Duality in Computer Science *

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Abstract

This is a paper on Stone duality in computer science with special focus on topics with applications in formal language theory. In Section 2 we give a general overview of Stone duality in its various forms: for Boolean algebras, distributive lattices, and frames. For distributive lattices, we discuss both Stone and Priestley duality. We identify how to move between the different dualities and which dual spaces carry the Scott topology. We then focus on three themes.

The first theme is additional operations on distributive lattices and Boolean algebras. Additional operations arise in denotational semantics in the form of predicate transformers. In verification they occur in the form of modal operators. They play an essential rôle in Eilenberg's variety theorem in the form of quotient operations. Quotient operations are unary instantiations of residual operators which are dual to the operations in the profinite algebras of algebraic language theory. We discuss additional operations in Section 3.

The second theme is that of hyperspaces, that is, spaces of subsets of an underlying space. Some classes of algebras may be seen as the class of algebras for a functor. In the case of predicate transformers the dual functors are hyperspace constructions such as the Plotkin, Smyth, and Hoare powerdomain constructions. The algebras-for-a-functor point of view is central to the coalgebraic study of modal logic and to the solution of domain equations. In the algebraic theory of formal languages various hyperspace-related product constructions, such as block and Schützenberger products, are used to study complexity hierarchies. We describe a construction, similar to the Schützenberger product, which is dual to adding a layer of quantification to formulas describing formal languages. We discuss hyperspaces in Section 4.

The final theme is that of "equations". These are pairs of elements of dual spaces. They arise via the duality between subalgebras and quotient spaces and have provided one of the most successful tools for obtaining decidability results for classes of regular languages. The perspective provided by duality allows us to obtain a notion of equations for the study of arbitrary formal languages. Equations in language theory is the topic of Section 5.

Categories and Subject Descriptors F [*3;4*]: 2;3

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1. Introduction

In 1936, M. H. Stone initiated duality theory in logic by presenting a dual category equivalence between the category of Boolean algebras and the category of compact Hausdorff spaces having a basis of clopen sets. Stone's duality and its variants are central in making the link between syntactical and semantic approaches to logic. In computer science this link is central as the two sides correspond to specification languages and to spaces of computational states, respectively. The ability to translate faithfully between these two worlds has often proved itself to be a powerful theoretical tool as well as a handle for making practical problems decidable. A prime example is the seminal work (Abramsky 1991) linking program logic and domain theory via Stone duality. Other examples include the work (Plotkin 1980) and (Smyth 1983) on predicate transformers, and (Goldblatt 1989) identifying extended Stone duality as the setting for completeness issues for Kripke semantics in modal logic. Applications of Stone duality in logic and computer science generally need more than just basic Stone duality. For example, Abramsky's work needs Stone or Priestley duality for distributive lattices and the dualisation of additional structure in the form of functors. Applications in modal logic require a duality for Boolean algebras or distributive lattices endowed with additional operations. Thus much work in duality theory has been spawned to answer questions and solve problems coming from semantics both in computer science and logic.

In contrast, Stone duality has not played a direct rôle in more algorithmic areas of computer science until recently. Profinite topology is a central tool in the algebraic theory of automata (Almeida 1994) and, as was observed as early as 1937 by Birkhoff, profinite topological algebras are based on Stone spaces. However, the connection was not used until much more recently, first in an isolated case (Pippenger 1997) and then more structurally starting with (Gehrke, Grigorieff & Pin 2008). This work has led to applications within regular language theory (Branco & Pin 2009; Kufleitner & Lauser 2011, 2012) as well as explorations of the general mechanism of finite recognition (Gehrke 2016; Adamek et al. 2015; Bojańczyk 2015; Daviaud, Kuperberg & Pin 2016). Further, a number of articles in coalgebraic logic have started exploring the duality theoretic content of the related notion of minimisation (Kozen et al. 2013; Bonchi et al. 2014).

The realisation that finite recognition and the profinite methods used in language theory fit within the setting of Stone/Priestley duality also has as consequence that one can explore methods of *compact* rather than just finite recognition (Gehrke, Grigorieff & Pin 2010). Initial efforts in this direction have focused on connections with circuit complexity, which plays a rôle in the search for lower bounds in complexity theory.

In (Furst, Saxe & Sipser 1984) it was shown by a combinatorial argument that the language PARITY, consisting of all bit words with an odd number of 1s, is not in AC^0. AC^0 is a circuit complexity class, that is, its members are specified by sequences of Boolean circuits, one for each input length, identifying which words of the given length are accepted. For each n, AC^n is the class of languages given by families of Boolean circuits for which the size of the circuits is polynomial in the length of the input word and the depth of the circuit is of order $\log^n$ in the length of the word. The class AC is the union of the hierarchy AC^n over all n. The ACC hierarchy is obtained by adding gates that can count modulo q for each q. Clearly PARITY is in ACC^0, so the result of Furst, Saxe, and Sipser separates AC^0 from ACC^0. We have the following chain of inclusions (for suitable restrictions of the circuit classes, so called uniform circuit classes):

$$\mathrm{PSPACE} \geq \mathrm{NP} \geq \mathrm{P} \geq \mathrm{AC} \ldots \geq \mathrm{AC}^1 \geq \mathrm{NL} \geq \mathrm{L} \geq \mathrm{ACC}^0 > \mathrm{AC}^0$$

where L stands for logarithmic space. Here the only known-to-be *strict* inclusion is the last one. Most results in the field are proven using complexity theoretic and algorithmic methods. However, there are a few connections with the topo-algebraic tools of the theory of regular languages, most famously, the result of (Barrington, Straubing &Thérien 1990) which characterises the regular languages belonging to AC^0 in terms of finite recognition. A number of related conjectures (Straubing 1994, Chapter IX) rely on the connection with logic: as is the case in the theory of regular languages, many computational complexity classes have been given characterisations as finite model classes of appropriate logics (Immerman 1998). For example, $\mathrm{AC}^0 = \mathrm{FO}[\mathcal{N}]$ and $\mathrm{ACC}^0 = (\mathrm{FO} + \mathrm{MOD})[\mathcal{N}]$ where $\mathcal{N}$ is the set of all numerical predicates, FO is usual first-order logic, and MOD stands for the modular quantifiers (one for each remainder modulo each q, which count the number of true instances (in a finite word) of a formula modulo q). The presence of *non-regular numerical predicates* is what brings one beyond the scope of the profinite algebraic theory of regular languages. Thus this area is a prime candidate for exploring the generalised methods of recognition afforded by Stone duality. Some essential tools from regular languages which one wants to generalise are: the notions of recognition and syntactic recogniser (see Section 3.2), a construction yielding a recogniser for the language corresponding to a quantified formula from a recogniser of the unquantified formula (see Section 4.3), and the notion of profinite equations (see Section 5.2).

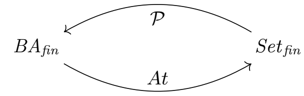
The layout of the paper is outlined in the abstract. We have chosen to organise topics according to their duality theoretic kinship rather than their home among computer science topics. We hope this will make clear that the methods and duality theoretic ideas applied in semantics, in the theory of regular languages and beyond in the exploration of circuit classes are very closely related thus affording an opportunity for making connections across these traditional boundaries.

2. Stone duality: Behaviour versus specification

The basic dichotomy between behaviour of physical computing systems and specification languages or program logic is closely related to Stone's duality between distributive lattices and Stone spaces. On one side of the duality, we have state spaces and state transformers; on the other side, properties (or predicates) and predicate transformers. The correspondence is expressed as a contravariant equivalence of categories, allowing one to pass back and forth between these two points of view, without loss of information. The magic ingredient which allows one, in more complicated cases than the finite, to get this tight correspondence is – topology!

2.1 Stone duality for finite Boolean algebras

At the most basic level, namely that of finite sets and finite Boolean algebras, on the physical systems or geometric side, we just have finite sets. Properties of points are subsets and thus the dual algebra of observables is simply the powerset. From the powerset, one recovers the set, or at least an isomorphic copy, by taking the atoms. These are the minimal elements above the bottom of the Boolean algebra. For any set X we have $At(\mathcal{P}(X)) = \{\{x\} \mid x \in X\} \cong X$ and for a finite Boolean algebra, the fact that $B \cong \mathcal{P}(At(B))$ follows as each element of B splits into a unique join of atoms.



The really powerful fact about duality is that morphisms transform bijectively to morphisms *in the opposite direction*. Thus a homomorphism $h\colon A \to B$ between finite Boolean algebras corresponds to a function $f\colon At(B) \to At(A)$ given by adjunction:

$$f(x) \leq a \qquad \iff \qquad x \leq h(a), \tag{1}$$

where $a \in A$ and $x \in At(B)$. In this correspondence, quotients of a Boolean algebra B correspond to subsets of its dual space X. Even better, there is a Galois connection between subsets of $B \times B$ and subsets of X. That is, for $x \in X$ and $a, b \in B$, we define the relation

$$x \Vdash a \approx b \quad \iff \quad (x \leq a \iff x \leq b).$$

Then the maps

$$\begin{array}{rcl} \Phi\ : \mathcal{P}(B \times B) & \leftrightarrows & \mathcal{P}(X) :\ \Psi \\ R & \mapsto & \{x \mid \forall (a,b) \in R\ x \Vdash a \approx b\} \\ \{(a,b) \mid \forall x \in Y\ x \Vdash a \approx b\} & \leftarrow\!\shortmid & Y \end{array}$$

satisfy

$$Y \subseteq \Phi(R) \quad \iff \quad R \subseteq \Psi(Y)$$

and have as images, respectively, the Boolean algebra congruences of B and the subsets of X. Thus, given a 'specification' that two properties $a, b \in B$ must be equated, $\Phi(\{(a,b)\}) \subseteq X$ yields the 'phase space' in which this specification holds. Conversely, every subset of X is given by a set of such specifications via Φ.

2.2 Stone duality for BAs and DLs

Stone's insight was that this duality may be extended to arbitrary Boolean algebras (BAs) — and even to arbitrary bounded distributive lattices (DLs), that is, distributive lattices with a top and a bottom element. In order to encompass infinite BAs and DLs we need to move to 'generalised elements' in the form of filters. A subset F of a DL A is a *filter* provided

- F is an upset, i.e., $a \in F$ and $a \leq b$ implies $b \in F$;
- F is non-empty, or equivalently, $1 \in F$;
- F is closed under finite meets i.e., $a, b \in F$ implies $a \wedge b \in F$.

Among these generalised elements we want those which correspond to atoms in the Boolean case and join-prime elements in the DL setting. To this end, a filter F is said to be *prime* provided

- If $a \vee b \in F$ then $a \in F$ or $b \in F$.

and, if A is a BA, it is said to be an *ultrafilter* provided

- For all $a \in A$ either $a \in F$ or $\neg a \in F$.

Ideals, prime ideals, and *ultraideals* are defined order dually, that is, by swapping upsets and downsets, $\top$ and $\bot$, and $\wedge$ and $\vee$. Given a subset $S \subseteq A$, we denote the complement of S by S^c.

Proposition 1. *Let A be a DL and $F \subseteq A$. The following conditions are equivalent:*

(1) F (F^c) is a prime filter (ideal);

(2) The characteristic function $\chi_F\colon A \to 2$ is a homomorphism.

If A is a Boolean algebra, then these are also equivalent to

(3) F (F^c) is an ultrafilter (ultraideal).

For an arbitrary DL, A, its *Stone dual space*, $St(A)$, is based on the set X of all prime filters of A, and by a Zorn's Lemma argument the map

$$\eta_A\colon A \to \mathcal{P}(X)$$
$$a \mapsto \ \widehat{a} \ = \ \{F \mid a \in F\}$$

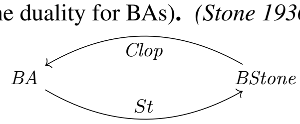
is a bounded lattice embedding. However, it is surjective only when A is finite. Accordingly Stone equips X with the topology σ_A generated by the basis $\mathcal{B}_A = \{\widehat{a} \mid a \in A\}$. Topological spaces typically do not have minimal bases. In Stone spaces however, the opens of the form $\widehat{a}$ are all topologically compact, thus they must all belong to any basis closed under finite unions. Accordingly, we may recover an isomorphic copy of A, in the form of $\mathcal{B}_A$, as the minimum basis closed under finite unions or as the set of compact-opens of $St(A)$.

