(L), D^\vee(L), u, l)$,
- an admissible morphism $h : L_1 \to L_2$ to the pair $\mathcal{D}(h) := (D^\wedge(h), D^\vee(h))$.

We show that $\mathcal{D}$ is a well-defined full and faithful functor.

For objects, note that $\mathcal{D}(L)$ is a doubly dense adjoint pair by Corollary 4.15 and Proposition 4.18 in the previous section.

Let $h : L_1 \to L_2$ be an admissible morphism. We need to show that $\mathcal{D}(h)$ is a morphism of **daDL**, i.e., that $u_{L_2} \circ D^\wedge(h) = D^\vee(h) \circ u_{L_1}$ and $l_{L_2} \circ D^\vee(h) = D^\wedge(h) \circ l_{L_1}$. Since $D^\wedge(L_1)$ is join-generated by the image of L_1, and both $u_{L_2} \circ D^\wedge(h)$ and $D^\vee(h) \circ u_{L_1}$ are join-preserving, it suffices to note that the diagram commutes for elements in the image of L_1. This is done by the following diagram chase:

$$u_{L_2} \circ D^\wedge(h) \circ \eta^\wedge_{L_1} = u_{L_2} \circ \eta^\wedge_{L_2} \circ h = \eta^\vee_{L_2} \circ h = D^\vee(h) \circ \eta^\vee_{L_1} = D^\vee(h) \circ u_{L_1} \circ \eta^\wedge_{L_1},$$

where we have used that $\eta^\wedge$ is a natural transformation and that $u_L \circ \eta^\wedge_L = \eta^\vee_L$. The proof that $l_{L_2} \circ D^\vee(h) = D^\wedge(h) \circ l_{L_1}$ is similar.

It remains to show that the assignment $h \mapsto \mathcal{D}(h)$ is a bijection between $\mathbf{L_a}(L_1, L_2)$ and $\mathbf{daDL}(\mathcal{D}(L_1), \mathcal{D}(L_2))$. If $(h^\wedge, h^\vee) : \mathcal{D}(L_1) \to \mathcal{D}(L_2)$ is a **daDL**-morphism, then $h^\wedge$ maps lattice elements to lattice elements. That is, the function $h^\wedge \circ \eta^\wedge_{L_1} : L_1 \to \mathcal{D}(L_2)$ maps into $\mathrm{im}(\eta^\wedge_{L_2}) = \mathrm{im}(l_{L_2})$, since

$$h^\wedge \circ \eta^\wedge_{L_1} = h^\wedge \circ l_{L_1} \circ u_{L_1} \circ \eta^\wedge_{L_1} = l_{L_2} \circ h^\vee \circ u_{L_1} \circ \eta^\wedge_{L_1}.$$

We may therefore define $h : L_1 \to L_2$ to be the function $(\eta^\wedge_{L_2})^{-1} \circ h^\wedge \circ \eta^\wedge_{L_1}$. Note that this function is equal to $(\eta^\vee_{L_2})^{-1} \circ h^\vee \circ \eta^\vee_{L_1}$, since

$$(\eta^\vee_{L_2})^{-1} \circ h^\vee \circ \eta^\vee_{L_1} = (\eta^\vee_{L_2})^{-1} \circ h^\vee \circ u_{L_1} \circ l_{L_1} \circ \eta^\vee_{L_1} = (\eta^\vee_{L_2})^{-1} \circ u_{L_2} \circ h^\wedge \circ \eta^\wedge_{L_1} = (\eta^\wedge_{L_2})^{-1} \circ h^\wedge \circ \eta^\wedge_{L_1},$$

