Topological Duality and Algebraic Completions

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Abstract In this chapter we survey some developments in topological duality theory and the theory of completions for lattices with additional operations paying special attention to various classes of residuated lattices which play a central role in substructural logic.
We hope this chapter will serve as an introduction and invitation to these subjects for researchers and students interested in residuated lattices, substructural logics, and the algebraic approach to proof theory developed and promoted in great part by Hiroakira Ono.

In honour of Hiroakira Ono

1 Introduction

Substructural logic is first and foremost a proof theoretic subject, but Hiroakira Ono has long been promoting an algebraic approach to the subject [52]. This approach has been very successful as witnessed by the book [14] by Ono and his co-authors, and by the very active research front in what is by now called *algebraic proof theory* [4]. This approach uses the universal algebraic study of classes, and in particular varieties, of algebras in order to advance our knowledge of substructural logics. As in other algebraic studies of logic, the appropriate algebras are at least partially ordered (with the order coming from the consequence relation of the logic) and mostly they are even lattice-ordered (with the meet and join coming from (one version of) conjunction and the disjunction of the logic, respectively). Unlike, what happens in intermediate and modal logics, however, mostly, these lattices are *not necessarily*

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distributive. This is what sets them apart from an algebraic point of view. Other connectives then are responsible for additional operations, and the most basic in the substructural setting is the binary fusion (another version of conjunction) and its residuals (corresponding to implication). Thus, the basic algebraic structures of substructural logic are *lattice-ordered residuated lattices*.

In algebraic treatments of logic in general, it is not only the universal algebraic structure that is exploited, but also other parts of lattice theory, such as *order completions* and *topological duality*. In the substructural logic or algebraic proof theory literature completions, mainly in the form of MacNeille completion, have played a central role. In algebraic treatments of logic for which the appropriate lattices are distributive (such as intermediate and modal logics), a more prominent role has been played by Stone duality. One problem with MacNeille completion is that it destroys many equational properties (in particular it may destroy distributivity). A problem with duality in the substructural setting is that duality for non-distributive lattices is much more complicated than it is for distributive lattices. In the presence of quantifiers, e.g. for first-order logic, neither MacNeille completion nor duality is ideal and one encounters the problems of both settings.

In [44] Jónsson and Tarski introduced canonical extensions, a completion of Boolean algebras with operators, for which they were able to prove a fairly general result on preservation of equational properties. At the level of the underlying lattice, this canonical extension is nothing other that the powerset of the Stone dual space described in the language of lattice completions. This algebraic formulation of duality is what allowed them, more than a decade earlier than Kripke did this for modal operators, to identify what the duals of additional operations on Boolean algebras should be [45]. In the original formulation, canonical extensions were only available for Boolean algebras with additional operations. However, in recent years, this restriction has been lifted and we now have a robust notion of canonical extension for bounded lattices with additional operations. And, very surprisingly, the theory essentially works the same with or without distributivity - even though the associated duality theory is vastly more complicated.

As mentioned above, canonical extension is a completion, like MacNeille completion. However, they differ for most lattices. Another advance of recent years is the full analysis of the so-called Δ_1-completions [23]. These are completions in which every element is a meet of joins and a join of meets of elements from the original lattice. Both the MacNeille completion and the canonical extension are Δ_1-completions, but there are many others. An expectation expressed by a number of researchers (and already seen in a few cases [22, 49]) is that various parametric choices of Δ_1-completions other than the MacNeille completion and the canonical extension are the appropriate tool for studying various propositional and first-order logics. For this expectation to be verified, we still need a better understanding of these completions and results known for specific completions have to be generalised to the parametric setting (or at least to other completions of interest). In this sense, Δ_1-completions may be seen as more accommodating setting for studying duality theory — a kind of *algebraic duality theory*. It makes most sense for these tools and methods to be explored first in concrete settings where a tool is needed to solve a specific problem. Accordingly, it is my hope that this chapter will serve as an introduction and invitation for substructural logicians to consider 'algebraic duality theory' in the form of canonical extensions and Δ_1-completions as part of their tool kit in solving problems in substructural logic.

The chapter is organised as follows. In Section 2 we present basic results on completions of posets and lattices in particular. We start with one sided completions: these are easily available, have free constructions in various combinations (for posets, semilattices, lattices viewed in complete join semilattices, complete meet semilattices, directedly complete partial orders...) and are generally abstractly characterised by compactness type properties. A classical representation theorem for these allows us to construct these as closure systems of appropriate ideals or filters. In addition, we introduce residuated maps on posets, a central tool in substructural logic, and we show how the classical representation theorem for one-sided completions is a consequence of basic facts on residuation.

We introduce Δ_1-completions of which MacNeille completion and canonical extension are special instances and provide a general representation theorem for Δ_1-completions as Galois closed sets of polarities based on a pair of closure systems on ideals and filters, respectively. This representation is closely related to MacNeille completion and one may in fact toggle between these via a smaller extension, first identified by Ghilardi and Meloni, known as the intermediate structure. It is possible that applications of MacNeille completion in algebraic proof theory actually only use the intermediate structure and do not require the completeness properties of the MacNeille completion. For this reason, it may be interesting for substructural logicians to know about this smaller extension of a polarity.

An important conceptual point of the abstract algebraic approach to completions and duality first introduced by Jónsson and Tarski in the case of canonical extensions is that the completions are characterised by appropriate *abstract properties*. This allows us to work with completions having the appropriate properties rather than cumbersome set theoretic constructions consisting of sets of sets.

In Section 3 we treat the extension of maps on lattices to their canonical extensions with special focus on additional algebraic operations and homomorphisms. These extensions come in two flavours, known as σ- and π-extensions, which depend on whether one first extends to the meet closure and then by joins or vice versa. These two extensions are in general different and one works best with join preservation, the other with meet preservation. By 'working better', we both mean has better equational preservation properties and, as we will see in Section 5, yields an extension that can be encoded as a Kripke relation. In this context, the appropriate notion of join preservation needed for an n-ary operation is preservation of join *in each coordinate*. This holds for fusion (and order variants of this holds for its residuals), but it is not join preserving on its binary domain. This makes he theory for residuated lattices more subtle than the theory of unary modal operators. We treat extension of binary residuated families of maps in particular.

Section 4 is on canonicity. A property is said to be *canonical* provided it is preserved by taking the canonical extension. Canonicity results are mainly based on results about extensions of maps commuting with composition of maps. We first survey the main results of this kind and give an overview of existing canonicity results: for lattices with operators, for a selection of particular equational properties of interest in substructural logic, for closures of classes of algebras under various constructors such as subalgebras, homomorphic images, and Boolean products. Finally we survey several results such as the canonicity of finitely generated varieties and the relation between preservation of universal first order properties under MacNeille completion and canonical extension.

We mainly focus our attention on not necessarily distributive but bounded lattices. The theory is most developed under the additional assumption of distributivity but a large part of the theory goes through without significant additional difficulty and this part is mainly what is covered in this chapter. However, a number of questions remain open about the generalisation from distributive to non-distributive lattices, such as the topological treatment of extensions of maps and the right categorical setting for understanding canonical extension as a free construction, and even more about the generalisation from lattices to posets, such as identifying the 'right' choice of ideals and filters for the notion of canonical extension.

A restriction we are making that is not so natural in the substructural setting is that our lattices and posets must be *bounded*. Anyone familiar with duality theory knows that this is a necessary restriction in order not to have to make exceptional provisions for the empty set and the whole space. This does not mean that duality and canonical extensions cannot be applied to unbounded structures. However, it means that when treating these, one way or the other, one has to add a new top and bottom, apply the theory to these extended structures and then take the bounds back off before interpreting the results. This of course creates complications, e.g. the original structure may be in some variety while, once a top and bottom have been added, there is no way to extend the structure and stay in the variety (as is for example the case for ℓ-groups). Thus dealing with the bounds problem is a necessity though we do not address it here. Some work in this direction may be found in [48]. Also, as there is not a good blanket solution to this difficulty, it is better dealt with case by case as needed as has always been done in topological duality theory.

Section 5 treats connections with topological duality and relational semantics. On the one hand we attempt to explain the idea of *algebraic duality theory* as introduced by Jónsson and Tarski in their seminal two part work on canonical extensions of Boolean algebras with operators [44, 45] and show how this program goes through for bounded lattices with operators and their order variants. This leads to a Jónsson-Tarski style representation theorem for bounded lattices with operators which encodes duality theory in algebraic form. We spell out in particular the *representation theorem in the case of bounded residuated lattices*. Finally we explain how the topological dualities of Hartung and of Urquhart can be derived from the canonical extension. We finish with some concluding remarks.

Essentially none of the material in this chapter is new. The paper draws heavily on the papers [20, 23, 17, 30, 25, 9, 15]. It also owes a lot to current work of the author on a book with H. A. Priestley. I also want to thank Sam van Gool for a careful reading of a first draft and helpful comments leading to great improvements

of the chapter. The focus and choice of topics has been guided by usefulness, or potential usefulness, in the study of substructural logics.

2 Completions

Let P be a poset. A *poset extension* of P is a poset Q and an order embedding $e\colon P \to Q$. That is, a map so that, for all $x, y \in P$, we have $e(x) \leq e(y)$ if and only if $x \leq y$. A poset extension $e\colon P \to C$ of P is a *completion* of P provided C is a complete lattice. Given a completion $e\colon P \to Q$, we may identify P with $e(P)$ so that e may be regarded as the inclusion map from $e(P)$ into C. In doing this, we will refer to C as being a completion of P, without explicit reference to e. This will lighten the notation.

A subset S of a poset Q is *join-dense* in Q, or equivalently Q is *join-generated* by S, provided every element of Q is the join of some (possibly empty) set of elements in S. Order dually, S is *meet-dense* in Q, equivalently Q is *meet-generated* by S, provided, for each $x \in Q$,

$$x = \bigwedge\{y \in S \mid y \geq x\}.$$

Given a poset Q and $S \subseteq Q$, the *join-closure* of S in Q is the subset $\{\bigvee T \mid T \subseteq S \text{ and } \bigvee T \text{ exists in } Q\}$. The *meet-closure* is defined order dually. A completion C of a poset P is said to be a *join-completion*, respectively *meet-completion*, provided P is join-dense, respectively meet-dense, in C.

It is reasonable to consider mainly completions of a poset P with the property that (the image of) P generates the completion as a complete lattice. Within such completions one can consider a complexity hierarchy according to how many times one has to alternate closing by joins and meets (and in what order). The first level of this hierarchy are the one-sided completions, namely the join- and meet-completions of P, respectively.

2.1 Join- and meet-completions

One-sided completions are easy to understand. This is mainly because free such completions exist in most settings. Thus there are free completions of posets by finite meets or joins, by directed meets or joins, and by arbitrary meets or joins. One can also freely complete join-semilattices, meet-semilattices, and lattices by directed or arbitrary joins or meets. These free constructions can all be described as posets of appropriate ideals or filters. Here are the details of the construction of the free join-completion of a poset.

A *downset* of a poset P is a subset $U \subseteq P$ satisfying $p \in P$, $q \in U$ and $p \leq q$ implies $p \in U$. Given a subset $S \subseteq P$, the downset generated by S is given by ${\downarrow}S = \{p \in P \mid \exists s \in S \text{ with } p \leq s\}$. For a singleton set $\{p\}$ we write ${\downarrow}p$ rather than ${\downarrow}\{p\}$.

An embedding $e\colon P \to C$ is said to be *hyper-compact*[2] provided $p \in P$ and $S \subseteq P$ with $e(p) \leq \bigvee e(S)$ implies that there is $s \in S$ with $p \leq s$.

Theorem 1 (Free $\bigvee$-completion of posets). *Let P be a poset. The free $\bigvee$-completion of P is given by the poset $\mathscr{D}(P)$ of all downsets of P ordered by inclusion with the embedding of P given by $p \mapsto {\downarrow}p$.*

Given an order-preserving map $f\colon P \to C$ where C is a complete lattice, the unique $\bigvee$-preserving extension $\mathscr{D}(f)\colon \mathscr{D}(P) \to C$ is given by $\mathscr{D}(f)(U) = \bigvee f(U)$.

Further, among completions of P, the free $\bigvee$-completion is characterized up to isomorphism by the fact that it is a hyper-compact join-completion.

Other variants of join-completion satisfy corresponding theorems with the notions of ideal and compactness adjusted accordingly. Some of these variants are summarized in Table 2.1.