In the case of BAs we get a simple description of the duality.

Definition 2. A topological space (X, σ) is said to be a *Boolean (Stone) space* provided it is compact, Hausdorff, and has a basis of clopen (i.e., simultaneously closed and open) subsets. We denote by $BStone$ the category of Boolean Stone spaces with continuous maps and by $Clop(X)$ the Boolean algebra of clopen subsets of a topological space X.

We will mostly refer to these spaces as Boolean spaces while we reserve the name Stone spaces for the DL variant. Note though that (Johnstone 1982) calls Boolean spaces Stone spaces.

Theorem 3 (Stone duality for BAs)**.** *(Stone 1936)*



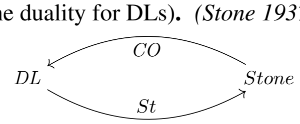
is a dual equivalence of categories.

Here both St and $Clop$ act on maps by appropriately restricted pre-image: Given a DL homomorphism $h\colon A \to B$, the inverse image sends ultrafilters to ultrafilters and thus induces a map $St(h)\colon St(B) \to St(A)$, which is continuous with respect to the Stone topology. Similarly, given a continuous map $f\colon X \to Y$ between Boolean spaces, the inverse image sends clopens to clopens and thus induces a map $Clop(f)\colon Clop(Y) \to Clop(X)$, which is a Boolean algebra homomorphism.

Stone's duality for DLs is a bit more awkward to describe.

Definition 4. A topological space (X, σ) is said to be a *Stone space* provided it is compact, its collection of compact-open subsets is closed under finite intersections and forms a basis, and each prime filter of the lattice of compact-open subsets generates the neighbourhood filter of a unique point of X. These spaces are also known as *coherent* or *spectral* spaces in the literature. We denote by $Stone$ the category of Stone spaces with maps for which the pre-image of a compact-open is compact-open and by $CO(X)$ the collection of compact-open subsets of a topological space X.

Theorem 5 (Stone duality for DLs)**.** *(Stone 1937)*



is a dual equivalence of categories.

Given the restrictions made on maps in $Stone$, the duality for maps now works as in the Boolean case.

2.3 Priestley duality

Stone's duality for DLs does not involve a full subcategory of topological spaces and the class of spaces is difficult to describe. (Priestley 1970) finds a remedy for this by moving to Nachbin's *ordered topological spaces*.

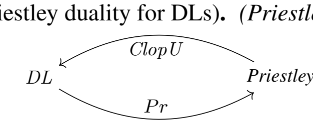
Definition 6. A *Priestley space* is a triple $(X, \leq, \pi)$, where $\leq$ is a partial order and π is a compact topology on X, so that

(TOD) $\forall x, y \in X$ ($x \nleq y$ implies $\exists V$ clopen upset of X with $x \in V$ and $y \notin V$).

This property is known as *Total Order Disconnectedness*. We denote by *Priestley* the category of Priestley spaces with maps that are continuous and order preserving and by $Clop\,U(X)$ the lattice of clopen upsets of an ordered topological space X.

Priestley duality is nicely described in category theoretic terms. Let 2 denote the two element lattice and let $\underline{2}$ denote the two element ordered topological space carrying the discrete topology and the usual order with $0<1$. Given a DL A, its Priestley space is $Pr(A) = Hom_{DL}(A, 2)$ viewed as a subspace of the product space $\underline{2}^A$. Given a Priestley space X, the dual lattice is $Hom_{\mathit{Priestley}}(X, \underline{2})$ viewed as a sublattice of the lattice 2^X. In order topological terms, the elements of $Hom_{\mathit{Priestley}}(X, \underline{2})$ correspond to the clopen upsets of X (by taking the pre-image of 1). When the duals are viewed as hom-sets, the dual of a morphism on either side is simply given by pre-composition. If we view the lattice dual to a Priestley space as the lattice of clopen upsets of the space, then the dual of a continuous and order preserving map is given by restricted pre-image as in Stone duality.

Theorem 7 (Priestley duality for DLs)**.** *(Priestley 1970)*



is a dual equivalence of categories.

Note that by Proposition 1.2, the underlying sets of $St(A)$ and $Pr(A)$ are in bijective correspondence. In fact, the categories *Priestley* and $Stone$ are not only equivalent but isomorphic. This isomorphism may be seen as part of the more general isomorphism between compact ordered spaces and so-called stably compact spaces, see (Lawson 2011) (or (Jung & Moshier 2006) for a bitopological treatment). Given a Priestley space $(X, \leq, \pi)$, the corresponding Stone space is $(X, \pi \cap U(X))$, where $U(X)$ denotes the lattice of upsets of X. Given a topological space (X, τ), its *specialisation order* is defined by $x \leq_\tau y$ provided y belongs to each open neighbourhood of x. Given a Stone space (X, σ), the co-compact dual topology of σ, denote it by σ^∂, is generated by the complements of compact-opens of X, and $(X, \leq_\sigma, \sigma \vee \sigma^\partial)$ is then the corresponding Priestley space. This isomorphism between $Stone$ and *Priestley* is simply the identity on maps.

To move more easily between the Stone and Priestley dualities, it is convenient to think of both the Stone dual and the Priestley dual of a DL A as based on a set X_A which comes with bijections $x \mapsto F_x$ and $x \mapsto h_x$ to the set of prime filters of A and to the set of homomorphisms from A to 2, respectively. It will also be convenient to denote $F_x^c = I_x$, that is, we also have a bijection $x \mapsto I_x$ to the set of prime ideals of A (cf. Proposition 1.1) which will be useful when discussing additional operations.

The Galois connection (Φ, Ψ) between specifications and subsets given in the finite setting lifts to arbitrary DLs simply by redefining the relation $x \Vdash a \approx b$ for $x \in X_A$ and $a, b \in A$ by

$$\begin{aligned} x \Vdash a \approx b \iff & \ (a \in F_x \iff b \in F_x) \\ \iff & \ (h_x(a) = 1 \iff h_x(b) = 1). \end{aligned}$$

The Galois closed sets are the DL congruences on one side and the subsets that are closed in the Priestley topology on the other.

2.4 The (Ω, Pt) adjunction and duality for sober spaces

In Stone and Priestley duality, finitary operations on the algebras of opens suffice because the compact members generate. Dualities for more general spaces necessarily require infinitary operations (or relations instead of operations). Such a duality was emerging simultaneously in Ehresmann's seminar in France (Papert & Papert 1958; Bénabou 1958) and in Canada (Bruns 1962; Thron 1962) before being studied by many authors, in particular (Isbell 1972) who advocated the study of locales as generalised spaces.

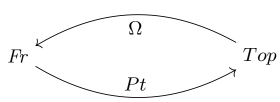
Definition 8. A *frame*, A, is a complete lattice satisfying the Join Infinite Distributive law

$$(JID) \qquad \forall a \in A\, \forall S \subseteq A \qquad a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}.$$

A frame homomorphism is a map between frames which preserves finite meets and arbitrary joins. We denote by *Fr* the category of frames with frame homomorphisms. The category of *locales* is the formal opposite of the category of frames.

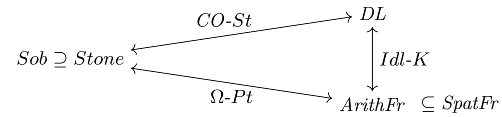
Given a topological space X we denote by $\Omega(X)$ the frame of opens of X. Also, note that if $f\colon X \to Y$ is a continuous map, then inverse image under f restricts to a map $\Omega(f)\colon \Omega(Y) \to \Omega(X)$ which is a frame homomorphism. Given a frame A, we denote by $Pt(A)$ the space whose underlying set is the set of frame homomorphisms from A to the frame 2 and whose opens are of the form $\widehat{a} = \{h : A \to 2 \mid h(a) = 1\}$ for $a \in A$. An alternative description of the points of A is as the *completely prime filters* of A, that is, the proper filters $F \subseteq A$ so that $\bigvee S \in F$ implies $S \cap F \neq \emptyset$. Given a frame homomorphism, $h\colon A \to B$, it is not hard to see that precomposition by h gives a continuous map from $Pt(B)$ to $Pt(A)$. Open set frames are *spatial*, that is, they satisfy $\Omega(Pt(A)) \cong A$ and the spaces of points of frames are *sober*, that is, the points of the frame of open sets of the space are each given by a unique element of the underlying set of the space.

Theorem 9 (The Ω-Pt adjunction)**.**



is a contravariant adjunction which restricts to a duality between the full subcategories consisting of spatial frames *and* sober spaces*, respectively.*

The last property in the definition of a Stone space is equivalent to sobriety, thus Stone spaces are compact sober spaces in which the family of compact-open sets is closed under binary intersection and forms a basis. The Ω-Pt duality is not directly a generalisation of Stone duality as, under the Ω-Pt duality, a Stone space is sent to its entire open set lattice rather than just to the lattice of compact-opens. However, DL embeds in *Fr* via the ideal (or free directed join) completion $A \mapsto Idl(A)$. A *compact* element k of a frame F is one such that, for every directed subset S of F, we have $k \leq \bigvee S$ implies $k \leq s$ for some $s \in S$. We denote by $K(F)$ the join-subsemilattice of compact elements of F. In $Idl(A)$ the compact elements are the principal ideals, thus we have $A \cong K(Idl(A))$. Finally, calling *arithmetic frames* those frames whose compact elements form a sublattice which generates the frame by directed joins, we obtain the following diagram which illustrates how to move back and forth between the Stone duality and the Ω-Pt duality.



Ω-Pt & *Stone*

This diagram is not entirely correct as it does not specify what happens with morphisms. Stone duality acts on maps in $Stone$, which is not a full subcategory of the category Sob of sober spaces with continuous maps. The Ω-Pt duality, on the other hand, works on the full subcategory given by the Stone spaces. To get the maps dual to the Stone maps we need to restrict the frame homomorphisms between arithmetic frames to those that carry compact elements to compact elements. Or alternatively, we need to weaken the notion of morphism on DL to correspond to lattice homomorphisms $A \to Idl(B)$, which in turn may be seen as certain approximable relations from A to B, see (Abramsky & Jung 1994, Definition 7.2.24).

2.5 Stone spaces carrying the Scott topology

A topology is a second-order structure relative to the points of the space – which is hard to justify computationally. However, (Scott 1972) introduced topology based on limits given by directed suprema of compact elements which has strong computational content. In the wake of pointfree topology, taking frames, or locales, as the point of departure, the axioms of topology have been argued to have a natural place in computer science as an abstract version of semi-decidability (Smyth 1983) or semi-observability (removing the commitment to effectivity) (Abramsky 1987, Chapter 2.3).

Definition 10. Let X be a partially ordered set. The *Scott topology* on X is the collections of upsets $U \subseteq X$ such that if $\bigvee S \in U$ for some directed subset $S \subseteq X$, then $S \cap U \neq \emptyset$. A *directedly complete partially ordered set (dcpo)* is a partially ordered set in which every directed subset has a supremum. An *algebraic domain* is a dcpo in which every element is the directed supremum of the compact elements below it.

If one wants to specify algebraic domains using either the 'geometric logic' of frames or the finitary logic of DLs and BAs, the pertinent question is which algebraic domains, equipped with their Scott topology, are sober spaces and which are Stone spaces. *Any* algebraic domain, equipped with its Scott topology, is sober so such spaces can always be specified in geometric logic. It is a Stone space if and only if it satisfies property M (Minimal upper bounds property): the common upper bounds of any finite set of compact elements is the upset of a finite set of compact elements (Plotkin 1981, Chapter 8 p. 41). The DLs dual to these Stone spaces are characterised by the fact that each element is a finite join of join prime elements.[1] As we shall outline in Section 4.2, these Stone spaces and their dual lattices are crucial for the solution of domain equations as developed by Scott, Plotkin, Smyth, Larsen, Winskel, and others and culminating in Abramsky's Domain Theory in Logical Form, which formulates the method in terms of Stone duality.