where we have used that, for any lattice L, $l_L \circ \eta^\vee_L = \eta^\wedge_L$ and $u_L \circ \eta^\wedge_L = \eta^\vee_L$. So, since $(\eta^\wedge_{L_2})^{-1} \circ h^\wedge \circ \eta^\wedge_{L_1} = h = (\eta^\vee_{L_2})^{-1} \circ h^\vee \circ \eta^\vee_{L_1}$, it is clear that h is a homomorphism, since the left-hand-side preserves $\wedge$ and the right-hand-side preserves $\vee$. It remains to show that h is admissible, i.e., that h sends join-admissible subsets to join-admissible subsets, and meet-admissible subsets to meet-admissible subsets. Note that, by the adjunction in Corollary 4.14, if a function $k : L \to D$ admits a homomorphic extension $\widehat{k} : D^\wedge(L) \to D$, then k is $(\wedge, a\vee)$-preserving, since it is equal to the composite $\widehat{k} \circ \eta^\wedge_L$. In particular, the map $\eta^\wedge_{L_2} \circ h$ is $(\wedge, a\vee)$-preserving, its homomorphic extension being $h^\wedge$. It follows from this that h sends join-admissible subsets to join-admissible subsets, since join-admissible subsets are the *only* subsets whose join is preserved by $\eta^\wedge_{L_2}$. The proof that h preserves meet-admissible subsets is similar.
Now, since $h^\wedge \circ \eta^\wedge_{L_1} = \eta^\wedge_{L_2} \circ h$, we have that $h^\wedge = D^\wedge(h)$, since $D^\wedge(h)$ was defined as the unique homomorphic extension of $\eta^\wedge_{L_2} \circ h$, and similarly $h^\vee = D^\vee(h)$. We conclude that $(h^\wedge, h^\vee) = \mathcal{D}(h)$, so $h \mapsto \mathcal{D}(h)$ is surjective.
It is clear that if $h \neq h'$, then $D^\wedge(h) \neq D^\wedge(h')$, so $\mathcal{D}(h) \neq \mathcal{D}(h')$. Hence, the assignment $h \mapsto \mathcal{D}(h)$ is bijective, as required. □

Example 5.5. Not all daDLs are isomorphic to ones of the form $(D^\wedge(L), D^\vee(L), u, l)$. This may easily be seen by taking any distributive lattice D and considering the daDL given by $(F_\vee(D, \wedge), F_\wedge(D, \vee), f, g)$, where $F_\vee(D, \wedge)$ is the free join-semilattice generated by the meet-semilattice reduct of D viewed as a distributive lattice, and $F_\wedge(D, \vee)$ is defined order dually, and f and g both are determined by sending each generator to itself. Such a daDL is not of the form we are interested in since the meet and join envelopes of any distributive lattice both are equal to the lattice itself since all joins are admissible.

So, the category $\mathbf{L_a}$, that we will be most interested in, is a proper subcategory of **daDL**, but we start by giving a description of the topological duals of the objects of **daDL**. To this end, let (D, E, f, g) be a doubly dense adjoint pair. If X and Y are the dual Priestley spaces of D and E respectively, then it is well-known that an adjunction (f, g) corresponds to a relation R satisfying certain properties. In our current setting of *doubly dense* adjoint pairs, it turns out that it suffices to consider the topological reducts of the Priestley spaces X and Y (i.e., forgetting the order) and the relation R between them. Both the Priestley orders of the spaces X and Y and the adjunction (f, g) can be uniquely reconstructed from the relation R, as we will prove shortly. The dual of a doubly dense adjoint pair will be a *totally separated compact polarity* (TSCP), which we define to be polarity (X, Y, R), where X and Y are Boolean spaces and R is a relation from X to Y, satisfying certain properties (see Definition 5.6 for the precise definition).

We now first fix some useful terminology for topological polarities, regarding the closure and interior operators induced by a polarity, its closed and open sets, and its associated quasi-orders.[4]

[4]The reader may note the close similarity with Galois connections in the proof of Theorem 2.3. Also note, however, that we are dealing with a *covariant* adjunction here rather than a contravariant one, due to the fact that, the adjunction in a doubly dense adjoint pair coming from a lattice is covariant.

Let X and Y be sets and $R \subseteq X \times Y$. Then we obtain a closure operator $\overline{(\ )}$ on X given by

$$\overline{S} := \{x \in X \mid R[x] \subseteq R[S]\} \quad \text{for } S \subseteq X.$$

The subsets S of X satisfying $\overline{S} = S$ we will call *R-closed*. The R-closed subsets of X form a lattice in which the meet is intersection and join is the closure of the union. We of course also obtain an adjoint pair of maps:

$$\mathcal{P}(X) \underset{\Diamond}{\overset{\Box}{\leftrightarrows}} \mathcal{P}(Y)$$

given by

$$\Diamond S = R[S] = \{y \in Y \mid \exists x \in S \; xRy\}$$

and

$$\Box T = \left(R^{-1}[T^c]\right)^c = \{x \in X \mid \forall y \in Y (xRy \implies y \in T)\}.$$