Base	Type	Elements	Compactness
posets	$\bigvee$	downsets	$e(p) \leq \bigvee e(S) \implies \exists s \;\; p \leq s$
posets	$\bigvee^{\uparrow}$	order ideals	$e(p) \leq \bigvee^{\uparrow} e(S) \implies \exists s \;\; p \leq s$
posets	$\vee$	finitely generated downsets	$e(p) \leq e(s) \vee e(t) \implies p \leq s$ or $p \leq t$
bounded lattices	$\bigvee = \bigvee^{\uparrow}$	lattice ideals	$e(p) \leq \bigvee e(S) \implies \exists F$ finite $p \leq \bigvee F$

Here *order ideals* are (up-)directed downsets, i.e. downsets in which each finite subset has an upper bound and lattice ideals are downsets closed under finite joins.

Order dual statements are valid for the corresponding kinds of meet completions. Note though that filters, the order duals of ideals, are ordered by the *reverse* of the order of inclusion since $S \supseteq T$ implies $\bigwedge S \leq \bigwedge T$. As already mentioned in the introduction, the fact that the completions are characterised by the appropriate versions of denseness and compactness means that we don't have to work with the set-theoretic incarnations as families of ideals or filters. Instead we can work with abstract completions having the appropriate properties. This is particularly useful in the case of meet completion where the order reversal tends to hinder the order theoretic intuitions. Accordingly, we will denote the completions by $F_{\bigvee}(P), F_{\bigwedge}(P)$ etc. and think of them as join- or meet-completions with the appropriate compactness property.

Notice that if we consider (non-directed) join or meet completions

Background on adjoint pairs A central notion in substructural logic is that of an adjoint pair of maps on posets. This notion also plays a central role in the basic theory of completions and of topological duality. A pair of maps $f\colon P \rightleftarrows Q\colon g$ between posets is called an *adjoint pair* provided, for all $p \in P$ and for all $q \in Q$ we have

[2] compactness is the weaker property which, under the same assumptions, implies the existence of a finite subjoin which gets above $e(p)$ (see 3). The term hyper-compactness was coined by Marcel Erné as it is an extremal form of compactness.

$$f(p) \leq q \iff p \leq g(q).$$

One can show that a pair of maps $f\colon P \rightleftarrows Q\colon g$ is an adjoint pair if and only if the following properties hold:

1. f and g are both order-preserving;
2. $p \leq gf(p)$ for all $p \in P$;
3. $fg(q) \leq q$ for all $q \in Q$.

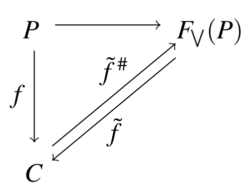
One can also show that a map in an adjoint pair is injective if and only if the other is surjective. When (f,g) is an adjoint pair, we call f the *lower* or left adjoint and g the *upper* or right adjoint. It should be clear that adjoints interdetermine each other, e.g. $f(p)$ is the minimum of all the $q \in Q$ such that $p \leq g(q)$. Given f, we denote its upper adjoint, if it exists, by $f^{\#}$, and its lower adjoint, if it exists, by $f^{\flat}$.

For a function $f\colon C \to D$ between complete lattices, one can show that f is a lower adjoint, or equivalently, has an upper adjoint if and only if f is $\bigvee$-preserving, and dually for upper adjoints. This has a nice consequence for understanding join- and meet-completions of posets.

Join-completions as closure systems Let P be a poset and $f\colon P \to C$ an order-preserving mapping into a complete lattice. The freeness of $F_{\bigvee}(P)$ provides a unique $\bigvee$-preserving extension $\tilde{f} : F_{\bigvee}(P) \to C$. It follows that $\tilde{f}$ has an upper adjoint

$$\tilde{f}^{\#}\colon C \to F_{\bigvee}(P).$$

The map $\tilde{f}^{\#}$ is injective if and only if $\tilde{f}$ is surjective, which is the case if and only if $f(P)$ join-generates C. Further, one can show that f is a join-completion of P if and only if the diagram below commutes also for $\tilde{f}^{\#}$ if and only if the image of P in $F_{\bigvee}(P)$ is included in the image of the embedding $\tilde{f}^{\#}$.



A closure operator or closure system on $F_{\bigvee}(P)$ is called standard provided each element of P is closed. There is then a three way correspondence between join-completions of P, standard closure operators on $F_{\bigvee}(P)$ (or equivalently on $\mathscr{D}(P)$), and standard closure systems on $F_{\bigvee}(P)$ (or equivalently on $\mathscr{D}(P)$). In this correspondence a join-completion $f\colon P \to C$ corresponds to the closure system $Im(\tilde{f}^{\#})$ and to the closure operator $\tilde{f}^{\#} \circ \tilde{f}$. Note that the *nuclei*, touted as possibly the most important notion in the theory of residuated lattices as applied in substructural logic [14, Section 3.4.11], are a residuation-enriched version of the closure operators $\tilde{f}^{\#} \circ \tilde{f}$ obtained by adjunction as explained above.

Theorem 2 ([1]). *Let P be a poset. Then*

(i) *there is a bijective correspondence between join-dense completions* (e,C) *of* P *and closure systems of down-sets of* P *which contain all principal down-sets;*
(ii) *there is a bijective correspondence between meet-dense completions* (e,C) *of* P *and closure systems of up-sets on* P *which contain all principal up-sets.*

Of course corresponding results hold for other arbitrary join- and meet-completions. For example, the meet-completions of a lattice L are in one-to-one correspondence with the closure systems on the complete lattice of all lattice filters of L.

2.2 Δ-completions

While join- and meet-completions are easy to come by, the situation is very different for two-sided completions: the inclusion of neither lattices, posets, or sets in the category of complete lattices with complete lattice homomorphisms has a left adjoint, and this remains so even if we restrict (and co-restrict) to distributive lattices or Boolean algebras, see [42, Proposition I.4.7 and I.4.10] for proof and further references. From an algebraic point of view, the reason that the two-sided completions fail to exist is that, by alternating joins and meets transfinitely, the candidate for the free completion may end up being a proper class. However, if we restrict the alternations of join- and meet-closure needed to generate the lattice, we get some viable constructions of completions though the freeness properties are not the best.

By a Δ_n-*completion*, we will mean a completion $e\colon P \to C$ in which each element is reachable both by a family of n alternations of join- and meet-closure, and by the dual alternation. Thus, the lowest level of Δ-completions are those that are simultaneously join- and meet-completions of the original poset. We will call this kind of completion a Δ_0-completion as in [30]. The next level are those satisfying the following density condition:

Definition 1. Let P be a poset and let $e\colon P \to C$ be a completion of P. We say that P is *dense* in C provided P is $\bigvee\bigwedge$- and $\bigwedge\bigvee$-dense in C or, in other words, every element of C is both a join of meets and a meet of joins of elements of $e(P)$. We will call a completion C in which P is dense a Δ_1-*completion* as in [30].

As stated in Theorem 2, join- and meet-completions of a poset correspond to closure systems of ideals and filters of the poset, respectively. A similar description is possible for Δ_1-completions. This description is based on the observation, already made by Birkhoff, that polarities allow one to construct complete lattices from given sets of join- and meet-generators.

For details and background on polarities and Galois connections beyond the basic information given here, see [15], [7, Chapters 3 and 7], and [11]. A polarity is a triple (X,Y,R) where X and Y are non-empty sets and R is a binary relation from X to Y. Such a polarity gives rise to a pair of maps, given by

$$\begin{array}{rcccl} (\quad)^R \quad : & \mathscr{P}(X) & \leftrightarrows & \mathscr{P}(Y) & : \quad {}^R(\quad) \\ & A & \mapsto & \{\, y \mid \forall x\ (x \in A \Rightarrow xRy)\,\} & \\ \{\, x \mid \forall y\ (y \in B \Rightarrow xRy)\,\} & & \leftarrow\!\shortmid & B. & \end{array}$$

They form a *Galois connection*, that is, a contravariant adjoint pair. Concretely this means that for all $A \subseteq X$ and all $B \subseteq Y$ we have

$$A \subseteq {}^R(B) \iff B \subseteq (A)^R.$$

The Galois closed subsets of X and of Y are, respectively,

$$\mathscr{G}(X,Y,R) = \{\, A \subseteq X \mid A = {}^R(A^R)\,\} = \{\, {}^R B \mid B \subseteq Y\,\},$$

$$\mathscr{G}(X,Y,R)^R = \{\, B \subseteq Y \mid B = ({}^R B)^R)\,\} = \{\, A^R \mid A \subseteq X\,\}.$$

There are natural maps from X and Y into $\mathscr{G}(X,Y,R)$ given by

$$\Xi \,:\, X \to \mathscr{G}(X,Y,R) \qquad \Upsilon \,:\, Y \to \mathscr{G}(X,Y,R)$$

$$x \mapsto {}^R(\{x\}^R) \qquad\qquad y \mapsto {}^R\{y\}.$$

The complete lattice $\mathscr{G}(X,Y,R)$ associated with the polarity (X,Y,R) may be characterised abstractly as stated in the following theorem.

Theorem 3 ([23]). *Let (X,Y,R) be a polarity. Then $\mathscr{G}(X,Y,R)$ is the unique (up to isomorphism) complete lattice equipped with mappings*

$$\Xi \,:\, X \to \mathscr{G}(X,Y,R) \qquad \textit{and} \qquad \Upsilon \,:\, Y \to \mathscr{G}(X,Y,R)$$

so that the following properties hold:

(i) For $x \in X$ and $y \in Y$, we have $\Xi(x) \leq \Upsilon(y)$ if and only if xRy;
(ii) $\mathscr{G}(X,Y,R)$ is join-generated by $Im(\Xi)$;
(iii) $\mathscr{G}(X,Y,R)$ is meet-generated by $Im(\Upsilon)$.

The fact that polarities provide a means of constructing complete lattices is very important in many applications. But the unicity part of the above theorem is no less central. We give here a stand-alone proof of this fact, stressing the abstract order theoretic nature of the characterisation.

Proposition 1. *Let X and Y be sets and C and C' complete lattices. Suppose further that the images of the maps $j : X \to C$ and $m : Y \to C$ are join- and meet-dense in C, respectively, and, similarly, that the images of the maps $j' : X \to C'$ and $m' : Y \to C'$ are join- and meet-dense in C', respectively. If*

$$\forall x \in X\ \forall y \in Y \quad (j(x) \leq m(y) \iff j'(x) \leq m'(y)) \tag{*}$$

then C and C' are isomorphic via an isomorphism that commutes with j and j' and with m and m'.

Proof. Define $f : C \to C'$ and $f' : C' \to C$ by

$$f(c) = \bigvee\{j'(x) \mid j(x) \le c\} \text{ and } f'(c') = \bigvee\{j(x) \mid j'(x) \le c'\}$$

for $c \in C$ and $c' \in C'$. It is clear that f and f' are order preserving. We show that they are inverses of each other. Let $c \in C$, $x \in X$, and $y \in Y$, then we have

$$\begin{aligned} c \le m(y) \quad &\iff \quad \forall x \in X \ (\ j(x) \le c \implies j(x) \le m(y)\ ) \\ &\iff \quad \forall x \in X \ \big(\ j(x) \le c \implies j'(x) \le m'(y)\ \big) \\ &\iff \quad f(c) \le m'(y) \end{aligned}$$

The first equivalence holds by join-density of $\mathrm{Im}(j)$, the second by the hypothesis $(*)$, and the third by definition of f. Now by symmetry of the hypotheses and for $c' = f(c)$, we thus also have

$$f(c) \le m'(y) \quad \iff \quad f'(f(c)) \le m(y).$$

Putting the two together and by meet-density of $\mathrm{Im}(m)$ in C it follows that $f'(f(c)) = c$. We thus conclude that $f' \circ f = id_C$, and by symmetry of the hypotheses we also have $f \circ f' = id_{C'}$ so that f and f' are mutually inverse order isomorphisms.