2.6 Stone spaces as profinite objects

By Birkhoff's theorem any algebra is the directed union of its finitely generated subalgebras and DLs are locally finite, that is, their finitely generated subalgebras are finite. Therefore any DL is the directed union of its finite sublattices. In category theoretic terms, this corresponds to saying that the category DL is the ind-completion (that is, the category of inductive systems, or filtered

[1] An element p of a DL A is *join prime* provided, for any finite $F \subseteq A$, we have $p \leq \bigvee F$ implies $p \leq a$ for some $a \in F$.

colimit objects) over the category DL_{fin} of finite DLs. By duality, it follows that $Stone$ is the pro-completion (that is, the projective systems or directed limit objects) over $Stone_{fin}$, which, by discrete duality is simply the category Pos_{fin} of finite posets with order preserving maps. That is, the category $Stone$ is equivalent to the category of *profinite posets*, i.e. the pro-completion of the category of finite posets (Speed 1972). This cuts down to an equivalence between the category $BStone$ of Boolean spaces and the category of *profinite sets*.

3. Additional structure

Fundamental to applications of Stone duality is the presence of additional structure. A classical example is supplied by the *Predicate transformers* of (Dijkstra 1975) and in particular the *weakest precondition*: Given a 'mechanism' S and a 'condition' U, the weakest precondition is the property that characterises all those initial states so that execution of S surely terminates and does so in a final state satisfying U.

Taking the basic point of view of denotational semantics with a state space X and properties being the opens $U \subseteq X$, we see that, for a fixed S, the weakest precondition is a map

$$wp(S, \_)\colon \Omega(X) \to \Omega(X).$$

Also, reading off the above definition of how it acts, it is clear that S, whatever its nature, gives rise to two structures on the set of states:

- $X_S = \{x \in X \mid S \text{ always terminates from } x\}$;
- $R_S \subseteq X_S \times X$ given by

$$x\, R_S\, x' \iff \text{starting in } x, S \text{ possibly terminates in } x'.$$

and then

$$\begin{aligned} wp(S,U) &= \{x \in X_S \mid xR_Sx' \implies x' \in U\} \\ &= (R_S^{-1}(U^c))^c = [R_S](U). \end{aligned} \tag{2}$$

The crucial preservation properties of this operation on $\Omega(X)$ as studied in (Plotkin 1980) are:

(Strictness) $wp(S, \emptyset) = \emptyset$;

(Dual Operator) $wp(S, U \cap V) = wp(S,U) \cap wp(S,V)$;

(Continuity) $wp(S, \bigcup \mathcal{D}) = \bigcup\{wp(S,U) \mid u \in \mathcal{D}\}$ whenever $\mathcal{D} \subseteq \Omega(X)$ is directed.

Defining an operator from a binary relation by the formula in (2) is precisely as in the semantics of (Kripke 1959) for *modal logic*. In modal logic, unary connectives $\Box$ (such as $[R_S]$) transform propositions and satisfy:

$$\Box(\top) \equiv \top \quad \text{and} \quad \Box(\varphi \wedge \psi) \equiv \Box(\varphi) \wedge \Box(\psi). \tag{3}$$

The realisation that operations as in (3) correspond via duality to certain binary relations dates back to (Jónsson & Tarski 1951-52). Jónsson and Tarski did not work directly with dual spaces but used a point-free setting, known as *canonical extensions*. A purely duality theoretic account, generalised to the setting of Priestley duality, may be found in (Goldblatt 1989).

Definition 11. Let A be a distributive lattice and X its Priestley dual space. An operation $f\colon A^n \to A$ is an *operator* provided f preserves binary join in each coordinate. That is, for all $a_1, \ldots, a_n, b \in A$ and all i with $1 \le i \le n$ we have

$$\begin{aligned} &f(a_1, \ldots, a_i \vee b, \ldots, a_n) \\ =&f(a_1, \ldots, a_n) \vee f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n). \end{aligned}$$

An operator is said to be *normal* provided it preserves the empty join in each coordinate. That is, for all $a_1, \ldots, a_n \in A$ and all i with $1 \le i \le n$ we have

$$f(a_1, \ldots, a_{i-1}, \bot, a_{i+1}, \ldots, a_n) = \bot.$$

In modal logic this is called an n-ary $\Diamond$-modality. A *normal dual operator*, obtained by swapping $\wedge$ for $\vee$ and $\top$ for $\bot$ in the definitions above, is an n-ary $\Box$-modality.

A relation $R \subseteq X \times X^n$ on a Priestley space is called an $\Diamond$-*relation* provided it satisfies:

(i) $\geq \circ R \circ \geq^n = R$;
(ii) $x\,R = \{\overline{x} \in X^n \mid x\,R\,\overline{x}\}$ is closed for all $x \in X$;
(iii) $R^{-1}[U_1 \times \ldots \times U_n]$ is clopen for all $U_1, \ldots, U_n \in ClopU(X)$.

The definition of a $\Box$-*relation* is obtained by turning around the order relation in (i) and (iii). Since Boolean algebras are dual to those Priestley spaces in which the order is the equality on X, a $\Diamond$-relation is the same as a $\Box$-relation and is given by just (ii) (point-closed) and (iii) (pre-images of clopen rectangles are clopen).

Theorem 12 (Extended Priestley duality)**.** *Let A be a distributive lattice and X its Priestley dual space. There is a one-to-one correspondence between n-ary normal operators on A and $(n+1)$-ary $\Diamond$-relations on X. Given a normal operator $f\colon A^n \to A$, the corresponding relation is given by*

$$x\,R_f\,\overline{x} \iff F_x \supseteq f(F_{x_1} \times \ldots F_{x_n}).$$

Given an $(n+1)$-ary $\Diamond$-relations on X, the corresponding normal operator is given by

$$f_R(\overline{a}) = b \iff R^{-1}[\widehat{a}_1 \times \ldots \times \widehat{a}_n] = \widehat{b}.$$

Order-dually, there is a one-to-one correspondence between n-ary normal dual operators on A and $(n+1)$-ary $\Box$-relations on X. Given a dual operator $g\colon A^n \to A$, the corresponding relation is given by

$$x\,S_g\,\overline{x} \iff I_x \supseteq f(I_{x_1} \times \ldots I_{x_n}).$$

Given an $(n+1)$-ary $\Box$-relations on X, the corresponding dual operator is given by

$$g_R(\overline{a}) = b \iff (R^{-1}[(\widehat{a}_1)^c \times \ldots \times (\widehat{a}_n)^c])^c = \widehat{b}.$$

Note that this theorem tells us that, viewed as operations on the clopens of the dual space, any normal operator/dual operator is given as in Kripke semantics. Specifically, a unary normal operator is given by

$$\Diamond(\widehat{a}) = \{x \in X \mid \exists y\ (xRy \text{ and } y \in \widehat{a}\},$$

and a unary normal dual operator is given by

$$\Box(\widehat{a}) = \{x \in X \mid \forall y\ (xRy \text{ implies } y \in \widehat{a}\}.$$

Let us briefly reconsider the predicate transformer $wp(S, \_)$ in the light of extended duality. Note that $wp(S, \_)$ is not normal as an operator on $\Omega(X)$ (or $\mathcal{P}(X)$). This is because the relation defining it is on $X_S \times X$ so that $\Box_{R_S}\colon \mathcal{P}(X) \to \mathcal{P}(X_S)$ so that $wp(S,X) = X_S \neq X$ in general. The usual way in duality theory of dealing with non-normality is to add a new top or bottom to the lattice (which amounts to adding a new (topologically isolated) bottom or top, respectively, to the space. Indeed, in (Plotkin 1980), since $wp(S, \top) = \top$ is not true in general, a new top is added to $\Omega(X)$ by adding a new bottom to the dual domain.

In addition, $wp(S, \_)$ is strict. This is a well-known axiom in modal logic (along with those for S4 and S5 etc). These all have *first-order correspondents*. That is, they hold of the modal operator if and only if the dual relation satisfies a certain first-order property. For strictness, the first-order correspondent is totality of the relation (indeed, the domain is all of X_S because X_S is by definition the set of states for which R_S surely terminates).

The continuity property of $wp(S, \_)$ falls outside the setting of Stone/Priestley duality – strictly speaking, the whole example does as it pertains to the Ω-Pt duality, but the parts discussed so far fit also within the finitary dualities. We will comment on $wp(S, \_)$ again in Section 4 since all the axioms for this operation are of such a form that the algebras may be seen as 'algebras for a functor' and then the particular duality is just a parameter. Before going there we treat an application that does not lend itself as readily to the algebras for a functor point of view.

3.1 Application of duality to automata and regular languages

While it is thoroughly established that Stone duality is a central tool in semantics, there have been very few direct links with more algorithmically focused areas of computer science. The recent realisation that the algebraic methods of automata theory may be seen as a special instance of Stone duality (Gehrke, Grigorieff & Pin 2008) has opened up a new perspective on this point. As I hope to be able to show in Sections 4.3 and 5, this connection allows for vast generalisations and provides an opportunity to import methods and insights from semantics into various areas of the theory of formal languages.

In this subsection, we show that the basic building blocks of algebraic automata theory may be viewed as instances of duality theory. For an in depth duality theoretic account and further results on the connection between topological algebra and Stone/Priestley duality, we refer to (Gehrke 2016).

The cornerstone of the algebraic approach to automata and regular languages is the fact that one may assign, to each automaton, and more specifically to each regular language L over a finite alphabet Σ, a finite monoid $\mathcal{S}(L)$, known as the *syntactic monoid* of L. This monoid is a quotient of the free monoid Σ^* and the quotient map $\varphi_L\colon \Sigma^* \to \mathcal{S}(L)$, known as the *syntactic morphism*, *recognises* L in the sense that $L = \varphi_L^{-1}(P)$ for some $P \subseteq \mathcal{S}(L)$.

From an extended duality point of view, the monoid operation on Σ^* yields operations on $\mathcal{P}(\Sigma^*)$. In modal logic, one would focus on the corresponding normal operator

$$\Diamond_\cdot(K, L) = \{w \in \Sigma^* \mid \exists u \in K \text{ and } v \in L \text{ with } uv = w\} = KL,$$

which is the well-known concatenation product of languages. But, in an automaton, the dynamics is given by (non-deterministically) 'multiplying by a letter', accordingly (as identified in Section 2.1 display (1)), on the dual, the pertinent operations should be *adjoint* to concatenation. A binary operation may or may not have either of two adjoints, called left and right *residuals*. The concatenation product of languages has both and they are given by

$$\forall H, K, L \quad (HK \subseteq L \iff K \subseteq H\backslash L \iff H \subseteq L/K).$$

Notice that residuation by singletons are the unary operations known as *quotients* in language theory

$$\{u\}\backslash L = \{v | uv \in L\} := u^{-1}L \text{ and } L/\{v\} = \{u | uv \in L\} := Lv^{-1}.$$

Now consider a finite state automaton $\mathcal{A} = (Q, \Sigma, \delta, I, F)$ with $\delta \subseteq Q \times \Sigma \times Q$ and $I, F \subseteq Q$ arbitrary and let L be the language recognised by $\mathcal{A}$, then, for $a \in \Sigma$, the language $a^{-1}L$ is recognised by $(Q, \Sigma, \delta, I', F)$, obtained from $\mathcal{A}$ by just changing the set of initial states to $I' = \{q \mid \exists q'((q, a, q') \in \delta \text{ and } q' \in I)\}$. Similarly La^{-1} is recognised by $(Q, \Sigma, \delta, I, F')$ where F' is obtained by moving forward along transitions labelled by a from F. Thus, for a language given by an automaton, closing under quotients by words just moves the initial and final states around in the 'underlying machine'. Accordingly, the set

$$Q(L) = \{u^{-1}Lv^{-1} \mid u, v \in \Sigma^*\} \text{ is } \textit{finite}.$$

Definition 13. A *(Boolean) residuation ideal* of $\mathcal{P}(\Sigma^*)$ is a (Boolean) sublattice which is closed under residuation with arbitrary denominators, that is, it is closed under the operations $S\backslash(\_)$ and $(\_)/S$ for all $S \subseteq \Sigma^*$. For $L \subseteq \Sigma^*$, we denote by $\mathcal{B}(L)$ the Boolean residuation ideal generated by L. Note that a residuation ideal is in particular a *residuation subalgebra* of $\mathcal{P}(\Sigma^*)$ (i.e. closed under $\backslash$ and $/$) but need not be closed under the concatenation product.