The relation with the closure operator on X is that $\overline{S} = \Box\Diamond S$. Note also that on points of X this yields a quasi-order given by

$$x' \leq x \iff R[x'] \subseteq R[x].$$

Similarly, on Y we obtain an interior operator

$$T^\circ = \{y \in Y \mid \exists x \in X \,[xRy \text{ and } \forall y' \in Y (xRy' \implies y' \in T)]\} = \Diamond\Box T$$

and a quasi-order on Y given by

$$y \leq y' \iff R^{-1}[y] \supseteq R^{-1}[y'].$$

The range of $\Diamond$ is equal to the range of the interior operator, and we call these sets R-open. This collection of subsets of Y forms a lattice isomorphic to the one of R-closed subsets of X. In this incarnation, the join is given by union whereas the meet is given by interior of the intersection. Note that the R-closed subsets of X as well as the R-open subsets of Y all are down-sets in the induced quasi-orders.

We are now ready to define the objects which will be dual to doubly dense adjoint pairs.

Definition 5.6. A *topological polarity* is a tuple (X, Y, R), where X and Y are topological spaces and R is a relation. A *compact polarity* is a topological polarity in which both X and Y are compact. A topological polarity is *totally separated* if it satisfies the following conditions:

- (i) *(R-separated)* The quasi-orders induced by R on X and Y are partial orders.
- (ii) *(R-operational)* For each clopen down-set U of X, the image $\Diamond U$ is clopen in Y; For each clopen down-set V of Y, the image $\Box V$ is clopen in X;
- (iii) *(Totally R-disconnected)* For each $x \in X$ and each $y \in Y$, if $\neg(xRy)$ then there are clopen sets $U \subseteq X$ and $V \subseteq Y$ with $\Diamond U = V$ and $\Box V = U$ so that $x \in U$, and $y \notin V$.

Remark 5.7. In the definition of totally separated topological polarities, the first property states that R separates the points of X as well as the points of Y. The second property states that R yields operations between the clopen downsets of X and of Y. Finally, the third property generalizes total order disconnectedness, well known from Priestley duality, hence the name total R-disconnectedness.

The following technical observation about total R-disconnectedness will be useful in what follows.

Lemma 5.8. *If a topological polarity (X, Y, R) is totally R-disconnected, then the following hold:*

- *If $x' \not\leq x$ then there is $U \subseteq X$ clopen and R-closed so that $x \in U$ and $x' \notin U$.*
- *If $y' \not\leq y$ then there is $V \subseteq Y$ clopen and R-open so that $y \in V$ and $y' \notin V$.*

Proof. Suppose that $x' \not\leq x$. By definition of $\leq$, there exists $y \in Y$ such that $x' \, R \, y$ and $\neg(x \, R \, y)$. By total R-disconnectedness, there exist clopen U and V such that $\Diamond U = V$, $\Box V = U$, $x \in U$ and $y \notin V$. We now have $x' \notin U$, for otherwise we would get $y \in \Diamond U = V$. Since $U = \Box V = \Box \Diamond U$, we get that U is R-closed, as required. The proof of the second property is dual. $\Box$

Now, given a daDL (D, E, f, g), we call its *dual polarity* the tuple (X, Y, R), where X and Y are the topological reducts of the Priestley dual spaces of D and E, respectively (which are in particular compact), and R is the relation defined by

$$x \, R \, y \iff f[x] \subseteq y,$$

where we regard the points of X and Y as prime filters of D and E, respectively. Conversely, given a totally separated compact polarity (X, Y, R), we call its *dual adjoint pair* the tuple $(D, E, \Diamond, \Box)$, where D and E are the lattices of clopen downsets of X and Y in the induced orders, respectively, and $\Diamond$ and $\Box$ are the operations defined above (note that these operations are indeed well-defined by item (ii) in the definition of totally separated).
The following three propositions constitute the object part of our duality for doubly dense adjoint pairs.