To finish we show that $f(j(x)) = j'(x)$ for all $x \in X$ and $f(m(y)) = m'(y)$ for all $y \in Y$. As was shown by the string of equivalences above, $c \le m(y)$ if and only if $f(c) \le m'(y)$, so for $c = j(x)$ we have

$$j(x) \le m(y) \quad \iff \quad f(j(x)) \le m'(y)$$

and applying $(*)$, we obtain

$$j'(x) \le m'(y) \quad \iff \quad f(j(x)) \le m'(y).$$

Since this holds for each $y \in Y$ and since $\mathrm{Im}(m')$ is meet-dense in C' it follows that $f(j(x)) = j'(x)$. Finally, for $y \in Y$ we have

$$\begin{aligned} f(m(y) &= \bigvee\{j'(y) \mid j(x) \le m(y)\} \\ &= \bigvee\{j'(y) \mid j'(x) \le m'(y)\} \\ &= m'(y). \end{aligned}$$

In the sense of the above proposition, double density uniquely determines a complete lattice while one-sided density does not. The other side of this coin is that free two-sided completions do not exist except in very restricted settings while one-sided completions are unproblematic.

The intermediate structure.

For some applications, the full completion, $\mathscr{G}(X,Y,R)$, is not necessary, but just the union of the images of X and Y. This structure was first studied by Ghilardi and Meloni in the setting of Heyting algebras [33]. It also played a central rôle in our analysis of unicity of the canonical extension in [9].

Recent applications of MacNeille completions pioneered by Ono and his collaborators take advantage of the close relation between proof theoretic sequents and polarities. While the associated complete lattices of Galois closed sets are used in these applications, the main purpose actually seems to be to get at the corresponding intermediate structures (or at most at the lattice generated by the intermediate structure). The following proposition is implicitly part of what was proved in Proposition 1 above. See also [23, Proposition 3.1]

Proposition 2. *Let C be a complete lattice and $\Xi : X \to C$ and $\Upsilon : Y \to C$ maps whose images are $\bigvee$- and $\bigwedge$-dense in C, respectively. Then the order on* $\mathrm{Im}(\Xi) \cup \mathrm{Im}(\Upsilon)$ *is uniquely determined by the order from* $\mathrm{Im}(\Xi)$ *to* $\mathrm{Im}(\Upsilon)$ *by*

(i) $\forall x_1, x_2 \in X \quad \Xi(x_1) \leq \Xi(x_2) \iff \forall y \in Y\ (\Xi(x_2) \leq \Upsilon(y) \implies \Xi(x_1) \leq \Upsilon(y))$;
(ii) $\forall y_1, y_2 \in Y \quad \Upsilon(y_1) \leq \Upsilon(y_2) \iff \forall x \in X\ (\Xi(x) \leq \Upsilon(y_1) \implies \Xi(x) \leq \Upsilon(y_2))$;
(iii) $\forall x \in X, y \in Y \quad \Upsilon(y) \leq \Xi(x) \iff$
$\forall x' \in X, y' \in Y\ [(\Xi(x') \leq \Upsilon(y) \text{ and } \Xi(x) \leq \Upsilon(y')) \implies \Xi(x') \leq \Upsilon(y')]$.

Definition 2. Given a polarity (X,Y,R), the *intermediate structure* for (X,Y,R) is the poset

$$\mathrm{Int}(X,Y,R) = \mathrm{Im}(\Xi) \cup \mathrm{Im}(\Upsilon).$$

By the previous proposition, it follows that the order on $\mathrm{Int}(X,Y,R)$ is the order reflection of the quasi-order on the disjoint union of X and Y given by:

$\forall x_1, x_2 \in X \quad x_1 \leq x_2 \iff \forall y \in Y\ (x_2Ry \implies x_1Ry)$;
$\forall x \in X, y \in Y \quad x \leq y \iff xRy$;
$\forall y_1, y_2 \in Y \quad y_1 \leq y_2 \iff \forall x \in X\ (xRy_1 \implies xRy_2)$;
$\forall x \in X, y \in Y \quad y \leq x \iff \forall x' \in X, y' \in Y\ [(x'Ry \text{ and } xRy') \implies x'Ry']$.

We now apply the characterisations of the complete lattice and of the intermediate structure given by a polarity to obtain characterisations of Δ_0- and Δ_1-completions of posets.

Dedekind-MacNeille completions

We begin with the lowest level of two-sided completions, namely the Δ_0-completions. The Dedekind-MacNeille completion was first developed by Dedekind in the special case of the poset of rational numbers. It was later generalised to arbitrary posets by MacNeille [46]. The characterisation as the unique doubly dense completion was first proved by Banaschevski and Bruns [2].

Theorem 4. *Let P be a poset. Then P has a unique simultaneously $\bigvee$- and $\bigwedge$-dense completion and it may be obtained as $\mathscr{G}(P,P,\leq)$.*

Proof. By Theorem 3, for all $p,q \in P$ we have $p \leq q$ if and only if $\Xi(p) \leq \Upsilon(q)$ and $\mathrm{Im}(\Xi)$ is $\bigvee$-dense in $\mathscr{G}(P,P,\leq)$ while $\mathrm{Im}(\Upsilon)$ is $\bigwedge$-dense in $\mathscr{G}(P,P,\leq)$. Thus, if we can show that $\Xi(p) = \Upsilon(p)$ for all $p \in P$, then it follows from Theorem 3 that $\mathscr{G}(P,P,\leq)$ is the unique doubly dense completion of P. Since $p \leq p$ it follows that $\Xi(p) \leq \Upsilon(p)$. Finally, since $p' \leq p \leq p''$ implies $p' \leq p''$, it follows that $\Xi(p) \geq \Upsilon(p)$.

We observe that, while the MacNeille completion may be seen as a special case of the Galois closed sets of a polarity, the Galois closed sets of a polarity may also be seen as the MacNeille completion of the much simpler intermediate structure.

Corollary 1. *Let (X,Y,R) be a polarity, then $\mathscr{G}(X,Y,R)$ is isomorphic to the MacNeille completion of* $\mathrm{Int}(X,Y,R)$.

MacNeille completions are very well adapted to the study of residuated structures and Boolean algebras, Heyting algebras, and residuated lattices are all closed under MacNeille completions. More generally, a restricted form of Sahlqvist correspondence goes through for residuated operations, see [47, 34]. On the other hand, MacNeille completions interact quite badly with morphisms in general, and as a consequence they also tend to destroy equational properties. More specifically, MacNeille completion is a reflector into complete lattices with complete lattice homomorphisms only for the category of posets with so-called cut-stable maps, see [10]. We note that, even for Boolean algebras and distributive lattices most homomorphisms are not cut-stable. With respect to equational properties, Funayama [13] showed that the MacNeille completion of a distributive lattice need not be distributive. See also [42, Chapter III, p.110] for a simpler example due to Cornish.

Δ_1-completions

Δ_1-completions, see Definition 1, are those in which each element is both a join of meets and a meet of joins of elements from the original poset. These provide a wider spectrum of choices of completions, including the canonical extension. We start with a general representation theorem for Δ_1-completions in the spirit of Theorem 2 for $\bigvee$- and $\bigwedge$-completions.

Given a Δ_1-completion, $e\colon P \to C$, and taking X to be the $\bigwedge$-closure of P in C (which is a $\bigwedge$-clompletion of P) and Y to be the $\bigvee$-closure of P in C (which is a $\bigvee$-clompletion of P), we obtain a polarity (X,Y,R) with R defined by xRy if and only if $x \leq y$ in C, whose Galois closed sets, by Theorem 3, must be isomorphic to C. Based on this idea it is straight forward to show the following theorem.

Theorem 5 (Theorem 3.4 [23]). *Let P be a poset. There is a one-to-one correspondence between Δ_1-completions of P and polarities $(\mathscr{F},\mathscr{I},R)$ where*

(i) $\mathscr{F}$ is a standard closure system of up-sets of P;

(ii) $\mathscr{I}$ *is a standard closure system of down-sets of* P*;*
(iii) the relation $R \subseteq \mathscr{F} \times \mathscr{I}$ *satisfies the following four conditions:*

(Pol 1) $\forall p \in P, x \in \mathscr{F}\ (p \in x \iff xRp)$*;*
(Pol 2) $\forall p \in P, y \in \mathscr{I}\ (p \in y \iff pRy)$*;*
(Pol 3) $\forall x, x' \in \mathscr{F}, y \in \mathscr{I}\ (x \supseteq x'Ry \Rightarrow xRy)$*;*
(Pol 4) $\forall x \in \mathscr{F}, y, y' \in \mathscr{I}\ (xRy \subseteq y' \Rightarrow xRy')$.

Here p*,* $\uparrow p$*, and* $\downarrow p$ *are identified for every* $p \in P$*.*

One may think that the relation R in the above theorem is uniquely determined by *(Pol 1)* through *(Pol 4)* but this is not the case in general, see [23] for further details.

Canonical extensions

Canonical extensions were first introduced by Jónsson and Tarski for Boolean algebras with operators [44]. Canonical extensions capture Stone duality for Boolean algebras in a completion theoretic form and are useful in the study of Stone duality, especially when it comes to the study of additional algebraic structure, such as additional operations. In their second paper on canonical extensions, Jónsson and Tarski, [45] develope relational semantics for Boolean algebras with operators thus preceding Kripke's work on semantics for modal algebras by more than a decade. Much later, the theory of canonical extensions was generalised to bounded distributive lattices [24] and, soon thereafter, also to arbitrary bounded lattices [20] and posets [9].

Canonical extensions may be defined abstractly by combining the compactness properties for directed join- and directed meet-completions and the Δ_1 density property of Definition 1.

Definition 3. Let P be a bounded poset. A *canonical extension* of P is a completion C of P with the following two properties :

(density) each $c \in C$ is both a meet of joins and a join of meets of elements from P;

(compactness) for all downwards directed $A \subseteq P$ and all upwards directed $B \subseteq P$, we have $\bigwedge A \leq \bigvee B$ in C implies that there are $a \in A$ and $b \in B$ with $a \leq b$.

Theorem 6. *Let* P *be a bounded poset. Then* P *has a canonical extension and it is unique up to an isomorphism commuting with the embedding of* P *in the canonical extension.*

Proof. In order that the density condition holds, by Theorem 5, the canonical extension must be given by a polarity $(\mathscr{F}, \mathscr{I}, R)$ as in the theorem. In order that the completion satisfies the $\bigvee^{\uparrow}$-compactness property, the associated join-completion needs to be given by the collection of all order ideals and, order dually, the associated meet-completion needs to be given by the collection of all order filters. In order to insure that these two compactness properties are preserved in the full completion it is necessary and sufficient that the relation R be given by FRI if and only if $F \cap I \neq \emptyset$ for $F \in \mathscr{F}$ and $I \in \mathscr{I}$. The uniqueness then follows immediately by the uniqueness part of Theorem 5.

For Boolean algebras and distributive lattices, the existence was first proved by showing that the power set, respectively, set of downsets, of the dual space of the structure yields a canonical extension. The proof using the polarity $(Filt, Idl, R)$ given above was first given in [20]. Note that this proof does not require Stone's Prime Filter Theorem and thus it shows that the canonical extension is available in a constructive setting. For a recent survey with more details, see [16].

Canonical extension for Boolean algebras and distributive lattices is a reflector (left adjoint to the forgetful functor) for the subcategories of complete and atomic Boolean algebras and that of doubly algebraic distributive lattices, respectively. For lattices and posets no such characterisations are known. For lattices, and for some posets, the canonical extension as defined above may be characterised as the largest two-sided completion in a hierarchy of iterated completions [30]. However, even this characterisation fails for posets in general, and in total generality there is not much that is 'canonical' about the canonical extension.

The canonical extension of L was denoted by L^σ by Jónsson, mainly because of the choice of extension for additional operations (which, as we will see in the next section, come in a join-favouring σ version and a meet-favouring π version). However, since the defining properties of canonical extension are self dual, the order dual of the canonical extension of L is equal, up to isomorphism, to the canonical extension of the order dual of L. For this reason, the canonical extension has since been re-baptised as L^δ. The $\bigvee$-completion of L in L^δ is, not surprisingly, isomorphic to the lattice of filters of L. We denote this subset by $F(L^\delta)$ and refer to its elements as *filter elements* of L^δ. One can show that $F(L^\delta)$ is closed under arbitrary meets as well as finite joins in L^δ. However, by density $F(L^\delta)$ is join-dense in L^δ so $F(L^\delta)$ is not closed under arbitrary joins unless $L^\delta = F(L^\delta)$, which does not happen so often. Order dually, the $\bigwedge$-completion of L in L^δ is isomorphic to the lattice of ideals of L and is a subframe of L^δ which is meet-dense in L^δ. We denote this subset by $I(L^\delta)$ and refer to its elements as *ideal elements* of L^δ.