Using the fact that the quotienting operations $u^{-1}(\_)$ are Boolean homomorphisms and that $S\backslash(\_) = \bigcap_{u \in S} u^{-1}(\_)$ (and similarly on the right), we obtain the following proposition.

Proposition 14. *If L is a language recognised by some automaton, then $\mathcal{B}(L)$ is the BA generated by $Q(L)$ and is thus finite.*

Now the following extended discrete duality result is pertinent.

Theorem 15. *There is a one-to-one correspondence between finite Boolean residuation ideals of $\mathcal{P}(\Sigma^*)$ and finite monoid quotients of Σ^*. In particular, for L recognised by an automaton, the dual of the residuation algebra $(\mathcal{B}(L), \backslash, /)$ is the syntactic monoid $\mathcal{S}(L)$ and the discrete dual of the embedding $\mathcal{B}(L) \hookrightarrow \mathcal{P}(\Sigma^*)$ is the syntactic morphism $\varphi_L\colon \Sigma^* \twoheadrightarrow \mathcal{S}(L)$.*

The idea of the proof is as follows: If $\mathcal{B} \hookrightarrow \mathcal{P}(\Sigma^*)$ is a finite Boolean residuation ideal, then, by discrete BA duality, the dual of $\mathcal{B}$ is a finite set X and the duals of the residuation operations $\backslash$ and $/$ are ternary relations on X. It is well-known that if two operations are related by adjunction (contravariant or not), then their dual relations are the same, up to rearrangement of the order of the coordinates. So $\backslash$ and $/$ give, up to rearrangement of the coordinates, *one* ternary relation $R \subseteq X^3$. Now, what seems like a strange thing from the perspective of extended duality theory happens: with a well-chosen order of the coordinates, R is *functional*, that is, it is the graph of a *binary operation* $\cdot\colon X^2 \to X$. This binary operation, which is the relation dual to the residuals, is the monoid operation of X. Finally, the fact that the embedding $\mathcal{B}(L) \hookrightarrow \mathcal{P}(\Sigma^*)$ has the property of a residuation ideal corresponds via duality to the quotient map $\Sigma^* \twoheadrightarrow X$ being a monoid morphism.

Quite a number of consequence may be derived from Theorem 15. For one, using the fact that a monoid morphism $\varphi\colon \Sigma^* \to F$ to a finite monoid may be seen as a special automaton, we obtain the classical result that a language $L \subseteq \mathcal{P}(\Sigma^*)$ is recognised by an automaton if and only if it is recognised by a finite monoid.

Another consequence of Theorem 15 is the universal property of the syntactic morphism: L is recognised by some finite monoid quotient $\varphi\colon \Sigma^* \to F$ if and only if L belongs to the residuation ideal $\mathcal{B}$ dual to φ, which in turn is equivalent to $\mathcal{B}(L)$ being contained in $\mathcal{B}$. But $\mathcal{B}(L) \hookrightarrow \mathcal{B}$, again by duality, is equivalent to φ_L factoring through φ. That is, L is recognised by φ if and only if φ_L factors through it.

A slightly more involved consequence of Theorem 15 is the following theorem.

Theorem 16. *(Gehrke, Grigorieff & Pin 2008) The dual of the Boolean residuation algebra $(Reg(\Sigma^*), \backslash, /)$ of all languages over Σ recognised by automata is the profinite completion $\widehat{\Sigma^*}$ of the free monoid Σ^*.*

The idea of the proof is as follows: given the equivalence of recognition by automata and by finite monoids, and the fact that recognition by a finite monoid quotient is equivalent to belonging to the dual residuation ideal, we see that $Reg(\Sigma^*)$ is given by

$$Reg(\Sigma^*) = \bigcup\{\varphi^{-1}(\mathcal{P}(F)) \mid \varphi \in Hom(\Sigma^*, F) \text{ and } F \text{ finite}\}.$$

Noting that this is a directed union, we see that $Reg(\Sigma^*)$ is in fact the filtered colimit of its finite residuation ideals, and thus, by extended Stone duality, the dual of $Reg(\Sigma^*)$ is the directed limit of the finite quotients of Σ^*. The latter is, by definition, the profinite completion of Σ^*. For a detailed proof along these lines of the following slightly more general theorem, see (Gehrke 2016, Theorem 4.5).

Theorem 17. *The profinite completion of any abstract algebra* including its operations *is the dual of the residuation algebra based on the recognisable subsets of the algebra.*

In (Gehrke 2016), a more general theorem is proved, namely that all topological algebras, of any type, based on Boolean spaces are, up to isomorphism, the extended Stone duals of certain Boolean residuation algebras thus obtaining *a duality between algebras and algebras*. This is surprising from the point of view of duality theory as an algebraic operation $f\colon X\times\ldots\times X\to X$ on the dual space of a BA A yields coalgebra structure on A in the form of a homomorphism $h\colon A\to A\oplus\ldots\oplus A$ where $\oplus$ is coproduct (which is not an easy construction to deal with for lattices and Boolean algebras). This, more direct duality theoretic approach was taken by Rhodes and Steinberg who independently introduced a bialgebraic and duality-theoretic approach to profinite semigroups (Rhodes & Steinberg 2009, Chapter 8). Their point of view, based on Boolean rings rather than Boolean algebras, identifies deep connections with classical algebra.

Related work in a coalgebraic setting has been very active in the last few years producing results, e.g. on minimisation for more general structures such as Markov processes, and generalisations of Eilenberg type theorems in various categorical settings (Bonchi et al. 2014; Kozen et al. 2013; Adamek et al. 2015). Further, in model theory and in the model theoretic and universal algebraic approach to Constraint Satisfaction Problems (CSP) a number of recent results use topological structure on term clones (Bodirsky & Pinsker 2015). See (Gehrke & Pinsker 2016) for the connection with Eilenberg-Reiterman theory.

3.2 Beyond regular languages

An exciting consequence of the fact that recognition and syntactic monoids are instances of Stone duality is that it opens the way for extending these notions to arbitrary languages. The foundations of such an extension were laid in (Gehrke, Grigorieff & Pin 2010) in terms of certain monoids equipped with uniform structure. Here we give an equivalent notion, developed in (Gehrke, Petrişan & Reggio 2016), which is better suited for applying the tools of duality theory.

When only regular languages are considered, all Boolean algebras considered are subalgebras of $Reg(\Sigma^*)$ and thus all pertinent spaces are quotients of the free profinite monoid. Once we broaden the scope and consider arbitrary languages, the ambient Boolean algebra is $\mathcal{P}(\Sigma^*)$ and thus its dual space plays a central rôle. The dual space of $\mathcal{P}(\Sigma^*)$, which we denote by $\beta(\Sigma^*)$, is also the *Stone-Čech compactification* of the discrete space on Σ^*. The embedding of Σ^* in $\beta(\Sigma^*)$ is given by identifying $w\in\Sigma^*$ with the point corresponding to the principal ultrafilter $F_w=\{S\subseteq\Sigma^*\mid w\in S\}$. The clopen sets of $\beta(\Sigma^*)$ are of course the $\widehat{L}=\{x\in\beta(\Sigma^*)\mid L\in F_x\}$ for $L\in\mathcal{P}(\Sigma^*)$ but this is also the topological closure $\overline{L}$ viewing L as a subset of $\beta(\Sigma^*)$ via the embedding of Σ^* in $\beta(\Sigma^*)$.

Given a homomorphism $h\colon A\to\mathcal{P}(\Sigma^*)$ of BAs, by duality, we get a continuous map $St(h)\colon\beta(\Sigma^*)\to X$, where X is the dual of A. In particular, if $f\colon\Sigma^*\to S$ is a set map then $f^{-1}\colon\mathcal{P}(S)\to\mathcal{P}(\Sigma^*)$ is a BA homomorphism and its dual $St(f^{-1})\colon\beta(\Sigma^*)\to\beta(S)$ is a continuous extension of f. The fact that $\beta(\Sigma^*)$ is also the Stone-Čech compactification of the discrete space Σ^* tells us a bit more: If $f\colon\Sigma^*\to X$ is any set map into *any* compact Hausdorff space X, then f has a unique continuous extension $\beta(f)\colon\beta(\Sigma^*)\to X$. In the case of $f\colon\Sigma^*\to S$ a set map, viewing f as mapping into $\beta(S)$, we obtain $\beta(f)\colon\beta(\Sigma^*)\to\beta(S)$, and $\beta(f)$ is of course one and the same map as $St(f^{-1})$. Thus we will write $\beta(f)$ for $St(f^{-1})$ in this setting.

Recall that, from the duality point of view, the syntactic monoid is the dual of the residuation ideal $\mathcal{B}(L)$ (Definition 13) and that, for a regular language, this is the Boolean algebra generated by $Q(L)$ (Proposition 14). Classes of languages of interest in complexity theory, such as Boolean circuit classes, are typically closed under the quotient operations $u^{-1}(\_)$ and $(\_)u^{-1}$ but not under residuation with respect to arbitrary denominators. Thus, it is appropriate to consider Σ^* as equipped with left and right actions of itself, whose components, for each $w\in\Sigma^*$, are

$$\lambda_w\colon\Sigma^*\to\Sigma^*,\ u\mapsto wu\quad\text{and}\quad\rho_w\colon\Sigma^*\to\Sigma^*,\ u\mapsto uw.$$

This is a *biaction* of Σ^* on itself in the sense that these actions are compatible in that $\lambda_u(\rho_w(v))=u(vw)=(uv)w=\rho_w(\lambda_u(v))$.

The duals of these actions are the *left quotients* and *right quotients* $u^{-1}(\_)$ and $(\_)u^{-1}$ and these are homomorphisms and compatible Σ^*-actions on $\mathcal{P}(\Sigma^*)$. Dualising again, we see that $\beta(\Sigma^*)$ is equipped with (compatible and continuous) left and right Σ^*-actions, which extend the action on Σ^*.

Given a Boolean subalgebra $\mathcal{B}$ of $\mathcal{P}(\Sigma^*)$ closed under left and right quotients by words, when dualising we get a quotient space $\tau\colon\beta(\Sigma^*)\twoheadrightarrow X$ and the duals of the restricted quotient operations yield compatible actions, which we also call $\beta(\lambda_w)$ and $\beta(\rho_w)$, respectively, and the following diagrams commute as the dual ones do

$$\begin{array}{ccc} \beta(\Sigma^*) & \xrightarrow{\beta(\lambda_w)} & \beta(\Sigma^*) \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau} \\ X & \xrightarrow{\beta(\lambda_w)} & X \end{array}\qquad\qquad \begin{array}{ccc} \beta(\Sigma^*) & \xrightarrow{\beta(\rho_w)} & \beta(\Sigma^*) \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau} \\ X & \xrightarrow{\beta(\rho_w)} & X. \end{array}$$

Taking $M=\tau[\Sigma^*]$ one may show that the dual of $\mathcal{B}$ and the quotient map $\tau\colon\beta(\Sigma^*)\twoheadrightarrow X$ belong to the following category.

Definition 18. A *Boolean space with an internal monoid* is a pair (X,M) consisting of

- a Boolean space X,
- a dense subspace M equipped with a monoid structure,
- a biaction of M on X with continuous components extending the biaction of M on itself.

A *morphism* between two Boolean spaces with internal monoids (X,M) and (Y,N) is a continuous map $f\colon X\to Y$ such that f restricts to a monoid morphism $M\to N$.

One may then show that morphisms are in fact also biaction-preserving maps.