Proposition 5.9. *If (D, E, f, g) is a doubly dense adjoint pair, then its dual polarity (X, Y, R) is compact and totally separated.*

Proof. Let (D, E, f, g) be a doubly dense adjoint pair, and let L be the lattice which is isomorphic to both the image of g in D and to the image of f in E.
The dual polarity (X, Y, R) is compact because the dual Priestley spaces of D and E are compact.
For R-separation, suppose that $x \neq x'$ in X. We need to show that $R[x] \neq R[x']$. Without loss of generality, pick $d \in D$ such that $d \in x$ and $d \notin x'$. Since L is join-dense in D and x is a prime filter, there exists $a \in L$ with $a \leq d$, such that $a \in x$. Note that $a \notin x'$ since $a \leq d$ and $d \notin x'$. It follows that $f(a) \notin f[x']$: if we would have $d' \in x'$ such that $f(a) = f(d')$, then we would get $d' \leq gf(d') = gf(a) = a$, contradicting that $a \notin x'$. By the prime filter theorem, there exists a prime filter $y \subseteq E$ such that $f[x'] \subseteq y$ and $f(a) \notin y$. Since we do have $f(a) \in f[x]$, it follows that $x' \, R \, y$ and $\neg(x \, R \, y)$, so $R[x'] \neq R[x]$, as required. The proof that R induces a partial order on Y is similar.
For R-operationality, it suffices to observe that, for any $d \in D$, we have $\Diamond \widehat{d} = R[\widehat{d}] = \widehat{f(d)}$ and, for any $e \in E$, we have $\Box \widehat{e} = \widehat{g(e)}$.
For total R-disconnectedness, suppose that $\neg(x \, R \, y)$. This means that $f[x] \not\subseteq y$, so there is $d \in D$ such that $d \in x$ and $f(d) \notin y$. Since $d \leq gf(d)$, we get $gf(d) \in x$, so we may put $U := \widehat{gf(d)}$ and $V := \widehat{f(d)}$. $\Box$

Proposition 5.10. *If (X, Y, R) is a totally separated compact polarity, then its dual adjoint pair is doubly dense.*

Proof. From what was stated in the preliminaries above, it is clear that we get an adjoint pair between the lattices of clopen downsets. We need to show that it is doubly dense.

To this end, let U be a clopen downset of X. We show that U is a finite union of clopen R-closed sets. First fix $x \in U$. For any $x' \notin U$, we have that $x' \nleq x$. By L-separation, pick a clopen R-closed set $U_{x'}$ such that $x \in U_{x'}$ and $x' \notin U_{x'}$. Doing this for all $x' \notin U$, we obtain a cover $\{U_{x'}^c\}_{x' \notin U}$ by clopen sets of the compact set U^c. Therefore, there exists a finite subcover $\{U_i^c\}_{i=1}^n$ of U^c. Let us write $V_x := \bigcap_{i=1}^n U_i$. We then get that $x \in V_x \subseteq U$, and V_x is clopen R-closed, since each of the U_i is. Doing this for all $x \in U$, we get a cover $\{U_x\}_{x \in X}$ by clopen R-closed sets of the compact set U, which has a finite subcover. This shows that U is a finite union of clopen R-closed sets.

The proof that clopen downsets of Y are finite intersections of clopen R-open sets is essentially dual; we leave it to the reader. □

Proposition 5.11. *Any totally separated compact polarity is isomorphic to its double dual.*

More precisely, if (X, Y, R) is a TSCP, let (X', Y', R') be the dual polarity of the dual adjoint pair of (X, Y, R). Then there are homeomorphisms $\phi : X \to X'$, $\psi : Y \to Y'$ such that $x\, R\, y$ iff $\phi(x)\, R' \psi(y)$.

Proof. Note that if (X, Y, R) is a TSCP, then X and Y with the induced orders are Priestley spaces: total-order-disconnectedness follows from Lemma 5.8 and the fact, noted above, that R-closed and R-open sets are downsets in the induced orders.

Therefore, by Priestley duality we have homeomorphisms $\phi : X \to X'$ and $\psi : Y \to Y'$, both given by sending points to their neighbourhood filters of clopen downsets. It remains to show that ϕ and ψ respect the relation R. Note that, by definition, we have $x'\, R'\, y'$ iff for any clopen downset U in x', we have that $R[U]$ is in y'. Suppose $x\, R\, y$, and that $U \in \phi(x)$. Then $x \in U$, so $y \in R[U]$, so $R[U] \in \psi(y)$. Conversely, suppose that $\neg(x\, R\, y)$. By total R-disconnectedness, we pick a clopen R-closed set U with $x \in U$ and $y \notin \Diamond U = R[U]$. This set U is a clopen downset which witnesses that $\neg(\phi(x)\, R'\, \psi(y))$. □

We can extend this object correspondence between daDL's and TSCP's to a dual equivalence of categories. The appropriate morphisms in the category of totally separated compact polarities are pairs of functions (s_X, s_Y), which are the Priestley duals of $(h^\wedge, h^\vee)$. The condition that morphisms in **daDL** make two squares commute (see Definition 5.3) dualizes to back-and-forth conditions on s_X and s_Y, as in the following definition.