We note that the boundedness restriction is necessary to get as clean a treatment of canonical extension as we give here. As is thoroughly familiar to anyone having dealt with Stone duality, in order to deal with non-bounded structures, the simplest approach is to add a top and bottom (to all structures whether they already have them or not) and then remove them when recapturing the structure from its dual. A similar approach is possible for canonical extensions. In [9] the boundedness assumption was not stated though some statements made there are false without this assumption. This was subsequently fixed in [23] where a counterexample was given. The recent paper [48] explores other choices of canonical extensions with particular attention payed to the non-bounded case and residuation. *In the remainder of this paper, we assume that all posets and lattices are bounded.*

As we will see in Section 4.3, canonical extensions have quite good properties relative to homomorphisms and preservation of identities. In fact, it has been shown in [21] that any first-order universal property, in particular any equational property, preserved by MacNeille completion is also preserved by canonical extension. One way to understand why MacNeille completion behaves better on the intermediate structure of $(Filt, Idl, R)$ than on the original structure is that extending, e.g., lattice homomorphisms to this structure lands one in the otherwise very restrictive category of cut-stable maps [30].

3 Extensions of maps

As mentioned above, the very *raison d'être* of canonical extensions is the study of duality theory for additional operations. The first definition of extensions of maps by Jónsson and Tarski remains essentially unchanged today, but their formulation was restricted to order preserving maps, whereas a slight generalisation, given below, allows one to extend arbitrary maps. In contrast with the canonical extension of lattices, which are uniquely characterised by an abstract property, the definition of extensions of maps is ad hoc. The work presented in [25], with which the fully general definition originates, also provided, for the first time, a natural setting for the extensions of maps on distributive lattices, showing that the two canonical choices for extending maps are upper and lower semicontinuous envelopes for certain topologies. The topological approach to extension of maps seems an important topic for arbitrary lattices as well though the situation there is not as clear cut. The topological analysis of extensions of maps played a crucial rôle in [21] and the topological theory in the setting of arbitrary lattices was developed further in the survey article [31] and in the Ph.D. thesis of Jacob Vosmaer [56], and we refer the reader interested in this topic there.

From hereon out we restrict our attention to (bounded but not necessarily distributive) lattices as this is a pertinent setting for the topics of this book in which the theory is well developed. Many open questions remain about how much of the theory exposed here generalises to posets or to other Δ_1-completions.

Definition 4. Let K and L be lattices, $f : K \to L$ any function. We define maps f^σ and f^π from K^δ into L^δ by

$$f^\sigma(u) := \bigvee \left\{ \bigwedge \{ f(a) \mid a \in K \text{ and } x \le a \le y \} \mid F(K^\delta) \ni x \le u \le y \in I(K^\delta) \right\},$$
$$f^\pi(u) := \bigwedge \left\{ \bigvee \{ f(a) \mid a \in K \text{ and } x \le a \le y \} \mid F(K^\delta) \ni x \le u \le y \in I(K^\delta) \right\}.$$

Since each element of L is both a filter element and an ideal element of L^δ, it is easily seen that both f^σ and f^π extend f. We record here a few basic facts about these two extensions:

Proposition 3. *[20, Lemma 4.2,4.3,4.4] Let $f : L \to M$ be a map between lattices.*

1. *f^σ and f^π both extend f;*

2. *$f^\sigma \leq f^\pi$;*
3. *If f preserves or reverses order, then $f^\sigma = f^\pi$ on the intermediate structure;*
4. *If f preserves or reverses binary joins or meets, then $f^\sigma = f^\pi$.*

Crucial to the treatment of additional operations using these definitions is the fact that the canonical extension of a Cartesian product is the Cartesian product of the canonical extension so that, e.g. for a binary operation $f\colon L \times L \to L$, the extension $f^\pi\colon (L \times L)^\delta \to L^\delta$ is in fact a binary operation on L^δ since $(L \times L)^\delta \cong L^\delta \times L^\delta$.

Note that the definitions of the extensions of maps are self dual in the order *on the domain* of the map. Thus we can take the order dual of the domain, or of any coordinate of the domain, and still obtain the same extension. Further note that, if the map is order preserving, then the upper bounds of the intervals on which we are taking meets, and the lower bounds of the intervals on which we are taking joins play no role. Accordingly, for $f : K \to L$ order-preserving we have

$$f^\sigma(u) = \bigvee \left\{ \bigwedge \{ f(a) \mid x \leq a \in K \} \mid F(K^\delta) \ni x \leq u \right\},$$

or, in two tempi, for filter elements $x \in F(K^\delta)$ and $u \in L^\delta$

$$f^\sigma(x) = \bigwedge \{ f(a) \mid x \leq a \in K \}$$
$$f^\sigma(u) = \bigvee \left\{ f^\sigma(x) \mid F(K^\delta) \ni x \leq u \right\}.$$

Given a map $f : K \to L$ between lattices, we say f is *smooth* provided the extensions f^σ and f^π are equal. Not all maps are smooth but, as stated above, for maps that are binary join or meet preserving, or that turn binary joins into meets or vice versa, the two extensions agree. When the two extensions agree, we denote it by f^δ.

The crucial algebraic facts about the two extensions relative to joins and meets are as follows.

Proposition 4. *Let K, L, and M be lattices and $f\colon K \times L \to M$ an order preserving map.*

1. *If f preserves binary meets in the ith coordinate (or turns binary joins in the ith coordinate into meets), $i = 1$ or 2, then f^π preserves arbitrary non-empty meets (or turns arbitrary non-empty joins into meets) in that coordinate.*
2. *If f preserves binary joins in the ith coordinate (or turns binary meets in the ith coordinate into joins), $i = 1$ or 2, then f^σ preserves arbitrary non-empty joins (or turns arbitrary non-empty meets into joins)in that coordinate.*
3. *If $h\colon L \to M$ is a bounded lattice homomorphism, then h^δ is a complete lattice homomorphism.*

The facts in the above proposition may be found in most of the foundational papers on canonical extension. See e.g. [20, Lemma 4.3 and Proposition 4.6] for the lattice version.

Remark 1. Here it is maybe worthwhile pointing out an important difference in the hypotheses of Proposition 3.4 and Proposition 4.1 and 2 above. The former requires that f be join or meet preserving or reversing *on its domain as a whole*, whereas the latter only requires such behaviour in a particular coordinate of an n-ary operation. An operation which preserves binary joins in each coordinate is precisely what Jónsson and Tarski called an *operator*. If f is unary there is of course no difference between the two hypotheses, however, for binary operations, being binary join preserving is a much stronger property than being an operator. For example, the binary operation $\wedge$ on a non-trivial lattice is always meet preserving, it is never join preserving, but it is an operator (and thus join preserving in each coordinate) if and only if the lattice is distributive. An operator is said to be *normal* provided it preserves the bottom in each coordinate, that is, as soon as one coordinate is 0, then the tuple is sent to 0. Thus a normal operator preserves all finite joins in each coordinate. The order dual notions are called dual operators and normal dual operators preserve 1 in each coordinate. Finally, certain coordinates of the domain may need to be reversed in order to realise an operation as an operator or dual operator.

By Proposition 4.2 it follows that the σ-extension of an operator is a *complete operator*, that is, it preserves arbitrary non-empty joins in each coordinate. Also, the σ-extension of normal operator is again normal and thus actually completely join preserving in each coordinate. In particular, if $(L,\cdot,/,\backslash)$ is a residuated lattice, then $\cdot^\sigma$ preserves arbitrary joins in each coordinate and $/^\pi$ and $\backslash^\pi$ preserve arbitrary meets in their order preserving coordinate and send arbitrary joins to meets in their order reversing coordinate.

However, binary operators (and their variously flipped versions) are almost never smooth. In the case of a Heyting algebra, the residuated pair of operations is $\wedge$ and $\to$. The operation $\wedge$ happens to be smooth because it preserves binary meets. Being a lower adjoint it also preserves joins in each coordinate, that is, it is an operator (or, equivalently, the lattice is distributive). Similarly, the upper adjoint $\to\colon L^\partial \times L \to L$ is meet preserving in each coordinate (relative to the order on L this means that $\to$ sends joins in the first coordinate and meets in the second coordinate to meets). That is, with a flip in the first coordinate, $\to$ is a dual operator. It is however *not* smooth in general. See [16, Example 1] for a simple example of a Heyting algebra for which the two extensions of its implication are different.

3.1 Extension of residuated families of maps

A fundamental notion in the algebraic treatment of substructural logic is that of a residuated family of binary maps. A family of maps $\cdot\colon L\times L\to L$, $\backslash\colon L\times L\to L$, and $/\colon L\times L\to L$ is *residuated* provided $\backslash$ and $/$ are parametrised upper adjoints of the unary maps obtained from $\cdot$ by holding, respectively, the right and left coordinate fixed. That is, we have for all $a,b,c\in L$

$$\begin{aligned} a\cdot b\le c &\iff b\le a\backslash c\\ &\iff a\le c/b. \end{aligned}$$

The fact that by choosing the correct extension of these operations, residuated families lift to residuated families on the canonical extension is fundamental to the application of duality in the area. This fact is true in the most general setting of posets and varying proofs may be found in varying settings (posets, lattices, distributive lattices) in the literature, see e.g. [9]. We include a slightly generalised version of the proof given in [16] for Heyting algebras.

Proposition 5. *Let* $(L, \cdot, \backslash, /)$ *be a lattice equipped with a residuated family, then so is* $(L^\delta, \cdot^\sigma, \backslash^\pi, /^\pi)$.

Proof. Let $x, x' \in F(L^\delta)$, and $y \in I(L^\delta)$. Using the fact that $\cdot\colon L \times L \to L$ and $\backslash$, viewed as a map from $L^\partial \times L$ to L, are order preserving, we have

$$x \cdot^\sigma x' = \bigwedge\{a \cdot a' \mid x \le a \in A \text{ and } x' \le a' \in A\},$$
$$x' \backslash^\pi y = \bigvee\{a' \backslash b \mid x' \le a' \in A \ni b \le y\}.$$

By compactness, then residuation in L, and compactness again, we have

$$\begin{aligned} x \cdot^\sigma x' \le y &\iff \exists a, a', b \in L\ (x \le a, x' \le a', b \le y \text{ and } a \cdot a' \le b) \\ &\iff \exists a, a', b \in A\ (x \le a, x' \le a', b \le y \text{ and } a' \le a \backslash b) \\ &\iff x' \le x \backslash^\pi y. \end{aligned}$$

Now for $u, u', v \in L^\delta$, again by order monotonicity of the operations, we have

$$u' \backslash^\pi v = \bigwedge\{x' \backslash^\pi y \mid F(L^\delta) \ni x' \le u' \text{ and } v \le y \in I(L^\delta)\},$$
$$u \cdot^\sigma u' = \bigvee\{x \cdot^\sigma x' \mid F(L^\delta) \ni x \le u \text{ and } F(L^\delta) \ni x' \le u'\}.$$

Finally, using the join-density of $F(L^\delta)$ in L^δ and the meet-density of $I(L^\delta)$ in L^δ, we obtain

$$\begin{aligned} & u \cdot^\sigma u' \le v \\ \iff & \forall x, x' \in F(A^\delta)\, \forall y \in I(A^\delta)\, \big[(x \le u, x' \le u', \text{and}\, v \le y) \Rightarrow x \cdot^\sigma x' \le y\big] \\ \iff & \forall x, x' \in F(A^\delta)\, \forall y \in I(A^\delta)\, \big[(x \le u, x' \le u', \text{and}\, v \le y) \Rightarrow x \le x' \backslash^\pi y\big] \\ \iff & u \le u' \backslash^\pi v. \end{aligned}$$

The proof for $/$ is right-left symmetric to the one for $\backslash$.

When we talk of canonical extension for lattices equipped with a residuated family, $(L, \cdot, \backslash, /)$, we assume that it is given by $(L^\delta, \cdot^\sigma, \backslash^\pi, /^\pi)$.

4 Canonicity

A central property in algebraic logic is that of canonicity: A class of algebras is *canonical* provided it is closed under canonical extensions. This property is by no means necessary in order for canonical extensions and duality to be useful in studying a class of algebras (and thereby the corresponding logic as is done in algebraic logic). For example, the papers [29, 28] develop dualities for varieties of so-called double quasioperator algebras even though these are not canonical in general. In fact, a fundamental example of a variety of double quasioperator algebras is the variety of MV-algebras which is non canonical [27]. Canonical extensions were also used to study sheaf representations of MV-algebras in [19]. However, when a class is canonical, then the duality for the properties defining the class reduces to a study of these in a discrete (non-topological) duality available between relational structures and certain complete lattices with additional operations. We will make this statement more precise in Section 5.