Definition 19. Let Σ be a finite alphabet, let $L\in\mathcal{P}(\Sigma^*)$ be a language, and let $f\colon(\beta(\Sigma^*),\Sigma^*)\to(X,M)$ be a morphism of space with internal monoids. We say that L (or $\widehat{L}$) is *recognised by the morphism* f if there is a clopen $C\subseteq X$ such that $\widehat{L}=f^{-1}(C)$. Moreover, the language L is *recognised by the space* (X,M) if there is a morphism $(\beta(\Sigma^*),\Sigma^*)\to(X,M)$ recognising L. Similarly, we say that a morphism (or a space) recognises a Boolean algebra if it recognises all its elements.

With this definition of recognition, each $L\in\mathcal{P}(\Sigma^*)$ has a *syntactic space* and a *syntactic morphism* which is the space with an internal monoid and the quotient morphism dual to the Boolean subalgebra of $\mathcal{P}(\Sigma^*)$ closed under quotients generated by L. As in the regular setting, the syntactic morphism of L has a universal property in that it factors through any morphism recognising L (Gehrke, Grigorieff & Pin 2010, Section 3).

We will see in each of the following sections how this extended notion of recognition can be applied to study classes of not necessarily regular languages.

4. Hyperspaces

4.1 Algebras and coalgebras for a functor

Some categories of DLs, BAs, or frames with additional operations may be identified as the category of all *algebras for a functor* on DLs, BAs, or frames, respectively. In such cases, the dual category consists of the *coalgebras* for the dual functor.

Definition 20. Let $\mathcal{C}$ be a category and $F\colon \mathcal{C} \to \mathcal{C}$ a functor. An F-algebra (or an algebra for the functor F) is a pair (A, f) where A is an object from $\mathcal{C}$ and $f\colon F(A) \to A$ is a morphism from $\mathcal{C}$. A homomorphism $h\colon (A, f) \to (B, g)$ of F-algebras is a morphism $h\colon A \to B$ in $\mathcal{C}$ so that $h \circ f = g \circ F(h)$.

An F-coalgebra (or a coalgebra for the functor F) is a pair (A, f) where A is an object from $\mathcal{C}$ and $f\colon A \to F(A)$ is a morphism from $\mathcal{C}$. A morphism $h\colon (A, f) \to (B, g)$ of F-coalgebras is a morphism $h\colon A \to B$ in $\mathcal{C}$ so that $F(h) \circ f = g \circ h$.

A DL expansion (DLE) is an algebra $(A, (f^A)_{f \in \sigma})$ where A is a DL and σ is a signature of additional operations. A class of such algebras may be seen as algebras for a functor on DL whenever it is given by a set Σ of *rank* 1 identities.

An identity $s(\overline{x}) \approx t(\overline{x})$ in the combined signature of DLs and σ, is said to have rank 1 provided there is no nesting of operations from σ in the equation and each variable occurrence is in the scope of exactly one operation symbol from σ. Thus all the equations mentioned in Section 3 (strictness, normality, operator, ...) are of rank 1. Heyting algebras (HA) are DLs expanded by a single binary operation, namely implication. Quite a number of equational properties of HAs are of rank 1, e.g.

$$
\begin{aligned}
x \to x &\approx 1 \\
x \to (y \wedge z) &\approx (x \to y) \wedge (x \to z) \\
(x \vee y) \to z &\approx (x \to z) \wedge (y \to z),
\end{aligned}
$$

but, crucially, some are not, e.g. $1 \to x \approx x$.

Given a set Σ of rank 1 equations in a DLE signature σ, a functor F on DL is obtained as follows. For a DL A, one defines $F(A)$ by generators and relations. The generating set is

$$G(A) = \{f(\overline{a}) \mid f \in \sigma, \overline{a} \in A^{ar(f)}\}$$

(here the $f(\overline{a})$ are just formal objects). Thus we take the free DL on G, call it $F_{DL}(G)$. The relations are the ones in

$$R(A) = \{(s(\overline{a}), t(\overline{a})) \mid s(\overline{x}) \approx t(\overline{x}) \in \Sigma, \overline{a} \in A^{|\overline{x}|}\}$$

where, in 'plugging-in' $\overline{a}$ in terms s and t, any DL operations in the scope of operations from σ are carried out in A, and any outside are interpreted in $F_{DL}(G)$. For example, if we would want to see DLs expanded with an implication satisfying just the one equation $x \to (y \wedge z) \approx (x \to y) \wedge (x \to z)$, we would have, for a DL A

$$F(A) = F_{DL}(a \to b \mid a, b \in A)/\theta(R(A))$$

where $\theta(R(A))$ is the DL congruence generated by

$$R(A) = \{(a \to d, (a \to b) \wedge (a \to c)) \mid a, b, c, d \in A \text{ and } d = b \wedge c\}.$$

For a DL homomorphism $h\colon A \to B$ we obtain a DL homomorphism from $F_{DL}(G(A))$ to $F_{DL}(G(B))/\theta(R(B))$ by

$$f(\overline{a}) \;\mapsto\; [f(h(a_1), \ldots, h(a_{ar(f)}))]_{\theta(R(B))}.$$

Then by noticing that it factors through $\theta(R(A))$, we obtain a DL homomorphism $F(h)\colon F(A) \to F(B)$.

Now given a DLE $(A, (f^A)_{f \in \sigma})$ satisfying a set Σ of rank 1 equations, we obtain a corresponding F-algebra on A. To see this, define a map from $F_{DL}(G(A))$ to A by $f(\overline{a}) \mapsto f^A(\overline{a})$ and notice that, since A satisfies Σ, it factors through $\theta(R(A))$ and thus gives a DL homomorphism from $h\colon F(A) \to A$. Conversely, given an F-algebra (A, h) we obtain a DLE satisfying Σ by defining $f^A(\overline{a}) := h([f(\overline{a})]_{\theta(R(A))})$ and these assignments are inverse to each other. Similarly one may show that the DLE homomorphisms between two DLEs are exactly the F-algebra homomorphisms between the corresponding F-algebras thus establishing an isomorphism between the category of DLEs satisfying Σ and the category of F-algebras. For more details see (Bonsangue & Kurz 2006; Kurz & Rosicky 2012).

Now given a functor F on a category which is involved in a duality with another category, we of course get a dual functor T on the dual category by pre- and post-composing with the two functors of the duality. Further, it is easy to see that the dual of an F-algebra based on A is a T-coalgebra based on the dual of A.

Example 21 (Modal algebras and the Vietoris functor)**.** Modal algebras $(B, \Box)$, that is, BAs with a single unary normal dual operator are given by rank 1 axioms and are thus algebras for a functor. Following the recipe given above it is not hard to see that the corresponding functor on BA is

$$F(B) = \text{ free BA over the finite meet semilattice reduct of } B$$

The dual of F, which must send X to a space homeomorphic to $St(F(Clop(X)))$, may be seen as the restriction of the *Vietoris functor*, first defined in (Vietoris 1922), to Boolean spaces. The Vietoris functor sends a Boolean space X to the space

$$\mathcal{V}(X) = \{F \subseteq X \mid F \text{ is closed in } X\}$$

equipped with the topology generated by the subbasis of sets $\Box U$ and $\Diamond U$ for U clopen in X, where

$$\Box(U) = \{F \in \mathcal{V}(X) \mid F \subseteq U\}$$

and

$$\Diamond(U) = \{F \in \mathcal{V}(X) \mid F \cap U \neq \emptyset\} = (\Box(U^c))^c$$

The action of $\mathcal{V}$ on continuous maps is simply given by forward image. As it will be useful later, we also observe that $\mathcal{P}_{fin}(X)$, the set of finite subsets of X, is dense in $\mathcal{V}(X)$.

The connection between the Vietoris functor and modal algebra goes back to (Esakia 1974). See also (Johnstone 1982, Chapter III.4) as well as (Abramsky 2005; Venema & Vosmaer 2014) for overviews of the rôle of the Vietoris construction in computer science and logic.

Coalgebra is a natural setting for studying and modelling transition systems. What's more, by varying the functor as well as the underlying category, one can obtain a uniform theory of a wide range of transition systems in computer science, such as probabilistic systems, quantum systems etc. From this point of view, the coalgebras come first, and if the underlying category is in a duality, then the algebras for the dual functor provide a logic for specification for the coalgebras, much as outlined above. The ensuing area of *coalgebraic logic* is an active area with strong ties to (extended) Stone/Priestley duality (Cîrstea et al. 2011).

4.2 Hyperspaces in denotational semantics

The problem of modelling weakest precondition in denotational semantics lends itself well to the algebras/coalgebras for a functor point of view. The relational semantics, as in (2), may be captured on (spatial) frames by the axioms of strictness, being a dual operator, and continuity, and, when adding a bottom to the domains, we may assume normality. Now as these axioms are all of rank 1, we may see these frames with additional structure as the algebras for a functor – or via Ω-Pt duality, we may see the denotational semantics of weakest precondition directly as the coalgebras for the dual functor on sober spaces. Since the axioms required are closely related to those of modal logic, it is not surprising that the appropriate functor is a kind of hyperspace construction closely related to

Vietoris. This was first realised and worked out for flat domains in (Plotkin 1980) and generalised and fully identified as a topological phenomenon in (Smyth 1983).

Let A be a frame. The functor F for which F-algebras correspond to frame expansions $(A, \Box)$ satisfying

$$\Box\bot \approx \bot \qquad \Box\top \approx \top \qquad \Box(a \wedge b) \approx \Box a \wedge \Box b \tag{4}$$

(i.e. weakest precondition without the continuity property) is given by

$$F(A) = F_{Fr}(\{\Box a \mid a \in A\})/\theta(R(A))$$

where

$$\begin{aligned} R(A) = &\{(\Box\bot^A, \bot), (\Box\top^A, \top)\} \\ &\cup \{(\Box c, \Box a \wedge \Box b) \mid a, b, c \in A \text{ with } c = a \wedge b\} \end{aligned}$$

(Frames are not (finitary) algebras because of the arbitrary joins, but free objects relative to geometric theories exist, see (Johnstone 1982, Chapter II.2.11) and (Vickers 1993) for a general introduction). It is not hard to see that in this case $F(A) = F_{\bigvee^+}((A, \wedge))$ where $F_{\bigvee^+}$ denotes the free completion of A (as either a poset or a meet semilattice) by non-empty joins. Concretely, This may be obtained as the frame of non-empty downsets of A. Denote the set of proper filters of A by $Filt_{pr}(A)$. Since completely join prime filters must be witnessed by the elements of A which they contain, and since all elements of A except $\bot^A$ are completely join prime in $F_{\bigvee^+}((A, \wedge))$, we have:

Proposition 22. *(Smyth 1983, Theorem 4.1) Let A be a frame then $Pt(F_{\bigvee^+}((A, \wedge))) \cong Filt_{pr}(A)$ equipped with the topology generated by the sets*

$$\widehat{a} = \{\mathcal{F} \mid a \in \mathcal{F}\}.$$

Now let X and Y be sober spaces. Noticing that $F_{\bigvee^+}((A, \wedge))$ is a spatial frame and applying Proposition 22 in the case $A = \Omega(Y)$ we obtain

$$\begin{aligned} \{\Box\colon \Omega(Y) \to \Omega(X) \mid \Box \text{ satisfies (4)}\} &\leftrightarrow Hom_{Fr}(F_{\bigvee^+}(A), \Omega(X)) \\ &\leftrightarrow [\, X \,,\, Filt_{pr}(\Omega(Y))\,] \end{aligned}$$

where $[\_\,,\_]$ stands for the set of continuous functions from the first space in question to the second. Finally, if we also want the $\Box$ operations to be Scott continuous, then it is not difficult to see that the space of points needs to be cut down to the proper filters which are Scott open, but by the celebrated theorem of (Hofmann & Mislove 1981), these correspond to the non-empty compact subsets of $\Omega(Y)$ with the topology generated by the $\Box(U) = \{K | K \subseteq U\}$ for $U \in \Omega(Y)$, which is the *Smyth powerdomain*.