Definition 5.12. A *morphism* in the category **TSCP** of totally separated compact polarities from (X_1, Y_1, R_1) to (X_2, Y_2, R_2) is a pair (s_X, s_Y) of continuous functions $s_X : X_1 \to X_2$ and $s_Y : Y_1 \to Y_2$, such that, for all $x \in X_1$, $x' \in X_2$, $y \in Y_1$, $y' \in Y_2$:

(forth) If $x\, R_1\, y$, then $s_X(x)\, R_2\, s_Y(y)$,

($\Diamond$-back) If $x'\, R_2\, s_Y(y)$, then there exists $z \in X_1$ such that $z\, R_1\, y$ and $s_X(z) \leq x'$,

($\Box$-back) If $s_X(x)\, R_2\, y'$, then there exists $w \in Y_1$ such that $x\, R_1\, w$ and $y' \leq s_Y(w)$.

The conditions on these morphisms should look natural to those familiar with back-and-forth conditions in modal logic. More detailed background on how these conditions arise naturally from the theory of canonical extensions can be found in [8, Section 5].

Theorem 5.13. *The category* **daDL** *is dually equivalent to the category* **TSCP**.

Proof. The hardest part of this theorem is the essential surjectivity of the functor which assigns to a daDL its dual polarity. We proved this in Proposition 5.11. One may then either check directly that the assignment which sends a **daDL**-morphism $(h^\wedge, h^\vee)$ to the pair (s_X, s_Y) of Priestley dual functions between the spaces in the dual polarities is a bijection between the respective **Hom**-sets, or refer to [8, Section 5] for a more conceptual proof using canonical extensions. □

In particular, combining Theorem 5.13 with Proposition 5.4, the category $\mathbf{L_a}$ of lattices with admissible homomorphisms is dually equivalent to a full subcategory of **TSCP**. The task that now remains is to identify which totally separated compact polarities may arise as duals of doubly dense adjoint pairs which are isomorphic to ones of the form $(D^\wedge(L), D^\vee(L), u_L, l_L)$ for some lattice L (not all doubly dense adjoint pairs are of this form; cf. Example 5.5).
Given any daDL (D, E, f, g), there is an associated lattice, namely, $L = Im(g) \cong Im(f)$ and this lattice embeds in D meet-preservingly and in E join-preservingly. We write $i : L \hookrightarrow D$ and $j : L \hookrightarrow E$ for the embeddings of L into D and E, respectively. These images generate D and E, respectively, because of the double denseness. However, the missing property is that i and j need not preserve admissible joins and meets, cf. Example 5.5. We will now give a dual description of this property.
To do so, we will use the *canonical extension* of the adjunction $f : D \leftrightarrows E : g$ and of the embeddings i and j. For the definitions and the general theory of canonical extensions of maps we refer to [10, Section 4]. All maps in our setting are either join- or meet-preserving, so that they are smooth and the σ- and π-extensions coincide. We therefore denote the unique extension of a (join- or meet-preserving) map h by h^δ. Thus, we have maps $f^\delta : D^\delta \leftrightarrows E^\delta : g^\delta$, $i^\delta : L^\delta \to D^\delta$ and $j^\delta : L^\delta \to E^\delta$. For our dual characterization, we will need the following basic fact, which is essentially the content of Remark 5.5 in [10].