From an algebraic point of view, canonicity is closely related to the more technical question of when canonical extension for maps commutes with composition of maps. In fact, it is particularly important to identify whole clones (classes containing all projections and closed under composition) with this property. This topic is best understood using a number of different topologies (at least nine if not twelve of them in the lattice setting!), however, as mentioned above, these topics are out of scope for this chapter, and we refer the reader to [25, 31, 56] for a treatment with this point of view.

Consider the diagrams

$$
\begin{array}{ccc}
A & \xrightarrow{\;f\;} & B \\
{\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
C & \xrightarrow[\;k\;]{} & D
\end{array}
\qquad\qquad
\begin{array}{ccc}
A^{\delta} & \xrightarrow{\;f^{\sigma}\;} & B^{\delta} \\
{\scriptstyle h^{\sigma}}\big\downarrow & & \big\downarrow{\scriptstyle g^{\sigma}} \\
C^{\delta} & \xrightarrow[\;k^{\sigma}\;]{} & D^{\delta}
\end{array}
$$

Given that the first diagram commutes, we want to be able to conclude that the second diagram also commutes. An obvious way to prove this is to show that

$$g^{\sigma} f^{\sigma} = (gf)^{\sigma} = (kh)^{\sigma} = k^{\sigma} h^{\sigma}.$$

Here the second equality holds by hypothesis. The question is therefore, when canonical extensions distribute over composition, i.e., under what conditions it is the case that $(gf)^{\sigma} = g^{\sigma} f^{\sigma}$?

An identity $s = t$ holding in an algebra A means that the term functions s^A and t^A are equal. These are elements of the clone of the algebra A and thus they are generated from the basic operations and the projections by closure under composition. Now suppose each basic operation on A^{δ} is given by the σ-extension of the corresponding operation on A, and suppose that the σ-extension the projections between powers of A are the corresponding projections of A^δ, and suppose that σ-extension commutes with composition on the clone of term functions on A, then if the equation $s=t$ holds in A, it follows by Noetherian induction that $s=t$ also holds in A^δ. This is the basic idea underlying algebraic canonicity proofs.

We note that sometimes, as in the case of residuated lattices, some operations are extended by σ and some by π. Also, often, only one inequality of an identity is non-trivial. In these cases, canonicity may sometimes be provable by a lax version of the above schema of the following form:

$$g^\sigma f^\sigma \leq (gf)^\sigma \leq (kh)^\pi \leq k^\pi h^\pi.$$

Note that the center inequality always holds by Proposition 3.2. In [25], for distributive lattices, a method based on topology was developed that may allow one to conclude that the two other inequalities hold. It would be interesting to explore further to which extent these methods generalise to the lattice and poset case and to other Δ_1-completions.

The lattice operations are usually not thought of as interpreted in the canonical extension by the extension of the corresponding lattice operations on L but by the lattice operations on L^δ. However, one can show that $\vee^\sigma = \vee^\pi$ is the join operation on L^δ and similarly for the meet [20, Lemma 5.1].

Note also, that the compositionality question raised above also has functoriality of canonical extensions for lattices with additional operations as a special case: If $A = B^n$, $C = D^n$, g is a lattice homomorphism, and $h = g^{[n]}$ is n-fold coordinate-wise g, then the above question is equivalent to whether a homomorphism $g\colon (B,f) \to (D,k)$ extends to a homomorphism $g^\delta\colon (B^\delta, f^\sigma) \to (D^\delta, k^\sigma)$.

The basic facts about composition and canonical extension for maps are as follows, where *Scott continuity* means continuity with respect to the Scott topology on both the domain and the codomain, which in turn reduces simply to the order theoretic property of preserving directed joins. Of course, order dual statements hold for dual Scott continuity and filtered meets. In the lattice setting these results may be found in [20, Lemma 4.5].

Proposition 6. *Let $f\colon K \to L$ and $g\colon L \to M$ be order preserving maps between lattices.*

1. $(gf)^\sigma \leq g^\sigma f^\sigma \leq g^\sigma f^\pi, g^\pi f^\sigma \leq g^\pi f^\pi \leq (gf)^\pi$ *with equality on the intermediate structure.*
2. *If g^σ is Scott continuous, then $(gf)^\sigma = g^\sigma f^\sigma$.*
3. *If g^π is dually Scott continuous, then $(gf)^\pi = g^\pi f^\pi$.*
4. *If f preserves binary meets, then $(gf)^\sigma = g^\sigma f^\sigma$.*
5. *If f preserves binary joins, then $(gf)^\pi = g^\pi f^\pi$.*

We are now ready to see how these basic facts lead to canonicity results.

4.1 Canonicity for LOs

The first application of Proposition 6 is to varieties of lattices with an equational base consisting of terms involving only operators, or only dual operators, but not a mix of the two. This is essentially an application of Proposition 6.1 and 2. and goes back, in the setting of Boolean algebras, to the original paper of Jónsson and Tarski [44]. The proof, identifying Scott continuity as the essential ingredient, goes back to the first article in the distributive lattice setting [24].

Let L be a lattice. It is not hard to see that for a projection $\pi_i^L\colon L^n \to L$, we have

$$(\pi_i^L)^\delta = \pi_i^{L^\delta}$$

and this map is completely join preserving and thus in particular Scott continuous. Now denote by $\sigma\text{-}Scott(L)$ the clone of all operations on L whose σ-extension is Scott continuous and by $Scott(L^\delta)$ the clone of all operations on L^δ that are Scott continuous. Then $(\ )^\sigma\colon \sigma\text{-}Scott(L) \to Scott(L^\delta)$ is a well-defined mapping. Since Scott continuous functions are in particular order preserving, and since the composition of Scott continuous functions is again Scott continuous, as a consequence of Proposition 6.1 we obtain the following theorem.

Theorem 7. *Let L be a lattice. Then $\sigma\text{-}Scott(L)$ is a clone on L and*

$$(\ )^\sigma\colon \sigma\text{-}Scott(L) \to Scott(L^\delta)$$

is a clone homomorphism.

This theorem is particularly interesting due to the following two observations. By Proposition 4.2, the σ-extension of an operator is a complete operator, and therefore, given that it has only finite arity, it is in fact Scott continuous. The set of all operators on L is *not* a clone as it is not closed under composition (except for the case of the trivial lattice with only one element). Thus knowing that $(gf)^\sigma = g^\sigma f^\sigma$ for operators f and g is not sufficient to prove a canonicity theorem for equations involving operators. Such a theorem is however a corollary of Theorem 7.

Corollary 2. *[20, Theorem 6.3] Let L be a lattice with additional operations. Let $s = t$ be an identity holding in L. If all of the basic operations comprising s and t interpret to operators of L, then $s = t$ holds in L^σ. Order dually, if all of the basic operations comprising s and t interpret to dual operators of L, then $s = t$ holds in L^δ.*

Note that $\wedge$ of a lattice is an operator if and only if $\vee$ is a dual operator if and only if the lattice is distributive. Thus for non-distributive lattices with dual operators, Corollary 2 above does not cover equations involving $\vee$. This of course does not mean that such equations are necessarily non-canonical. Already in the 1970's a careful syntactic calculus extending Jónsson and Tarski's original version of Corollary 2 was developed by Sahlqvist in the Boolean setting [53]. This calculus was analysed algebraically by Jónsson in the Boolean case in [43]. In [26] the Sahlqvist calculus was extended to the negation-free unary distributive setting while showing that the canonicity part of Sahlqvist theory depends on a careful analysis of order reversal together with the results stated in Proposition 4 above. Work is still going on in this direction, both in the substructural setting and from an algorithmic point of view, see e.g. [54, 5]. Rather than describing the full Sahlqvist calculus, we will treat a few equations often occurring in substructural logic to illustrate the ideas behind this calculus.

4.2 Canonicity of axioms from substructural logic

We now consider a few axioms that are prominent in substructural logic. The canonicity results given in this section are not difficult and are mainly given for illustration purposes. We do not claim that they are new but have not made the effort to check whether they happen to occur somewhere in the literature.

Proposition 7. *Let $(L,\cdot)$ be a lattice with a binary operator. Further, let* 0 *and* 1 *be elements of L. The following identities for $(L,\cdot,0,1)$ are canonical. That is, if they are satisfied by $(L,\cdot,0,1)$ then they are also satisfied by $(L^\delta,\cdot^\sigma,0,1)$.*

1. $a\cdot(b\cdot c)=(a\cdot b)\cdot c$;
2. $a\cdot b=b\cdot a$;
3. $a\leq a^2$;
4. $a^2\leq a$;
5. $a^{n+1}=a^n$;
6. $a\leq 1$;
7. $0\leq a$.

Proof. This follows by the result on operator canonicity stated in Corollary 2.

Combining this proposition with Proposition 5, we obtain the following corollary.

Corollary 3. *Let $\mathscr{V}$ be the variety of lattices equipped with a residuated family of operations. Then the subvarieties of $\mathscr{V}$ given by any combination of the equations in Proposition 7 are canonical.*

Once one combines operators and dual operators as in the *McKinsey axiom*, $\Box\Diamond a\leq\Diamond\Box a$, [36], or an order reversing isomorphism like the negation on a Boolean algebra with operators, canonicity may fail. See e.g. [31, Example 5] for a full description of the classical example of the *Gödel-Löb axiom*, $\Diamond a\leq\Diamond(\neg\Diamond a\wedge a)$, for Boolean algebras equipped with a unary finite join preserving operation. These classical examples of non-canonicity are caused by having an operator in the scope of a dual operator below the inequality and a dual operator in the scope of an operator above the inequality as in the McKinsey axiom (a similar mechanism is at play in the Gödel-Löb axiom where the order reversal yields both a ‘negative’ and a ‘positive’ branch with bad behaviour, see [26, Section 3] for a precise description of positive and negative branches and ‘bad behaviour’).

In the setting of substructural logic, there are a few common axiom that squirt this situation, but they are nevertheless canonical. We start with a simple application of the results in Proposition 6.

Proposition 8. *Let* $(L,\cdot,\backslash,/,0)$ *be a lattice equipped with a residuated family and a constant* 0*. Define derived operations* $\sim a = a\backslash 0$ *and* $-a = 0/a$*. Then the equations*

$$-\sim a = a \quad \textit{and} \quad \sim -a = a$$

are canonical for L.

Proof. We consider the equation $-\sim a = a$. First we note that it is easy to verify that$-^{\pi}u = 0/^{\pi}u$ and $\sim^{\pi} u = u\backslash^{\pi}0$. Also, since both $\sim$ and $-$ turn finite joins into finite meets we have $\sim^{\sigma}=\sim^{\pi}$ and $-^{\sigma} = -^{\pi}$. Denote the order dual of L by L^{∂}. Define $g\colon L^{\partial} \to L$ by $g(a) = -a$ and $f\colon L \to L^{\partial}$ by $f(a) = \sim a$. Then, if $-\sim a = a$ holds in L, then we have

$$id_{L^{\delta}} = id_L^{\delta} = (gf)^{\pi} = g^{\pi} f^{\pi} = g^{\delta} f^{\delta}.$$

The third equality holds by either of Proposition 6.3 or 5. Thus $u = 0/^{\pi}(u\backslash^{\pi}0)$ on L^{δ}.

The canonicity of the equation $\sim -a = a$ follows by right-left symmetry.

Proposition 9. *Let D be a distributive lattice and let* $(\cdot,\backslash,/)$ *a residuated family on D. Then the equations*

$$(a\backslash b)\vee(b\backslash a) = 1 \quad \textit{and} \quad (a/b)\vee(b/a) = 1$$

are canonical for D.

Proof. We claim that the identity $(a\backslash b)\vee(b\backslash a) = 1$ holding in D is equivalent to the following property holding for all $x,x' \in F(D^{\delta})$ and all $y,y' \in I(D^{\delta})$

$$(x \le y') \text{ and } (x' \le y) \implies (x\backslash^{\pi}y)\vee(x'\backslash^{\pi}y') = 1. \tag{1}$$

To see this, suppose $(a\backslash b)\vee(b\backslash a) = 1$ holds in D and let $x,x' \in F(D^{\delta})$ and $y,y' \in I(D^{\delta})$ with $x \le y'$ and $x' \le y$. Then by compactness there are $a,b \in D$ with $x \le a \le y'$ and $x' \le b \le y$. Thus we have

$$(x\backslash^{\pi}y)\vee(x'\backslash^{\pi}y') \ge (a\backslash b)\vee(b\backslash a) = 1$$

so that the property (1) holds. Conversely, suppose (1) holds and let $a,b \in D$. Then, we see that $(a\backslash b)\vee(b\backslash a) = 1$ simply by choosing $x = a = y'$ and $x' = b = y$.