Another area where duality, and in particular duality for functors, has played a central rôle, is the search for solutions to domain equations. Scott's original example of a domain satisfying $Y = [Y \to Y]$ is obtained as a (special) directed limit of finite posets. Thus it is a Stone space (see Section 2.6) even though this was not used explicitly. One may view the constructors, such as function space, sums and products, and compositions of these as functors, and then solutions are fixed points of such functors. An algebra for a functor, $h\colon F(A) \to A$, may be seen as a pre-fixed-point (Smyth & Plotkin 1982), and in fact, it is not hard to see that an initial (or free) F-algebra must be a fixed point for F. So the question becomes whether such functors have initial algebras (or terminal coalgebras – depending on the direction one considers for the morphisms). Clearly, the function space $[\_\,,\_]$ is the main problem, and early on (Plotkin 1976) identified a category of domains, the SFP domains, which is closed under function space. (Larsen & Winskel 1991) realised that dual objects (in the form of information systems) made the existence of fixed points via countable colimits unproblematic. While these early contributions may in retrospect be seen as hinging on duality, the duality was not used or identified explicitly. This development culminated in the *Domain Theory in Logical Form* (DTLF) of (Abramsky 1991), where a uniform result on solving domain equations was identified as arising via Stone duality.

The idea of DTLF is the following: As discussed in Section 2.5, the algebraic domains satisfying property M (also known as $2/3SFP$ domains) are Stone spaces in their Scott topology – the SFP domains form a subcategory of these, and the dual category of DLs is equipped with a functor dual to the function space construction on SFP. It is a construction akin to adding an implication-like operation, i.e it is order reversing in one coordinate, but on the level of the DLs it is not contravariant. The algebras for this functor are finitary algebras and thus the existence of free algebras, and hence fixed points of the functor is unproblematic.

Apart from placing the previous work on domain equations in a uniform and conceptually simple environment, the casting of DTLF in Stone duality identified domains, which are denotational semantic models of computation, as *dual* to the corresponding DLs (presented by Abramsky in a generators and relation format akin to sequent calculi for logics) – which he identified as a kind of programme logic for specification. Thus, DTLF provides a mathematically precise result identifying behavioural models as dual to logics for specification.

This is by no means the end of that story. The handbook chapter (Abramsky & Jung 1994) has greatly expanded the theory laying the groundwork for a theory based on *continuous domains* rather than merely algebraic ones. However, fully generalising DTLF to the continuous setting, and in particular giving an account of systems with probabilistic effects in this vein, is still an ongoing topic of research (Jung 2013).

4.3 Schützenberger products

The theory of formal languages is related to logic through Büchi's *logic on words*. A word $w \in \Sigma^*$ over a finite alphabet Σ may be seen as a relational structure based on the set $\{0, \ldots, |w| - 1\}$.[2] This structure is equipped, at least, with a unary predicate for each letter $a \in \Sigma$, which holds at i if and only if $w_i = a$. In addition the words may be considered as equipped with various other predicates such as (uniform) numerical predicates, which are simply predicates on the natural numbers that, by restriction, also live on the domain $\{0, \ldots, |w| - 1\}$ of w viewed as a structure. Now given a sentence Φ (in a language interpretable over words as structures), the set L_Φ of all words satisfying Φ is a language over Σ. Büchi's result on finite words shows that the languages given by monadic second order (MSO) sentences in the logical language with the letter predicates and the numerical predicate $<$ are precisely the languages recognised by automata. In descriptive complexity theory, characterisations of many complexity classes beyond the regular setting in terms of corresponding logic fragments have been given (Immerman 1998).

Sentences of a logical calculus are built up from atomic formulas by application of logical connectives. Thus one can build up the corresponding classes of formal languages by understanding the effect of adding a layer of connectives. This is particularly interesting for quantifiers. In the theory of regular languages, as well as their interaction with Boolean circuit classes, the connection between recognition by a block-product of monoids (a form of bilateral semidirect product) and quantification has served as a central tool in the study of classes given by logic fragments (Straubing 1994), see also (Tesson & Thérien 2007) for an introductory survey.

The material in the remainder of this section comes from the paper (Gehrke, Petrişan & Reggio 2016). In particular, we introduce

[2] $|w| \in \mathbb{N}$ denotes the length of the word $w = w_0 \cdots w_{|w|-1} \in \Sigma^*$.

the unary Schützenberger product for spaces with internal monoids and show that it is *dual to adding a layer of quantification on the language side* (Theorem 26).

Given a formula Φ with one free first-order variable x, the set of word models of Φ may naturally be given as a subset of the set $\Sigma^* \otimes \mathbb{N}$ of *words in* Σ^* *with a marked spot* defined by

$$\Sigma^* \otimes \mathbb{N} := \{(w,i) \in \Sigma^* \times \mathbb{N} \mid i < |w|\}.$$

Such a marked word (w,i) is a model of Φ provided w satisfies Φ under the interpretation in which x points to the ith position.

The set $\Sigma^* \otimes \mathbb{N}$ does not have a suitable monoid structure, only a Σ^*-biaction structure: For $v \in \Sigma^*$, the components of the left and right actions are given by

$$\lambda_v(w,i) := (vw, i + |v|),$$
$$\rho_v(w,i) := (wv, i).$$

However, $\Sigma^* \otimes \mathbb{N}$ embeds in the free monoid over the extended alphabet $\Sigma \times 2$ via the map

$$\gamma_1 \colon \Sigma^* \otimes \mathbb{N} \to (\Sigma \times 2)^*$$
$$(w,i) \mapsto w^{(i)},$$

where $w^{(i)}$ is defined by it being a word of the same length as w and

$$(w^{(i)})_j := \begin{cases} (w_j, 0) & \text{if} \quad i \neq j < |w| \\ (w_i, 1) & \text{if} \quad i = j. \end{cases}$$

This is an embedding of sets with biactions of Σ^* if we define the left and right actions of Σ^* on $(\Sigma \times 2)^*$ as given by the monoid embedding $\gamma_0 \colon \Sigma^* \to (\Sigma \times 2)^*, w \mapsto w^0$, where w^0 has the same length as w and

$$(w^0)_j := (w_j, 0) \quad \text{for each} \quad j < |w|.$$

This allows us to define L_Φ as a language in the extended alphabet $\Sigma \times 2$: It is of the set of words which are images under γ_1 of marked words satisfying Φ. Thus L_Φ is always a subset of the language $\mathrm{Im}(\gamma_1) = (\Sigma \times \{0\})^*(\Sigma \times \{1\})(\Sigma \times \{0\})^*$, see (Straubing 1994) for more details. Now consider the span

$$\Sigma^* \xleftarrow{\ \pi\ } \Sigma^* \otimes \mathbb{N} \xrightarrow{\ \gamma_1\ } (\Sigma \times 2)^*$$

where π is the projection on the first coordinate. Then $L_{\exists x \Phi}$ is obtainable from L_Φ via this span in the sense that

$$L_{\exists x \Phi} = \pi[\gamma_1^{-1}(L_\Phi)].$$

The following problem is fundamental in the study of formal languages given by logic fragments: Given a space with an internal monoid recognising L_Φ, we would like to identify a space with an internal monoid recognising $L_{\exists x \Phi}$. Assume that a language $L \subseteq (\Sigma \times 2)^*$ is recognised by a morphism of Boolean spaces with internal monoids

$$\tau \colon (\beta(\Sigma \times 2)^*, (\Sigma \times 2)^*) \to (X, M).$$

Let $\mathcal{B}$ be the dual of X. We obtain the following diagram

$$\mathcal{P}(\Sigma^*) \underset{\pi[\_]}{\overset{\pi^{-1}}{\rightleftarrows}} \mathcal{P}(\Sigma^* \otimes \mathbb{N}) \xleftarrow{\ \gamma_1^{-1}\ } \mathcal{P}((\Sigma \times 2)^*) \xleftarrow{\ \tau^{-1}\ } \mathcal{B}$$

and we define $L_\exists = \pi[\gamma_1^{-1}(L)]$. Since forward image under π is a normal operator, so is $(\_)_\exists \circ \tau^{-1}$. Therefore the dual of this operation is a $\Diamond$-relation, or equivalently, given by a continuous map into the Vietoris space of X. This map

$$\xi_1 \colon \beta(\Sigma^*) \to \mathcal{V}(X) \tag{5}$$

is given by the composition $\tau \circ \beta\gamma_1 \circ (\beta\pi)^{-1}$, or equivalently as the unique continuous extension of the map $\xi_1 \colon \Sigma^* \to \mathcal{P}_{fin}(M)$ defined for $w \in \Sigma^*$ by

$$\xi_1(w) = \tau[\gamma_1[\pi^{-1}(w)]].$$

The set of $L_\exists$ for $L = \tau^{-1}(C)$ for some C clopen in X, is not closed under intersections nor under complements. More importantly, the BA generated by these languages *is not closed under the quotient operations*. However, using the map γ_0 one can describe the Boolean algebra with quotients generated by the languages $L_\exists$, for L coming from the Boolean algebra $\mathcal{B}$ in terms of the embeddings of $\mathcal{B}$ and of the $L_\exists$'s in $\mathcal{P}((\Sigma \times 2)^*)$. By duality this yields a construction of a recognising space with an internal monoid based on the Cartesian product of X and $\mathcal{V}(X)$.

Definition 23. Let (X, M) be a Boolean space with an internal monoid. The *unary Schützenberger product* of (X, M) is the pair $(\Diamond X, \Diamond M)$, where $\Diamond X$ is the space $\mathcal{V}(X) \times X$ equipped with the product topology and $\Diamond M$ is the bilateral semidirect product $\mathcal{P}_{fin}(M) * M$ of the monoids $(\mathcal{P}_{fin}(M), \cup)$ and $(M, \cdot)$. Explicitly, the underlying set of $\Diamond M$ is the Cartesian product $\mathcal{P}_{fin}(M) \times M$, and the operation on $\mathcal{P}_{fin}(M) * M$ is given by

$$(S, m) * (T, n) := (S \cdot n \cup m \cdot T, m \cdot n).$$

The biaction of $\Diamond M$ on $\Diamond X$ is given by

$$(S, m)(K, x) := (Sx \cup mK, mx)$$

and

$$(K, x)(T, n) := (Kn \cup xT, xn).$$

Here $Sx = \{m'x \mid m' \in S\}$ and $mK = \{mx' \mid x' \in K\}$ are both defined using the left action of M on X. Similarly the right action of $\Diamond M$ on $\Diamond X$ is defined from the right action of M on X.

Lemma 24. *Let* (X, M) *be a Boolean space with an internal monoid. The unary Schützenberger product* $(\Diamond X, \Diamond M)$ *is a Boolean space with an internal monoid and the projection* $\pi_2 \colon \Diamond X \to X$ *onto the second component is a morphism of Boolean spaces with internal monoids.*

We have the following result which shows that the unary Schützenberger product recognises $L_\exists$ whenever the original space recognises L.

Proposition 25. *If* $\tau \colon (\beta(\Sigma \times 2)^*, (\Sigma \times 2)^*) \to (X, M)$ *is a morphism of Boolean spaces with internal monoids recognising* L_Φ, *then there is a morphism* $\xi \colon (\beta(\Sigma^*), \Sigma^*) \to (\Diamond X, \Diamond M)$ *with first coordinate equal to* ξ_1 *of (5) so that* ξ *recognises* $L_{\exists x.\Phi}$ *and the following diagram commutes.*

$$\begin{array}{ccc} \beta(\Sigma^*) & \xrightarrow{\ \xi\ } & \Diamond X \\ {\scriptstyle \beta\gamma_0}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \beta(\Sigma \times 2)^* & \xrightarrow{\ \tau\ } & X \end{array}$$

In fact, not only does the unary Schützenberger product recognise existential projections, it is minimal in doing so in the sense of the following theorem – which, for technical reasons, is expressed in terms of semigroups rather than monoids.

Given a Boolean space with an internal semigroup (X, S), let $\mathcal{B}(X, \Sigma)$ denote the Boolean algebra generated by the languages in $\mathcal{P}(\Sigma^+)$ recognised by (X, S). Further, for a subset $\mathcal{S}$ of a BA, denote by $<\mathcal{S}>_{BA}$ the Boolean subalgebra generated by $\mathcal{S}$.