Proposition 5.14. *Let $f : D \leftrightarrows E : g$ be an adjunction between distributive lattices. Then the following hold:*

- (i) *$f^\delta : D^\delta \leftrightarrows E^\delta : g^\delta$ is an adjunction;*
- (ii) *The image of g^δ forms a complete $\bigwedge$-subsemilattice of D^δ which is isomorphic, as a completion of L, to L^δ;*
- (iii) *The image of f^δ forms a complete $\bigvee$-subsemilattice of D^δ which is isomorphic, as a completion of L, to L^δ.*

Proof. Item (i) is proved in [10, Proposition 6.6]. The image of an upper adjoint between complete lattices always forms a complete $\bigwedge$-subsemilattice. To see that the image of g^δ is isomorphic to L^δ as a completion of L, it suffices by Theorem 2.4 to check that the natural embedding $L \hookrightarrow \mathrm{im}(g^\delta)$ (given by the composition $L \hookrightarrow D \hookrightarrow D^\delta$) is compact and dense. Neither of these properties is hard to verify. The proof of item (iii) is order-dual to (ii). □

Let M be a finite subset of the lattice L. Recall that by Lemma 4.2, M is join-admissible if and only if, for each $x \in J^\infty(L^\delta)$, we have $x \leq \bigvee M$ implies $x \leq m$

for some $m \in M$. In order to translate this to a dual condition, it is useful to get a dual characterization of the elements of $J^\infty(L^\delta)$. In the following Lemma, we will use the fact that the relation R can be alternatively defined using the lifted operation f: regarding X as $J^\infty(D^\wedge(L)^\delta)$ and Y as $J^\infty(D^\vee(L)^\delta)$, we have that $x\,R\,y \iff y \leq f^\delta(x)$.

Lemma 5.15. *Let (D, E, f, g) be a daDL, (X, Y, R) its dual polarity, $L \cong \mathrm{im}(g) \cong \mathrm{im}(f)$, with $i : L \hookrightarrow D$ and $j : L \hookrightarrow E$ the natural embeddings. Then the following hold:*

- (i) *$J^\infty(D^\delta) \subseteq i^\delta[F(L^\delta)]$ and $M^\infty(E^\delta) \subseteq j^\delta[I(L^\delta)]$.*
- (ii) *For all $x \in X = J^\infty(D^\delta)$, the following are equivalent:*
  - (a) *$x \in i^\delta[J^\infty(L^\delta)]$,*
  - (b) *$R[x] \neq R[\{x' \in X \mid x' < x\}]$.*
- (iii) *For all $y \in Y = J^\infty(E^\delta)$, the following are equivalent:*
  - (a) *$\kappa(y) \in j^\delta[M^\infty(L^\delta)]$,*
  - (b) *$R^{-1}[y] \neq R^{-1}[\{y' \in Y \mid y' > y\}]$.*

Proof. (i) Let $x \in X = J^\infty(D^\delta)$. Then $x \in F(D^\delta)$, so x is equal $\bigwedge F$ for some $F \subseteq D$. For each $d \in F$, since $\mathrm{im}(i) = \mathrm{im}(g)$ is join-dense in D, we may pick $S_d \subseteq L$ such that $d = \bigvee i(S_d)$. Let us write Φ for the set of choice functions $F \to \bigcup_{d \in F} S_d$. Then, by distributivity of D^δ, we have

$$x = \bigwedge F = \bigwedge\{\bigvee i(S_d) \mid d \in F\} = \bigvee\{\bigwedge_{d \in F} i(\phi(d)) \mid \phi \in \Phi\}.$$

Since x is completely join-irreducible in D^δ, we get that $x = \bigwedge_{d \in F} i(\phi(d))$ for some $\phi \in \Phi$. Since i^δ is completely meet-preserving, we get $x = i^\delta(x')$ where $x' := \bigwedge_{d \in F} \phi(d) \in F(L^\delta)$. The proof that $M^\infty(E^\delta) \subseteq j^\delta[I(L^\delta)]$ is order-dual.

(ii) For the direction (a) $\Rightarrow$ (b), suppose that $R[x] = R[\{x' \in X \mid x' < x\}]$. By definition of R, we then get that $f^\delta(x) = \bigvee_{x' < x} f^\delta(x')$ holds in E^δ. Since f^δ is lower adjoint to g^δ by Proposition 5.14(i), we get that

$$x \leq g^\delta f^\delta \left(\bigvee_{D^\delta} \{x' \in X \mid x' < x\} \right). \tag{$\star$}$$

By item (i), we have that $X \subseteq i^\delta[L^\delta]$, so the right-hand-side of this inequality is equal to $i^\delta(\bigvee_{L^\delta}\{v \in (i^\delta)^{-1}(X) \mid i^\delta(v) < x\})$. It follows from injectivity of i^δ that if $x = i^\delta(u)$, then $u \leq \bigvee_{L^\delta}\{v \in (i^\delta)^{-1}(X) \mid i^\delta(v) < x\}$. Then u is actually equal to the join on the right-hand-side, so u is not join-irreducible.