Finally, we use distributivity to show that (1) is equivalent to the identity $(a\backslash b)\vee(b\backslash a) = 1$ holding in D^{δ}. Clearly, if $(a\backslash b)\vee(b\backslash a) = 1$ holds in D^{δ} and $x,x' \in F(D^{\delta})$ and $y,y' \in I(D^{\delta})$ with $x \le y'$ and $x' \le y$, then

$$(x\backslash^\pi y) \vee (x'\backslash^\pi y') \geq (x\backslash y) \vee (y\backslash x) = 1$$

so that (1) holds. On the other hand, if (1) holds, D is distributive, and $u, v \in D^\delta$ then

$$\begin{aligned}(u\backslash^\pi v) \vee (v\backslash^\pi u) &= (\bigwedge\{x\backslash^\pi y \mid x \leq u, v \leq y\}) \vee (\bigwedge\{x'\backslash^\pi y' \mid x' \leq v, u \leq y'\}) \\ &= \bigwedge\{(x\backslash^\pi y) \vee (x'\backslash^\pi y') \mid x \leq u \leq y', x' \leq v \leq y\} \\ &= 1.\end{aligned}$$

where the variables x and x' range over $F(D^\delta)$ and the variables y and y' range over $I(D^\delta)$. For the middle equality we use the fact that if D is distributive then D^δ is a dual frame – even without the use of non-constructive principles, see (the order dual of) [16, Theorem 3].

We finish this subsection with an axiom which is not canonical, even for distributive lattices equipped with residuated families. This is the *relativized law of double negation*:

$$(a \to b) \to b = a \vee b$$

Here $a \to b$ stands for $a\backslash b$ and b/a in the setting of a residuated family $(\cdot, \backslash, /)$ for which $\cdot$ is commutative. In the case where b is the bottom element of the lattice this becomes the usual law of double negation. What is interesting here is that this identity fails to be canonical for a different reason than the lack of compositionality as was the case for the MacKinsey axiom and the Gödel-Löb axiom in modal logic. As was shown in [28], if $\to : L^\partial \times L \to L$ is both an operator and a dual operator (as it is in the prelinear case), then the problematic direction of the above identity implies that the identity

$$(u \to^\pi v) \to^\sigma v \leq u \vee v$$

holds in L^δ. The problem is however, that $\to^\pi$ and $\to^\sigma$ are not equal in general, see [27, 28, 29] for more details.

Despite the fact that the above axiom is non-canonical, quite a well behaved duality theory may be developed based on the extended notion of canonicity using both σ- and π-extensions. This is the subject of [28] and [29]. This duality has been used, in the setting MV-algebras, to obtain a simple description of sheaf representations of MV-algebras and other structural results [18]. Also a preprint on the Priestley duality theory for MV-algebras is available [19].

4.3 Class operators and finitely generated varieties

The canonicity results discussed above are mainly based on the shape of (in)equations describing classes of lattices with additional operations. However, the basic compositionality results are also closely related to functoriality results for canonical extension, which in turn lead to canonicity results for various class operators.

Given an operational type τ, we will call *lattice expansion* of type τ (LE for short) any algebra that is a bounded lattice and has an additional operation of the appropriate arity for each operation symbol of the type τ. Further, we call a *monotonicity type*, a map $\varepsilon\colon \tau_n \to \{1,\partial\}^n$ which assigns to an operation symbol f of arity n in τ a tuple in $\{1,\partial\}^n$. Let L be a lattice expansion of type τ. We say that L is of monotonicity type ε provided, for each operation symbol f of arity n, we have that $f^L\colon L^{\varepsilon_1(f)} \times \ldots \times L^{\varepsilon_n(f)} \to L$ is order preserving. For example, a lattice equipped with a residuated family $(\cdot,\backslash,/)$ is an LE of type $(2,2,2)$ and its monotonicity type is ε where $\varepsilon(\cdot)=(1,1)$, $\varepsilon(\backslash)=(\partial,1)$, and $\varepsilon(/)=(1,\partial)$. Further, if L is an LE of type τ and $\alpha\colon \tau \to \{\pi,\sigma\}$, the α-canonical extension of L is the LE $(L^\delta,(f^{\alpha(f)})_{f\in\tau})$. For example, as we have seen in Proposition 5, the appropriate canonical extension for a lattice equipped with a residuated family is the one given by $\alpha\colon \cdot \mapsto \sigma, \backslash \mapsto \pi, / \mapsto \pi$. More generally, as we've seen in Proposition 4, the σ-extension is more appropriate for operators, while the π-extension is more appropriate for dual operators.

Functoriality of α-canonical extension

Canonical extension is a fairly well-behaved functor on lattices.

Proposition 10. *[20, Lemmas 4.4, 4.6, 4.9, and 4.5] Let $h\colon K \to L$ be a lattice homomorphism, then h is smooth and $h^\delta\colon K^\delta \to L^\delta$ is a complete lattice homomorphism and we have*

1. *h is injective if and only if h^δ is injective;*
2. *h is surjective if and only if h^δ is surjective.*

Furthermore, if $g\colon L \to M$ is also a lattice homomorphism, then $(gh)^\delta = g^\delta h^\delta$. Finally, $(\mathrm{id}_L)^\delta = \mathrm{id}_{L^\delta}$ so that $(\ )^\delta$ is a functor on the category of lattices with lattice homomorphisms.

Remark 2. In the restricted setting of distributive lattices, even more may be said: Let **DL** be the category of (bounded) distributive lattices with (bounded) lattice homomorphisms, and let $\mathbf{DL}^+$ be the category of completely distributive algebraic lattices with complete lattice homomorphisms, then $(\ )^\delta$ restricts to a functor from **DL** to $\mathbf{DL}^+$ which is left adjoint to the forgetful functor from $\mathbf{DL}^+$ to **DL**, and thus, in this restricted setting, canonical extension is just the witness of the fact that $\mathbf{DL}^+$ is a reflective subcategory of **DL**. However, this is no longer true for lattices nor for distributive lattices with additional operations, see [6, Proposition 6].

In the setting of distributive lattices canonical extension is also the profinite completion.This connection has been studied further by a number of authors. In particular, it was shown by Harding [39] that a variety of monotone lattice expansions of finite type is finitely generated if and only profinite completion and canonical extension agree for all algebras in the variety.

For lattices with additional operations, α-canonical extension is not functorial in general. As seen above, the compositionality of canonical extension for homomorphisms is not a problem. However, what remains a question is whether the canonical extension of a homomorphism of lattices with additional operations lifts to a homomorphism between the α-canonical extensions of the given lattices with additional operations. The following example, taken from [25], shows that this may indeed fail to be the case.

Example 1. Let B be a bounded dense chain, and let A be a bounded subchain that is dense in B, such that $B-A$ is also a dense subset of B (in the order theoretic sense). Let $g : B \to B$ be the unary operation on B that sends each element of A to 1 and each element of $B-A$ to 0. Then $(A, g\lceil A)$ is a subalgebra of (B,g) and $g_A = g\lceil A$ is the constant function equal to 1. Let $h : A \to B$ be the inclusion homomorphism, then it is not hard to show [25, Example 2.34], that $g_A^\sigma = g_A^\pi = 1$ whereas $g^\sigma(u) = 0$ for all $u \in B^\delta - B$ and thus, for $u \in A^\delta - A$, we have

$$h^\delta(g_A^\sigma(u)) = h^\delta(1) \neq 0 = g^\sigma(u) = g^\sigma(h^\delta(u)).$$

The fact that the above example concerns an embedding is no accident. By the first homomorphism theorem of universal algebra, all algebra homomorphisms decompose into a quotient map and an embedding into a subalgebra, and a key result of [25] showed that surjective homomorphisms between distributive lattices with additional operations always lift to the canonical extensions having as consequence that non-functoriality of canonical extension may be seen as a local property [25, Section 3]. This has as consequence that for any α there is a largest class of distributive lattices with additional operations of the given type, closed under taking subalgebras, on which α-canonical extension is a functor [25, Theorem 3.10]. Furthermore, it is shown there that this class is closed under homomomorphic images, subalgebras, and weak Boolean product. It is however, *not* closed under arbitrary products in general.

In the setting of general lattices with additional operations, the question of functoriality of α-canonical extension is even more complicated. The additional complications comes from the fact that the σ- and π-extensions of maps no longer necessarily have the universal properties of being the largest, respectively least, extensions with certain continuity properties. In [31] this is studied in the general lattice setting. The upshot is that canonical extension remains functorial on the two most important subclasses on which it is functorial in the distributive setting:

Theorem 8. *Let τ be a type of lattice expansions and α a corresponding canonical extension type. Then α-canonical extension is functorial on the following classes of lattice expansions of type τ:*

1. *The* monotone lattice expansions of type τ*: that is, those that have monotonicity type ε for some ε of type τ;*
2. *The lattice expansions of type τ that lie is some finitely generated variety.*

In the distributive setting, this was proved in Lemma 3.24 and Theorem 3.26 of [25]. In the lattice setting, (1) was proved in Theorem 5.2 of [20], while (2) was proved in Theorem 6 of [31]. Since all α-canonical extensions seem to work equally well, one may wonder how to choose, but this will become clearer in Section 5 which deals with the relation to duality and relational semantics.

Preservation of canonicity under class operators

By Birkhoff's variety theorem, varieties are precisely those clases closed under the operators H, S, and P which close a class under homomorphic images, subalgebras, and Cartesian products, respectively. An important observation, much used by Jónsson in his work in universal algebra, is that any Cartesian product may be seen as a Boolean product of all the ultraproducts obtainable from the given product. That is, $P \leq P_B P_\mu \leq HSP$ and thus $HSP = HSP_B P_\mu$ where P_B and P_μ are the class operators taking all possible Boolean products and ultraproducts of algebras of a given class, respectively.

For a class $\mathscr{K}$ of algebras denote by $\mathscr{K}^\alpha$, the class of α-canonical extensions of algebras in $\mathscr{K}$.

Theorem 9. *Let ε be a monotonicity type of lattice expansion and $\mathscr{K}$ a class of lattice expansions of monotonicity type ε. Further, let α be a compatible canonicity type. Then the following hold:*

1. *If $B \in H(\mathscr{K})$, then $B^\alpha \in H(\mathscr{K}^\alpha)$;*
2. *If $B \in S(\mathscr{K})$, then $B^\alpha \in S(\mathscr{K}^\alpha)$;*
3. *If $B \in P_B(\mathscr{K})$, then $B^\alpha \in P(\mathscr{K}^\alpha)$;*
4. *If $B \in P_\mu(\mathscr{K})$ implies $B^\alpha \in HSP(\mathscr{K})$, then $HSP(\mathscr{K})$ is α-canonical.*

In the distributive setting, this is a special case of Theorem 4.2 of [25] and the results leading up to it. In the general lattice setting this is Theorem 6.8 of [20]. Various parts of this theorem hold without the assumption of monotonicity, but then, at least for known results, some other restrictions have to be made. For example, (1) holds in full generality if we restrict to distributive lattices. For more information on the versions with weaker hypotheses, see [25] and [31].

For a finite algebra A and any α, we have that the α-canonical extension of A is A itself. In addition, we have that $P_\mu(A) = I(A)$, the isomophism class of A. Thus it is a corollary of Theorem 9.4 that the variety generated by any finite monotone lattice expansion is canonical. However, in the case of finitely generated varieties the monotonicity assumption is not needed.

Theorem 10. *[25, Corollary 4.6] [31, Section 5] Let τ be any signature of abstract algebras and α any canonical extension type compatible with τ. Any finitely generated variety of lattice expansions type τ is α-canonical.*

Relation to closure under MacNeille completion

The use of MacNeille completion rather than canonical extension goes back at least to Monk [47] who was the first to make a systematic study of MacNeille completion of lattices with additional operations. MacNeille completion has played a prominent role in recent work on the relation between an algebraic and a proof theoretic treatment of substructural logics [4]. Accordingly, it is interesting to understand the relationship between closure under canonical extension and MacNeille completion for classes of lattice expansions.