Theorem 26. *Let* (X, S) *be a Boolean space with an internal semigroup, and let* $\mathcal{B}(X, \Sigma \times 2)_\exists$ *denote the Boolean subalgebra of* $\mathcal{P}(\Sigma^+)$ *closed under quotients generated by the family*

$$\{L_\exists \mid L \in \mathcal{B}(X, \Sigma \times 2)\}.$$

Then

$$\mathcal{B}(\Diamond X, \Sigma) = <\mathcal{B}(X, \Sigma) \cup \mathcal{B}(X, \Sigma \times 2)_{\exists}>_{BA}.$$

Schützenberger did not consider the unary product given above but a closely related binary variant for pairs of finite monoids (Schützenberger 1965). This binary product was later generalised to an n-ary construction for arbitrary n in (Straubing 1981), but the unary version given there is trivial (i.e $\Diamond M := M$). In the (pro)finite monoid literature, Schützenberger product is mainly considered for recognition of concatenation products of languages, and block-products are used for recognition of existential projections (Straubing 1994, Lemma VI.1.2). Interestingly, as seen above, duality shows that unary Schützenberger product is naturally linked to quantification as it is dual to it.

In (Gehrke, Petrişan & Reggio 2016) a version for Boolean spaces with internal monoids of the binary Schützenberger product is also given and a generalisation of Reutenauer's Theorem stating that it recognises precisely the 'marked concatenation products' $L_1 a L_2$, where L_i is recognised by X_i and $a \in \Sigma$, is proved. Further, it is shown that the unary product is a quotient space of the binary product.

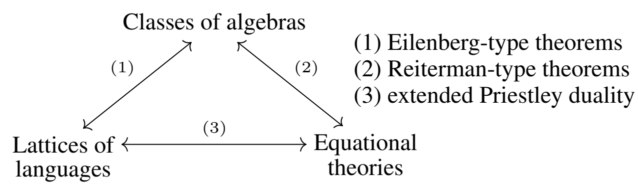
5. Equations

Syntactic monoids were introduced into the theory of regular languages early on by Myhill, and Rabin and Scott, and their power was established by Schützenberger's effective characterisation of star free languages based on syntactic monoids (Schützenberger 1965): Star free languages have a number of nice characterisations, e.g. as the languages given by expressions built up from the letters using concatenation and the Boolean connectives, or in terms of Büchi's logic on words as the model classes of first-order sentences in the logical language with the letter predicates and the numerical predicate $<$, but none of these descriptions allows one to decide whether or not the language given by an automaton is star free or not. Schützenberger showed that a language is star free if and only if its syntactic monoid is aperiodic (for all $m \in M$ there exists an n so that $m^{n+1} = m^n$). Combined with the fact that the syntactic monoid of a language is effectively computable given an automaton recognising it, it follows that star freeness is a decidable property.

Analysing the example of star free languages, one sees that it is essential that the corresponding class of monoids is given by an equation-like property which one can effectively check in a finite monoid. This property is easily seen to define a *pseudovariety*, that is, the finitary version of Birkhoff's varieties for abstract algebras: a class of similar finite algebras closed under homomorphic images, subalgebras, and *finite* products. Eilenberg's celebrated variety theorem (Eilenberg 1976) supplied a general framework in which to apply the strategy of Schützenberger's result by characterising those classes of regular languages for which the corresponding class of monoids is a pseudovariety and the method has proved very successful in producing decidability results. One may wonder whether all pseudovarieties are given by such nice properties as the one for aperiodicity. Here Reiterman's theorem (Reiterman 1982) gives a partial answer by supplying a generalisation of Birkhoff's variety theorem from universal algebra: It states that pseudovarieties of finite algebras are precisely the ones given by *profinite equations* (Reiterman 1982). Profinite equations being pairs of elements of the profinite completion of a free monoid. For example, aperiodicity is given by the profinite equation $x^{\omega+1} \approx x^{\omega}$. Here, $(\_)^{\omega}$ is the continuous function, available on any compact topological monoid, which sends an element to the unique idempotent in the closed semigroup generated by the element.

The combination of the theorems of Eilenberg and Reiterman allows the equational description of certain classes of regular languages and, in cases where researchers have found effective finite equational bases, this has lead to decidable criteria for membership in the corresponding classes. Given the success of the method, the search for generalisations applicable to more general classes of regular languages has been very active, e.g. (Pin 1995; Pippenger 1997; Straubing 2002).

The main result of (Gehrke, Grigorieff & Pin 2008) is that the composition of the Eilenberg and Reiterman theorems is a special instance of Stone duality for subalgebras and quotient spaces, and in this way it allows a significant generalisation of Eilenberg-Reiterman. For one, the direct duality route from lattices of languages to profinite equational theories is available also when the classes of finite algebras in the middle are not. Further, it allows a 'local' version (not *requiring* the consideration of all alphabets at once). Further, the only necessary requirement is closure under the lattice operations of finite intersection and union. All other desired specialisations of the theorem, up to the original Eilenberg-Reiterman combination, are obtained in a modular way by adding requirements. The relationship between the Eilenberg and Reiterman theorems and the Stone duality for subalgebras and quotient spaces may be illustrated by the following diagram.



5.1 Generalised equations for regular languages

The results presented here stem from (Gehrke, Grigorieff & Pin 2008). Detailed proofs in the more general setting of an arbitrary variety of abstract algebras is given in (Gehrke 2016, Section 4.4).

In Section 2 we saw that, underlying the duality between specification and behaviour, there is a Galois connection between sets of pairs of lattice elements and subsets of the dual space witnessing the duality between surjective lattice homomorphisms and (closed) subspace embeddings. There is a similar situation for injective lattice homomorphisms and quotient maps between the dual spaces yielding a duality between sublattices of a DL and those quasiorders on the dual space which correspond to its Priestley quotients. Given a Priestley space X, the Priestley quotients of X are given by quasiorders $\preceq$ on X extending the Priestley order of X and satisfying the following property:

$$\forall x, y\ (x \not\preceq y \implies \exists C \text{ clopen } \preceq\text{-upset with } x \in C \text{ and } y \notin C).$$

Such quasiorders are called *compatible quasiorders* on X. This is the source of the profinite equations as used in the theory of regular languages.

More specifically, the free profinite completion $\widehat{\Sigma^*}$ is the dual space of $Reg(\Sigma^*)$ (Theorem 16). Thus there is a one-to-one correspondence between the sublattices of $Reg(\Sigma^*)$ and the Priestley quotients of $\widehat{\Sigma^*}$.

Definition 27. A *profinite (lattice) equation* in the alphabet Σ is given by a pair of elements $x, y \in \widehat{\Sigma^*}$ and is denoted by $x \to y$. We say that $x \to y$ is satisfied by $L \in Reg(\Sigma^*)$, and write $L \Vdash x \to y$ provided one and then all of the following equivalent statements hold:

(1) $L \in F_y$ implies $L \in F_x$,
(2) $y \in \widehat{L}$ implies $x \in \widehat{L}$,
(3) $y \in \overline{L}$ implies $x \in \overline{L}$.

Now the duality between sublattices and quotient spaces may be stated as follows.

Theorem 28. *The maps*

$$\mathcal{P}(\widehat{\Sigma^*} \times \widehat{\Sigma^*}) \to \mathcal{P}(Reg(\Sigma^*))$$
$$E \;\mapsto\; \mathcal{L}_E = \{L \mid \forall (x,y) \in E\; L \Vdash x \to y\}$$

and

$$\mathcal{P}(Reg(\Sigma^*)) \to \mathcal{P}(\widehat{\Sigma^*} \times \widehat{\Sigma^*})$$
$$\mathcal{S} \;\mapsto\; Eq_{\mathcal{S}} = \{(x,y) \mid \forall L \in \mathcal{S}\; L \Vdash x \to y\}$$

establish a Galois connection whose Galois closed sets are the compatible quasiorders on $\widehat{\Sigma^*}$ and the bounded sublattices of $Reg(\Sigma^*)$, respectively.

We will say that $\mathcal{L} \subseteq Reg(\Sigma^*)$ is *defined by* a set E of equations provided $\mathcal{L} = \mathcal{L}_E$. The fact that the Galois closed sets of regular languages are exactly the sublattices becomes the following generalised Eilenberg-Reiterman theorem.

Corollary 29. *A collection of regular languages over Σ is a sublattice of $Reg(\Sigma^*)$ if and only if it can be defined by a set of profinite lattice equations.*

Noting that Boolean subalgebras of $Reg(\Sigma^*)$ are exactly those for which the corresponding compatible quasiorder is an equivalence relation and writing $x \leftrightarrow y$ for the conjunction $x \to y$ and $y \to x$, we get an equational description of the Boolean subalgebras of recognisable subsets. We call such $x \leftrightarrow y$ *profinite symmetric lattice equations*.

Corollary 30. *A collection of regular languages over Σ is a Boolean subalgebra of $Reg(\Sigma^*)$ if and only if it can be defined by a set of profinite symmetric lattice equations.*

The difference between the lattice case and the Boolean case is that we need an order relation in the lattice setting as in Priestley duality. This fact was rediscovered in the theory of formal languages and automata by Pin who introduced ordered monoids and an asymmetric notion of profinite identities (Pin 1995) without realising the connection with Priestley duality.

In the original Eilenberg theorem, not only is it necessary that the collections of recognisable sets be closed under Boolean complementation, they must also be residuation ideals and be 'closed under inverse images of morphisms' (see Definition 34 below). We now proceed to give Eilenberg-Reiterman theorems for each of these conditions separately.

Definition 31. A *profinite (algebra) equation* in the alphabet Σ is given by a pair of elements $x, y \in \widehat{\Sigma^*}$ and is denoted by $x \preceq y$. We say that $x \preceq y$ is satisfied by $L \in Reg(\Sigma^*)$ if and only if $L \Vdash sxt \to syt$ for all $s, t \in \widehat{\Sigma^*}$. Similarly, we denote the symmetric version by $x \approx y$.

Theorem 32. *A collection of regular languages over Σ is a (Boolean) residuation ideal of $Reg(\Sigma^*)$ if and only if it can be defined by a set of (symmetric) profinite algebra equations.*

So far our equations are 'local' in the sense that they are not invariant under substitution. The last ingredient of the original Reiterman theorem is this invariance. For this purpose we need the following concept.

Definition 33. A *class of regular languages* is an assignment $\Sigma \mapsto \mathcal{L}(\Sigma)$ for each finite alphabet Σ, where $\mathcal{L}(\Sigma) \subseteq Reg(\Sigma^*)$. We call such a class a *lattice class* provided $\mathcal{L}(\Sigma)$ is a sublattice of $Reg(\Sigma^*)$ for each finite alphabet Σ. Furthermore, a *class of equations* is an assignment $\Sigma \mapsto E(\Sigma)$ for each finite alphabet Σ, where $E \subseteq \widehat{\Sigma^*} \times \widehat{\Sigma^*}$. We say that a class $\mathcal{L}$ is given by a class of equations E provided, for each finite alphabet Σ, we have that $\mathcal{L}(\Sigma)$ is given by $E(\Sigma)$.

Thus Corollary 29 tells us that a class of regular languages is a lattice class if and only if it is given by some class of equations.

Notice that given finite alphabets Σ and Δ and a homomorphism $\sigma : \Sigma^* \to \Delta^*$, any regular language L over Δ has an inverse image under σ which is a regular language over Σ, where the recognising morphism is the pre-composition by σ of the recognising homomorphism for L. That is, σ induces a Boolean algebra homomorphism

$$Reg(\sigma) : Reg(\Delta^*) \to Reg(\Sigma^*), L \mapsto \sigma^{-1}(L).$$

The Stone dual of this homomorphism is a continuous function

$$\widehat{\sigma} : \widehat{\Sigma^*} \to \widehat{\Delta^*}.$$

Since it extends σ, it is in fact also the unique continuous extension of σ.

Definition 34. A lattice class $\mathcal{L}$ of regular languages is said to be *closed under inverse images of morphisms* provided, whenever Σ and Δ are finite alphabets and $\sigma\colon \Sigma^* \to \Delta^*$ is a homomorphism, then $L \in \mathcal{L}(\Delta)$ implies $\sigma^{-1}(L) \in \mathcal{L}(\Sigma)$.