Conversely, if $x \notin i^\delta[J^\infty(L^\delta)]$, then $(\star)$ must hold for x, from which it follows that $R[x] = R[\{x' \in X \mid x' < x\}]$, using adjunction and the definition of R again.

(iii) Order-dual to item (ii). □

Combining item (ii) of this Lemma with the characterization of join-admissibility in Lemma 4.2, we now get the following. A finite set $M \subseteq L$ being join-admissible corresponds to saying that, for each $x \in X$ with $R[x] \neq R[\{x' \in X \mid x' < x\}]$, we have $R[x] \subseteq R\left[\bigcup\{\widehat{m} \mid m \in M\}\right]$ implies $x \in \bigcup\{\widehat{m} \mid m \in M\}$. Note that in its dual incarnation this property does not really depend on M but only on the clopen down-set $\bigcup\{\widehat{m} \mid m \in M\}$. Accordingly, we make the following definition.

Definition 5.16. Let (X, Y, R) be a TSCP, and $U \subseteq X$ a clopen down-set. We say that U is *R-regular* provided that, for each $x \in X$ with $R[x] \neq R[\{x' \in X \mid x' < x\}]$, we have $R[x] \subseteq R[U]$ implies $x \in U$.
Order dually, we say that a down-set $V \subseteq Y$ is *R-coregular* provided that, for each $y \in Y$ with $R^{-1}[y] \neq R^{-1}[y' \in Y \mid y' > y]$, we have $R^{-1}[y] \subseteq R^{-1}[U]$ implies $y \in U$.

Recall that a clopen down-set $U \subseteq X$ is R-closed provided that, for each $x \in X$, we have $R[x] \subseteq R[U]$ implies $x \in U$. Thus it is clear that every R-closed clopen down-set in X is R-regular. Preserving admissible joins exactly corresponds to the reverse implication: as soon as U is R-regular it must also be R-closed. To sum up:

Proposition 5.17. *Let (D, E, f, g) be a daDL, and let (X, Y, R) be its dual polarity. Then the following are equivalent:*

(i) *There exists a lattice L such that $(D, E, f, g) \cong (D^\wedge(L), D^\vee(L), u_L, l_L)$;*
(ii) *The embedding $\mathrm{im}(g) \hookrightarrow D$ preserves admissible joins and the embedding $\mathrm{im}(f) \hookrightarrow E$ preserves admissible meets.*
(iii) *In (X, Y, R), all R-regular clopen downsets in X are R-closed, and all R-coregular clopen downsets in Y are R-open.*

Proof. The equivalence (i) $\iff$ (ii) holds by the results in Section 4.
Throughout the proof of the equivalence (ii) $\iff$ (iii), let us write L for the lattice $\mathrm{im}(g)$, in which meets are given as in D and $\bigvee_L S = gf(\bigvee_M S)$, for any $S \subseteq L$. In this proof, we regard L as a sublattice of D, suppressing the notation i for the embedding $L \hookrightarrow D$.
For the implication (ii) $\Rightarrow$ (iii), let U be an R-regular clopen downset in X. Since $\mathrm{im}(g)$ is dense in D, there exists $M \subseteq \mathrm{im}(g)$ such that $U = \bigcup_{m \in M} \widehat{m}$. We show that M is join-admissible in the lattice L, using Lemma 4.2. If $x \in J^\infty(L^\delta)$ and $x \leq \bigvee_L M = gf(\bigvee_D M)$, then $f^\delta(x) \leq f(\bigvee_D M)$. So, by definition of R and the fact that f is completely join-preserving, we get that $R[x] \subseteq R[\bigcup_{m \in M} \widehat{m}] = R[U]$. Since U is R-regular and $x \in J^\infty(L^\delta)$, we get that $x \in U$, so $x \leq m$ for some $m \in M$. So M is join-admissible, so (ii) implies that $\bigvee_L M = gf(\bigvee_D M) = \bigvee_D M$. That is, $\overline{U} = U$, so U is R-closed. The proof that R-coregular clopen downsets in Y are R-open is dual.
For the implication (iii) $\Rightarrow$ (ii), let $M \subseteq L$ be a join-admissible subset. Let $U := \bigcup_{m \in M} \widehat{m} \subseteq X$. Then U is clearly a clopen downset. We show that U is R-regular. Let $x \in X$ such that $R[x] \neq R[\{x' \in X \mid x' < x\}]$ and $R[x] \subseteq R[U]$. Then $x \in J^\infty(L^\delta)$ and $f^\delta(x) \leq f(\bigvee_D M)$, so $x \leq gf(\bigvee_D M) = \bigvee_L M$. So, since M is join-admissible, there exists $m \in M$ such that $x \leq m$. In particular, we have $x \in U$, as required. By the assumption (iii), we conclude that U is R-closed, i.e., $\overline{U} = U$, so that $\bigvee_L M = gf(\bigvee_D M) = \bigvee_D M$. The proof that $\mathrm{im}(f) \hookrightarrow E$ preserves admissible meets is dual. □