The following result, which gives an answer to this question was actually found as an algebraic generalisation of a model theoretic result on Kripke models of modal logic first proved by Fine [12], and later generalised by Goldblatt [35], that states that the so-called ultrafilter extension of a Kripke frame S can be obtained as a certain 'bounded morphic image' of some elementary extension of S.

As for canonical extension, we have two natural choices when extending monotone additional operations to the MacNeille completion: The *lower* extension, essentially using joins, and the *upper* extension, using meets, see [21, Definitions 2.7–2.11]. With the lower extension playing the role of the σ-extension and the upper extension the role of the π-extension, we then obtain a notion of α-MacNeille completion of a lattice expansion.

The following theorem uses the model theoretic notion of saturated extensions. This is outside the scope of this chapter and we refer the interested reader to look at the definition in the last part of [21, Section 2].

Theorem 11. *[21, Theorem 3.5] Suppose A is a monotone lattice expansion of cardinality κ and α is a completion type for A. Then for any κ^+-saturated non-standard extension A^*, the α-canonical extension of A can be embedded into the α-MacNeille completion of A^* via an embedding that preserves all joins and meets.*

Corollary 4. *[21, Theorem 3.6] Let $\mathscr{K}$ be a universal class (for instance, a variety) of monotone lattice expansions. If $\mathscr{K}$ is closed under α-MacNeille completions, then $\mathscr{K}$ is closed under α-canonical extensions.*

These results are stated and proved for classes of monotone lattice expansions, but this is mainly in order for various results on class operators as discussed above to be readily available. We expect, that this hypothesis may be weakened along the lines of subsequent results in [31].

In the study of substructural logics, as the connection from logic and proof theory involves both algebra and order, it is often most natural to work with other Δ_1-completions than the canonical extension [22, 49]. Accordingly it is an interesting question whether the above results hold for these intermediate Δ_1-completions as well as for the minimal one which is the MacNeille completion.

5 Connections with topological duality

The idea of Jónsson and Tarski in introducing canonical extension was to enable an *algebraic treatment of duality while still allowing access to the underlying relational frames*. This philosophy allows one to work with the underlying frame while keeping the topological information in the form of the canonical extension and is thus an alternative approach to the dualities, which is especially useful in dealing with additional operations on lattices and Boolean algebras. Here, we explain this algebraic approach to duality theory advocated by Jónsson and Tarski in [44, 45] which is based on a combination of canonical extension and 'discrete' duality for the extensions and show how it is also available for general lattices with additional operations. This leads to Jónsson-Tarski style representation theorems, which encode the information of the dualities in algebraic form. We spell this out in particular for residuated lattices.

Historically Jónsson and Tarski obtained canonical extension as an algebraic description of Stone duality. However, in retrospect, canonical extension can be obtained directly and in a choice-free manner as the unique compact and dense completion, which may be constructed e.g. as the Galois closed sets of the polarity obtained from the filters and ideals of the lattice equipped with the relation of non-empty intersection, see Theorem 6. If one chooses this point of view, where canonical extension comes first, then the duality can be obtained from it by adding points (via Stone's Prime Filter Theorem). This ability to derive the dualities from the canonical extension goes through unscathed to the setting of lattices and we explain how it works. However, topological duality theory for lattices is a fairly involved topic as one has to deal with several topologies and their interactions with a Galois connection. Further, the topological spaces involved are not very well behaved (e.g. not sober nor is their sobrification spectral in general [17, Examples 4.1 and 4.2]) and the duals of lattice homomorphisms are not functions but relations, which may be treated in a number of different ways. Accordingly, a full treatment of duality for lattices is beyond the scope of this chapter. The paper [17] also treats topological duality for lattices from the point of view of canonical extension, and [16] has a component, albeit mainly focussed on Heyting algebras, on this subject and we refer the reader to these two papers for a complementary treatment.

Part of the text below describing the connection to the topological dualities comes from an earlier, more detailed, version of [17] and I would like to acknowledge Sam van Gool's contribution in writing this text.

Algebraic duality theory

The schema of the algebraic approach to duality theory as introduced by Jónsson and Tarski in the papers [44, 45] goes as follows:

- Every Boolean algebra B has a unique canonical extension B^δ;
- Each n-ary normal operator f on B extends to an complete operator f^σ on B^δ;
- B^δ is isomorphic to the powerset of a set X;
- f^σ, viewed as an operation on $\mathscr{P}(X)$, is given as (inverse) relational image under an $n+1$-ary relation R on X.

That is, given a Boolean algebra equipped with a normal operator, (B, f), the canonical extension in which we choose the σ-extension of the operator yields an algebra which is, up to isomorphism, the *complex algebra* of a first-order structure (X,R) where $R \subseteq X \times X^n$. That is,

$$(B^\delta, f^\sigma) \cong (\mathscr{P}(X), R^{-1}[\,]).$$

The relation to duality comes from the fact that the set X may in fact be taken to be the set underlying the dual space of B and R is the so-called Kripke relation dual to f (except that Jónsson and Tarski showed its existence and role some 10 years before the work of Kripke).

The structure (X,R) thus obtained from (B,f) yields a representation theorem:

Theorem 12 (Jónsson-Tarski Representation Theorem for BAOs). *[45] Any Boolean algebra with operators embeds in the complex algebra of a relational structure*

$$(B,(f_i)_{i\in I}) \hookrightarrow (\mathscr{P}(X),(R_i^{-1}[\,])_{i\in I}).$$

Moreover, the structure $(X,(R_i)_{i\in I})$ is determined up to isomorphism by the fact that the above embedding is compact and dense and that $R_i^{-1}[\,]$ extends f_i in the manner of σ-extensions.

This is in fact more than a representation theorem and may be viewed as an algebraic form of duality: Instead of the dual being (X,R) where X is equipped with a topology generated by the image of the above embedding, we just 'equip' it with the above embedding itself. Then the usual topological requirements of compactness and zero dimensionality of the resulting space are replaced by the requirements of density and compactness of the embedding.

The theorem above, as well as the original papers of Jónsson and Tarski talk just about *operators*, that is, about join preservation in each coordinate. However, there is nothing primordial about this. By order duality, we obtain an order dual theorem for dual operators and we can also accommodate order reversal in various coordinates. We will say a bit about that below. In particular dual operators will require the use of π-extensions rather than σ-extensions in order to obtain extensions that fall under a discrete duality.

The remarkable fact is that, even though duality for lattice expansions is much more complicated than in the distributive setting, Jónsson-Tarski style representation theorems and thereby their algebraic formulation of duality theory go through without additional complications in the setting of general bounded lattice expansions.

The first bullet point in the Jónsson-Tarski schema goes through by Theorem 6:

Fact 1 *Every bounded lattice L has a unique dense and compact completion*

$$L \longrightarrow L^\delta.$$

The second bullet point goes through by Proposition 4.2:

Fact 2 *Each n-ary normal operator f on L extends to an n-ary complete normal operator f^σ on L^δ.*

The last two bullet points of the Jónsson-Tarski schema depend on the fact that when one takes the canonical extension of a Boolean algebra one obtains a very special Boolean algebra, namely a *complete and atomic Boolean algebra* (sometimes known as a CABA) and the extensions of morphisms are complete morphisms (by Proposition 4.3). The third bullet item relies on the fact that CABAs form a category dual to the category SET of sets and functions, and the fourth on the fact that under this duality, n-ary complete normal operators correspond to $n+1$-ary relations on the dual set.

In order to have this work in the general bounded lattice setting we need to discover what is special about canonical extensions of lattices and see whether there is an analogue of the duality between CABAs and SETs at work here as well. Generalising not only the first two bullet points of the Jónsson-Tarski approach but also the last two was first written down and worked out in [9]. Further details in this context of the discrete duality generalising the CABA-SET duality and its relation to work on relational semantics in modal logic may be found in [15]. We refer to these two papers for further details on the material exposed here.

We will need a few definitions. An element x of a complete lattice C is said to be *completely join irreducible* provided for all $S \subseteq C$ we have

$$x \leq \bigvee S \implies \exists s \in S \quad x \leq s.$$

We denote the set of all completely join irreducible elements of a complete lattice C by $J^\infty(C)$. Order dually one may define the completely meet irreducible elements of a complete lattice C, the set of which we denote by $M^\infty(C)$. The following theorem, which requires a non-constructive principle for its proof, is the incarnation of Stone's Prime Filter Theorem in this setting.

Theorem 13. *[20, Lemma 3.4] Let L be a bounded lattice and $u, v \in L^\delta$ with $u \nleq v$. Then there exist $x \in J^\infty(L^\delta)$ and $y \in M^\infty(L^\delta)$ with*

$$x \leq u \quad \text{and} \quad v \leq y \quad \text{and} \quad x \nleq y.$$

Remark 3. Let L be a distributive lattice. Then Stone's Prime Filter Theorem states that if F is a filter of L and I is an ideal of L and $F \cap I = \emptyset$, then there exists a prime filter F' of L with $F \subseteq F'$ and $I \cap F' = \emptyset$. We show that this is a special case of Theorem 13. Let F be a filter and I an ideal of a lattice L with $F \cap I = \emptyset$. Then $x = \bigwedge F \in F(L^\delta)$ and $y = \bigvee I \in I(L^\delta)$. Also, by compactness, $F \cap I = \emptyset$ implies that $x \nleq y$. Thus, by Theorem 13, there is $x' \in J^\infty(L^\delta)$ and $y' \in M^\infty(L^\delta)$ with $x' \leq x$ and $y \leq y'$ and $x' \nleq y'$. It is not hard to prove that if $x' \in J^\infty(L^\delta)$ and L is distributive, then $F' = {\uparrow}x' \cap L$ is a prime filter of L (see also Theorem 16 below). Finally, $x' \leq x$ implies $F \subseteq F'$ and $x' \nleq y$ implies $I \cap F' = \emptyset$.

Theorem 13 implies that the canonical extension of any lattice is a perfect lattice in the following sense:

Definition 5. Let C be a complete lattice. We call C a *perfect lattice* provided it is join generated by its completely join irreducible elements and meet generated by its completely meet irreducible elements. We denote the category of perfect lattices with complete lattice homomorphisms by L^{+}.

The category of perfect lattice is the appropriate generalisation of the category of CABAs in the schema of Jónsson and Tarski. Indeed, one may show that a perfect lattice is Boolean if and only if it is a CABA [15, Theorem 2.14]. The functors that provide the duality between CABA and SET are the powerset functor and the 'atom' functor. The generalisation of the powerset construction is obtained from Theorem 3: Any complete lattice C may be represented as $\mathscr{G}(X,Y,R)$ for various choices of (X,Y,R). In fact, it follows from Theorem 3 that for any $\Xi\colon X\to C$ whose image is join dense in C and any $\Upsilon\colon Y\to C$ whose image is meet dense in C, if we define R by xRy if and only if $\Xi(x)\le\Upsilon(y)$, then $C\cong\mathscr{G}(X,Y,R)$. This yields concrete representations of C but it is not a duality as there are many choices of Ξ and Υ.

However, for C a perfect lattice, there is a 'minimum' canonical choice: A subset of C join generates C if and only if it contains $J^\infty(C)$ and a subset of C meet generates C if and only if it contains $M^\infty(C)$. Also a natural choice for Ξ and Υ is to choose them injective (that is, one 'name' per join/meet generator). The latter of these two requirements corresponds to restricting our attention to *separated* polarities, that is, polarities (X,Y,R) so that

$$\forall x_1,x_2\in X\ \ (x_1\neq x_2\implies \{y\mid x_1Ry\}\neq\{y\mid x_2Ry\})$$

and

$$\forall y_1,y_2\in Y\ \ (y_1\neq y_2\implies \{x\mid xRy_1\}\neq\{x\mid xRy_2\}).$$

The former requirement, that is, that each element of X will be mapped to a completely join irreducible element in $\mathscr{G}(X,Y,R)$ and each element of Y will be mapped to a completely meet irreducible element in $\mathscr{G}(X,Y,R)$ corresponds to restricting our attention to *reduced* polarities, that is, polarities (X,Y,R) so that

$$\forall x\in X\,\exists y\in Y\ \big[(x,y)\notin R \text{ and } \forall x'\in X\ \big(x'<x\implies x'Ry\big)\big]$$

and

$$\forall y\in Y\,\exists x\in X\ \big[(x,y)\notin R \text{ and } \forall y'\in Y\ \big(y<y'\implies xRy'\big)\big]$$

where the quasi-orders, for which $<$ is a short-hand for '$\le$ but not $=$', on X and Y are induced by R via

$$x_1\le x_2 \text{ if and only if } \{y\mid x_1Ry\}\supseteq\{y\mid x_2Ry\}$$

and

$$y_1\le y_2 \text{ if and only if } \{x\mid xRy_1\}\subseteq\{x\mid xRy_2\},$$

respectively (note that (X,Y,R) is separated if and only if these quasi-orders are partial orders). Thus the objects of the category dual to the category of perfect lattices consists of those polarities that are separated and reduced, and the generalisation of the powerset functor sends such a polarity to its Galois closed sets. The generalisation of the atom functor sends a perfect lattice C to the polarity $(J^\infty(C), M^\infty(C), \leq)$ where $\leq$ stands for the binary relation from $J^\infty(C)$ to $M^\infty(C)$ obtained by restricting the partial order of the lattice C. The dual of a complete lattice homomorphism is a pair of relations. For more details, see [15, Definition 3.36].