A class E of equations is said to be *closed under substitution* provided, whenever Σ and Δ are finite alphabets and $\sigma\colon \Sigma^* \to \Delta^*$ is a homomorphism, then $x \to y \in E(\Sigma)$ implies $\widehat{\sigma}(x) \to \widehat{\sigma}(y) \in E(\Delta)$.

Theorem 35. *Let $\mathcal{L}$ be a lattice class of regular languages. Then $\mathcal{L}$ is closed under inverse images of morphisms if and only if it is given by some equational class which is closed under substitution.*

Closure under the lattice operations, Boolean complement, residuation, and inverses of morphisms are the hypotheses of the original Eilenberg theorem. As mentioned earlier, various generalisations have allowed the relaxation of certain of these hypotheses while keeping others. The treatment in (Gehrke, Grigorieff & Pin 2008), for which the duality theoretic components have been given above, is the first fully modular treatment and the first to allow the treatment of lattices of recognisable languages without any further properties.

The widened scope of Eilenberg-Reiterman theory has been applied within the theory of regular languages, see e.g. (Branco & Pin 2009; Kufleitner & Lauser 2011, 2012). Further, this work applies to finitary recognition beyond the setting of finite words and this has been explored in various directions in (Gehrke 2016; Adamek et al. 2015; Bojańczyk 2015).

5.2 Equations beyond regular languages

DLs and BAs of languages, $\mathcal{L} \subseteq \mathcal{P}(\Sigma^*)$, not contained in $Reg(\Sigma^*)$, can not be described by profinite equations, but they can be described by equations over the dual of $\mathcal{P}(\Sigma^*)$, which is $\beta(\Sigma^*)$.

Definition 36. A *(lattice) β-equation* in the alphabet Σ is given by a pair of elements $\mu, \nu \in \beta(\Sigma^*)$ and is denoted by $\mu \to \nu$. A language $L \in \mathcal{P}(\Sigma^*)$ satisfies $\mu \to \nu$ provided one and then all of the following equivalent statements hold:

(1) $L \in F_\nu$ implies $L \in F_\mu$,
(2) $\nu \in \widehat{L}$ implies $\mu \in \widehat{L}$,
(3) $\nu \in \overline{L}$ implies $\mu \in \overline{L}$.

As in the regular case, we use $\mu \leftrightarrow \nu$ as shorthand for the conjunction of $\mu \to \nu$ and $\nu \to \mu$; $\mu \preceq \nu$ as shorthand for $u\mu v \to u\nu v$ for all $u, v \in \Sigma^*$; $\mu \approx \nu$ as shorthand for $\mu \preceq \nu$ and $\nu \preceq \mu$.

As in the regular setting, it follows by duality that lattices of languages are precisely those subsets of $\mathcal{P}(\Sigma^*)$ that can be described by β-equations of the form $\mu \to \nu$, Boolean algebras those that can be described by β-equations of the form $\mu \leftrightarrow \nu$, lattices closed under quotients those that can be described by β-equations of the form $\mu \preceq \nu$, and Boolean algebras closed under quotients those that can be described by β-equations of the form $\mu \approx \nu$.

In the study of *Boolean circuit classes* one cannot expect decidability and recognition by finite algebras, but the compact recognition afforded by Stone/Priestley duality and β-equations may be useful in obtaining separation results. Boolean circuit classes are low level complexity classes studied in the search for lower bounds in complexity theory and since some of these have non-trivial intersections with the Boolean algebra of regular languages, these have been studied using the algebraic and profinite methods of automata theory (Straubing 1994). In particular, it follows from (Barrington, Straubing &Thérien 1990) and (Straubing 1991) that

$$\mathrm{FO}[\mathcal{N}] \cap \mathrm{Reg} = [\![\, (\mathrm{x}^{\omega-1}\mathrm{y})^{\omega+1} \approx (\mathrm{x}^{\omega-1}\mathrm{y})^{\omega} \text{ for } x, y \text{ words of the same length}]\!].$$

where $[\![E]\!]$ denotes the class of languages given by a set E of equations. This formula gives the profinite equations characterising the regular languages in $\mathrm{FO}[\mathcal{N}]$, the class of languages defined by sentences of first-order logic using arbitrary numerical predicates and the usual letter predicates. The proof makes use of the equality between $\mathrm{FO}[\mathcal{N}]$ and the circuit complexity class AC^0 consisting of the languages accepted by unbounded fan-in, polynomial size, constant-depth Boolean circuits (Straubing 1994, Theorem IX.2.1). See also (McKenzie, Thomas &Vollmer 2010) for similar results and problems. A medium term goal in this direction would be to find β-equations for full $\mathrm{FO}[\mathcal{N}]$ and prove the above result by projection without going through results from circuit complexity.

A first hurdle is the specification of even a single β-equation (other than the trivial ones between words) since all ultrafilters of $\mathcal{P}(\Sigma^*)$, other than the principal ones, are non-constructive. A solution to this problem has been given and, as a proof of concept, a complete set of β-equations for a fragment of $\mathrm{FO}[\mathcal{N}]$ has been given. These have then been used to obtain equations and decidability for the intersection of the fragment with the regular languages (Gehrke, Krebs & Pin 2016). In (Czarnetzki & Krebs 2016) the results have been generalised to obtain complete axiomatisations by β-equations for certain BAs of languages obtained by block product. Here we give a brief overview of the methods introduced in (Gehrke, Krebs & Pin 2016).

As one may have noticed even from the very few examples of axiomatisations by profinite equations given in this article, the operation $(\_)^\omega$ available on profinite monoids plays a central rôle. This operation produces the unique idempotent in the closed semigroup generated by an element. In the setting of $\beta(\Sigma^*)$ we have no continuous monoid structure and no hope of finding idempotents. However, $(\_)^\omega$ may be seen as a means of landing in a ‘reproducible infinite profinite position’ and this is what we have to describe in the $\beta(\Sigma^*)$ setting. For the application in (Gehrke, Krebs & Pin 2016), we only need a finite number of occurrences of the same infinite β-position (whereas in (Czarnetzki & Krebs 2016) infinitely many equivalent infinite positions are required). Let

$$\Sigma^* \otimes^k \mathbb{N} = \{(w, n_1, \ldots, n_k) \mid 0 \leq n_1 < n_2 \ldots < n_k < |w|\}$$

be the set of words over Σ with k (ordered) marked spots. We denote by π_0 the projection from $\Sigma^* \otimes^k \mathbb{N}$ on the word coordinate and by π_j the projection onto the jth marked position. We think of elements of $\beta(\Sigma^* \otimes^k \mathbb{N})$ as *generalised words with k marked positions*. We have the ‘generalised projection’ $\beta\pi_0 \colon \beta(\Sigma^* \otimes^k \mathbb{N}) \to \beta(\Sigma^*)$ to ‘generalised words’, and the $\beta\pi_j \colon \beta(\Sigma^* \otimes^k \mathbb{N}) \to \beta(\mathbb{N})$ for $1 \leq j \leq k$ to tell us which generalised positions are involved. However, a word of caution is required: an element of $\beta(\Sigma^* \otimes^k \mathbb{N})$ is not determined by these projections alone.

Proposition 37. *Let $\gamma \in \beta(\Sigma^* \otimes^k \mathbb{N})$ with $k \geq 1$. Then, for each $\alpha \in \beta(\mathbb{N})$, the following conditions are equivalent:*

(1) $\beta\pi_j(\gamma) = \alpha$ for each $j \in \{1, \ldots, k\}$;
(2) $\{\Sigma^* \times P^k \mid P \in F_\alpha\} \subseteq F_\gamma$.

Furthermore, these conditions hold for some α if and only if

(3) $\bigcup_{j=1}^n (\Sigma^* \times P_j^k) \in F_\gamma$ for each partition $\{P_1, \ldots, P_n\}$ of $\mathbb{N}$.

Proposition 37.3 tells us which generalised words with k marked spots have all k spots marking the same generalised position, and Proposition 37.2 tells us which of these γ have α in all k spots. The most important part of this proposition is that both

$$\{\Sigma^* \times P^k \mid P \in F_\alpha\}$$

and

$$\{\bigcup_{j=1}^n (\Sigma^* \times P_j^k) \mid \{P_1, \ldots, P_n\} \text{ is a partition of } \mathbb{N}\}$$

are *filter bases*. That is, the upsets they generate are *proper* filters. Stone's Prime Filter Theorem, the non-constructive principle needed in Stone duality, implies that every proper filter is contained in a prime filter, and thus Proposition 37 guarantees the existence of points with the property *(1)* of that proposition.

The BA treated in (Gehrke, Krebs & Pin 2016) is $\mathrm{FO}[\mathcal{N}_0, \mathcal{N}_1]$, where $\mathcal{N}_0$ and $\mathcal{N}_1$ are, respectively, the nullary and unary numerical predicates given by the subsets P of $\mathbb{N}$ as follows: the 0-ary predicate given by P is true in u if and only if $|u| \in P$ and the unary uniform predicate is true at i in u if and only if $i \in P$. Note that we do not consider $=$ as a logical symbol, so that each formula is equivalent to one of quantifier depth at most one (thus there is a close connection to the results in Section 4.3).

Let $\overline{a} \in \Sigma^k$. For $(u, \overline{n}) \in \Sigma^* \otimes^k \mathbb{N}$, define $u(\overline{a}@\overline{n})$ by

$$(u(\overline{a}@\overline{n}))_\ell = \begin{cases} u_\ell & \text{if } \ell \notin \{n_1, \ldots, n_k\} \\ a_j & \text{if } \ell = n_j. \end{cases}$$

Further, let $f_{\overline{a}} \colon \Sigma^* \otimes^k \mathbb{N} \to \Sigma^*$ be the function defined by $f_{\overline{a}}(u, \overline{n}) = u(\overline{a}@\overline{n})$.

Theorem 38. *(Gehrke, Krebs & Pin 2016, Theorems 3.2, 3.3 and 4.7) The Boolean algebra $\mathrm{FO}[\mathcal{N}_0, \mathcal{N}_1]$ is defined by the following two families of β-equations*

$$\beta f_{(a,b)}(\gamma) \approx f_{(b,a)}(\gamma)$$

where $\gamma \in \beta(\Sigma^* \otimes^2 \mathbb{N})$ and satisfies $\beta\pi_1(\gamma) = \beta\pi_2(\gamma)$ and

$$\beta f_{(a,a,b)}(\mu) \approx f_{(a,b,b)}(\mu)$$

where $\mu \in \beta(\Sigma^* \otimes^3 \mathbb{N})$ and satisfies $\beta\pi_1(\mu) = \beta\pi_2(\mu) = \beta\pi_3(\mu)$.

Given a complete axiomatisation E by β-equations of a Boolean algebra $\mathcal{B}$ of languages over Σ, it is not hard to see that one obtains a complete axiomatisation of $\mathcal{B} \cap Reg(\Sigma^*)$ by taking the set of profinite equations of the form $\tau(x) \to \tau(y)$ for $x \to y \in E$, where $\tau \colon \beta(\Sigma^*) \to \widehat{\Sigma^*}$ is the dual of the embedding $Reg(\Sigma^*) \hookrightarrow \mathcal{P}(\Sigma^*)$. Using this fact one may derive the following profinite axiomatisation.

Theorem 39. *(Gehrke, Krebs & Pin 2016, Theorems 5.16)*

$$\mathrm{FO}[\mathcal{N}_0, \mathcal{N}_1] \cap \mathrm{Reg} = [\![(x^{\omega-1}s)(x^{\omega-1}t) = (x^{\omega-1}t)(x^{\omega-1}s), \; (x^{\omega-1}s)^2 = (x^{\omega-1}s) \; \textit{for } x, s, t \textit{ words of the same length}]\!].$$

This theorem tells us that a regular language L is in $\mathrm{FO}[\mathcal{N}_0, \mathcal{N}_1]$ if and only if its syntactic monoid M satisfies the given equations. Using this fact one can show that membership in $\mathrm{FO}[\mathcal{N}_0, \mathcal{N}_1] \cap \mathrm{Reg}$ is decidable.

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