In the light of this proposition, we can now define a subcategory of TSCP's which will be dual to the category of lattices with admissible homomorphisms.

Definition 5.18. Let (X, Y, R) be a TSCP. We say that (X, Y, R) is *tight* if all R-regular clopen downsets in X are R-closed, and all R-coregular clopen downsets in Y are R-open. We denote by **tTSCP** the full subcategory of **TSCP** whose objects are the tight TSCP's.

We then obtain our topological duality theorem for lattices with admissible homomorphisms.

Theorem 5.19. *The category* $\mathbf{L_a}$ *of lattices with admissible homomorphisms is dually equivalent to the category* **tTSCP** *of tight totally separated compact polarities.*

Proof. By Proposition 5.4, we have that $\mathbf{L_a}$ is equivalent to a full subcategory of **daDL**. By Theorem 5.13, the category **daDL** is dually equivalent to **TSCP**. By Proposition 5.17, the image of $\mathbf{L_a}$ in **daDL** under this dual equivalence is **tTSCP**. □

Let us make a few closing remarks. In this paper, in light of the examples in Section 2, we set out to obtain a topological duality for lattices in which the spaces are nicer than those of Hartung's duality. Although the spaces obtained in our duality are as nice as can be (they are compact, Hausdorff and totally disconnected), this comes at the price of a rather complicated characterization. Therefore, we are inclined to draw as a negative conclusion that topology may not be the most opportune language to discuss 'duality' for lattices (unless the definition of a tTSCP can be simplified). Fortunately, the perspective of canonical extensions provides an alternative to topology: we have explained in Section 2 how canonical extensions can be viewed as a point-free version of Hartung's duality, and we have used them in Section 5 to reason about the topological dual spaces introduced in this paper.
On the positive side, we developed the theory of distributive envelopes in Section 4 of this paper. We see our methodology there as an example of the phenomenon that canonical extensions and duality may help to study lattice-based algebras, even when they do not lie in finitely generated varieties. As a case in point, the spatial representation of lattices in Section 2 gave us a concrete definition of a distributive envelope for an arbitrary lattice at the end of Section 3, which was then characterized algebraically in Section 4. This also led us to identify the $(\wedge, a\vee)$-morphisms between lattices, which are exactly the ones which have functional duals on the X-components of the dual spaces. We believe that canonical extensions may be used in a similar way for other varieties of algebras based on lattices, such as residuated lattices.
Let us mention one more possible direction for further work. For distributive lattices, the canonical extension functor is left adjoint to the inclusion functor of perfect distributive lattices into distributive lattices. However, this is known to be true for lattice-based algebras only in case all basic operations are both Scott and dually Scott continuous (see [2, Proposition C.9, p. 196] for a proof in the distributive setting). It follows from the results in Goldblatt [12] that the canonical extension functor for modal algebras (i.e., Boolean algebras equipped with a modal operator) can be viewed as a left adjoint. However, the codomain category that is involved here is not immediately obvious: it is not the category of 'perfect modal algebras' in the usual sense. We conjecture that the distributive envelope constructions developed in Section 4 of this paper may be used to define a category in which the canonical extension for lattices is a left adjoint. We leave the actual development of this line of thought to future research.

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