Now, the third bullet point in the schema of Jónsson and Tarski becomes:

Fact 3 *L^δ is isomorphic to $\mathscr{G}(X,Y,R)$ for a reduced and separated polarity. As in the Jónsson-Tarski representation theorem, (X,Y,R) is unique up to isomorphism.*

Remark 4. If L is a Boolean algebra then L^δ is a CABA, X is its atoms and Y is its co-atoms, and there is a bijection $\kappa\colon X \to Y$ so that xRy if and only if $y \neq \kappa(x)$ and then $\mathscr{G}(X,Y,R) \cong \mathscr{P}(X)$ (in fact, κ sends any atom to its complement which is a co-atom). If L is just distributive then there also is a bijection $\kappa\colon X \to Y$, but in this case we have xRy if and only if $y \nleq \kappa(x)$ and then $\mathscr{G}(X,Y,R) \cong \mathscr{D}(X,\leq)$, where the order on X is the one given above in the context of the definition of reduced polarities.

For the fourth bullet point, we need to understand the duals of complete normal operators under the duality between perfect lattices and separated and reduced polarities. Using Proposition 4, we see that if $f : L^n \to L$ is a normal operator, then, $f^\sigma : (L^\delta)^n \to L^\delta$ is a complete normal operator. As a consequence one readily sees that one can recover f^σ, and thus f, from its restriction to tuples with coordinates in $X_L = J^\infty(L^\delta)$. This motivates the following definition which is also responsible for the appearance of Kripke relations in the setting of modal algebras, and which, in the restricted setting of boolean algebras with operators, dates back to the second part of the seminal work of Jónsson and Tarski [45].

Definition 6. Given an n-ary normal operator $f : L^n \to L$ on a lattice, the relation dual to f is given by

$$R^j_f = \{(x,\bar{x}) \in X_L \times (X_L)^n \mid x \leq f^\sigma(\bar{x})\}.$$

Notice that in this definition it is not important that $x \in X_L = J^\infty(L^\delta)$. We may as well have used completely meet irreducible elements to describe the value of $f^\sigma(\bar{x})$ leading to an alternative choice of dual relation

$$R^m_f = \{(y,\bar{x}) \in Y_L \times (X_L)^n \mid y \geq f^\sigma(\bar{x})\}.$$

However, what is primordial in order to be able to recover an operation whose extension preserves arbitrary joins in each coordinate is that the $\bar{x}$'s include all tuples of completely join irreducible elements as we need to know the value of f^σ at these to be able to reconstruct f. Indeed, given a polarity (X,Y,R) and an additional relation $R' \subseteq Y \times X^n$, define

$$\begin{aligned}\mathscr{G}(R')\colon \mathscr{G}(X,Y,R)^n &\to \mathscr{G}(X,Y,R)\\ (A_1,\ldots,A_n) &\mapsto {}^R(\{y\in Y \mid \exists \overline{x}\in A_1\times\ldots\times A_n \ \ yR'\overline{x}\}).\end{aligned}$$

Then it is not hard to see that the isomorphism $L^\delta \cong \mathscr{G}(X_L,Y_L,R_L)$ carries the normal operator f^σ to the operation $\mathscr{G}(R^m_f)$, see [9, Proposition 5.6 and 5.7] for a proof of this discrete duality fact in the setting of binary operators. This is of course just the starting point of the treatment of maps in this duality, as one must characterise which relations arise as duals. The interested reader may find the required compatibility property in the setting of the R^m relations in [15, Definition 3.35]. The fact that it is dual is proved in the unary case in [15, Proposition 3.33]. Operations that preserve meets in each coordinate are of course treated order dually, swapping the use of the elements of the two parts X and Y of the dual structure. Similarly, one can use order duality to deal with operations such as implication that sends joins in the first coordinate as well as meets in the second coordinate to meets.

For homomorphisms, which preserve both join and meet, we get two relations, one from its join preserving nature as described above, and one from its meet preserving nature. These two, along with the properties witnessing that they come from one and the same function is the dual of a homomorphism as defined in [15, Definition 3.36]. If the lattices involved are distributive, then the two relations determine a continuous function. This is not the case for lattice homomorphisms in general, but it is true for some maps between lattices and the duality of [17] based of distributive envelopes identifies the natural setting in which this is the case.

Fact 4 *Let $f\colon L^n \to L$ be a normal operator on a bounded lattice, and (X,Y,R) the discrete dual of L^δ from Fact 3. Viewed as an operation on $\mathscr{G}(X,Y,R)$, the extension f^σ is equal to $\mathscr{G}(R')$ for a uniquely determined relation $R'\subseteq Y\times X^n$.*

As in the Boolean case, we now obtain a Jónsson-Tarski style representation theorem for bounded lattices with operators (LOs) which provides an algebraic version of duality much cleaner than actual topological dualities for such.

Theorem 14 (Representation Theorem for LOs). *[9] Any bounded lattice with operators embeds in the complex algebra of a relational structure*

$$(L,(f_i)_{i\in I}) \hookrightarrow (\mathscr{G}(X,Y,R),(\mathscr{G}(R_i))_{i\in I}).$$

Moreover, the structure $(X,Y,R;(R_i)_{i\in I})$ is determined up to isomorphism by the fact that the above embedding is compact and dense and that $\mathscr{G}(R_i)$ extends f_i in the manner of σ-extensions.

We spell this out in the particular case of bounded residuated lattices.

Theorem 15 (Representation Theorem for Residuated Lattices). *[9] Any bounded residuated lattice embeds in the complex algebra of a relational structure*

$$(L,\cdot,\backslash,/) \hookrightarrow (\mathscr{G}(X,Y,R),f_{R.},g^1_{R.},g^2_{R.}).$$

Moreover, the structure $(X,Y,R;R.)$ is determined up to isomorphism by the fact that the above embedding is compact and dense and that $f_{R.}$ extends $\cdot$ in the manner of σ-extensions.

Here $f_{R.}$ is $\mathscr{G}(R.)$ and $g^1_{R.}$ and $g^2_{R.}$ are obtained via order dual versions of the $\mathscr{G}(R')$ construction described above with a reversed coordinate applied to relations obtained from $R.$ by swapping the order of coordinates (see [9, Proposition 5.8]).

Dual spaces via canonical extension

The goal of Jónsson and Tarski, and later of modal logicians such as Goldblatt [35] and de Rijke and Venema [8], was to use duality in the algebraic form provided by canonical extensions to study additional operations on lattices and relational semantics for these. However, as mentioned in the introduction to this section, once we know that we can show existence (and uniqueness) of canonical extensions without going via the dualities, one may ask whether one can derive the dualities from the canonical extension. This is indeed the case and this fact is the justification for really considering the theory of canonical extensions as an alternative algebraic approach to duality theory. We outline in this process here. That is, suppose we are given the embedding $L \hookrightarrow L^\delta$, then we show how to define the topological dual space of L using just this embedding and the discrete duality between perfect lattices and reduced and separated polarities. We will mainly focus on the duality described by Hartung [41] though we will also say a few words about the relationship to the earlier duality of Urquhart [55].

Recall that the *filter elements* of L^δ are the elements in the meet closure of L in L^δ. These elements of L^δ form a poset isomorphic to the poset $Filt(L)$ of all lattice filters of L ordered by reverse inclusion under the correspondence $x \mapsto \uparrow x \cap L$ from $F(L^\delta)$ to $Filt(L)$ and $F \mapsto \bigwedge F$ from $Filt(L)$ to $F(L^\delta)$. Order dually, the *ideal elements* of L^δ are in one-to-one correspondence with the ideals of L. Now notice that the density property of L^δ implies that $J^\infty(L^\delta) \subseteq F(L^\delta)$ and order dually $M^\infty(L^\delta) \subseteq I(L^\delta)$, the set of ideal elements of L^δ. Thus the completely join and meet irreducibles of L^δ are in bijective correspondence with certain filters and ideals of L, respectively.

Theorem 16. *[20, Lemma 3.4, Proposition 3.5] Let L be a bounded lattice, $x \in F(L^\delta)$ a filter element of L^δ and $F \in Filt(L)$ the corresponding filter of L. Then the following conditions are equivalent:*

(i) $x \in J^\infty(L^\delta)$;

(ii) F is maximal among filters with respect to being disjoint from some ideal of L.

If in addition L is distributive, then these are also equivalent to:

(iii) F is a prime filter.

Of course we have the order dual equivalence between elements of $M^\infty(L^\delta)$ and ideals of L that are maximal with respect to being disjoint from some filter of L, and in the distributive setting these are precisely the prime ideals, in the Boolean setting they are the ultraideals.

Remark 5. The first duality for lattices was developed by Urquhart [55] and the dual space of a bounded lattice is based on the set of all pairs (I,F) consisting of an ideal and a filter of L, each maximal with respect to being disjoint from the other and a key contribution was his proof of a generalisation of the Stone Prime Filter Theorem (as in Theorem 13 above) in terms of these. It is of course not so strange that the canonical extension approach leads to the same points though we arrive at them as completely join/meet irreducibles of L^δ rather than via ideal-filter pairs.

We are now ready to define the topological dual of a lattice L from its canonical extension. In Hartung's incarnation [41], it is what he calls a standard topological context. That is, a polarity (X_L, Y_L, R_L) with certain properties, where X_L and Y_L are not just sets but topological spaces and R_L is a relation between them. Let

$$X_L = J^\infty(L^\delta) \quad \text{and} \quad Y_L = M^\infty(L^\delta).$$

For each $a \in L$ we define $\hat{a} = {\downarrow} a \cap J^\infty(L^\delta)$ and $\check{a} = {\uparrow} a \cap M^\infty(L^\delta)$. Let τ_X^L be the topology on X_L given by taking $\{\hat{a} \colon a \in L\}$ as a subbasis for the *closed sets* and τ_Y^L be the topology on Y_L given by taking $\{\check{a} \colon a \in L\}$ as a subbasis, again, for the *closed sets*. The sets X_L and Y_L and their basic closed sets, as they sit in L^δ, are illustrated in Figure 1. The relation R_L is simply the order of L^δ restricted to $J^\infty(L^\delta) \times M^\infty(L^\delta)$.

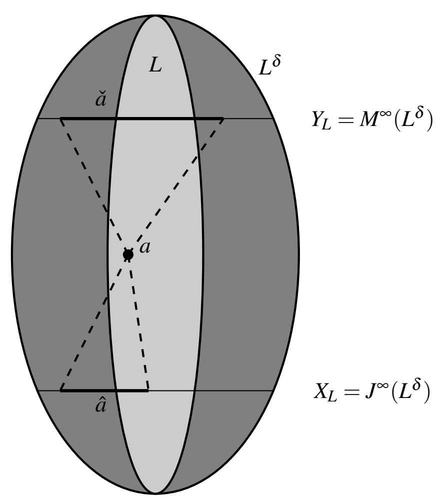


Fig. 1 A lattice L and its dual space, both inside L^δ.

Proposition 11. *[20, Remark 2.10], Let L be a lattice. The topological polarity*

$$((X_L, \tau_X^L), (Y_L, \tau_Y^L), R_L)$$

is isomorphic to Hartung's topological dual of L.

Remark 6. [41, Definition of standard topological context] As mentioned above the characterising properties of these duals are fairly involved. To wit: the closed sets of the two topologies need to be closed under Galois closure; the Galois closed closed sets of both topologies need to form subbases for the closed sets; the elements of X need to correspond bijectively with the completely join irreducible elements of the lattice of Galois closed sets given by the polarity (which will by the way be an isomorphic copy of the canonical extension of the dual lattice); similarly the elements of Y need to correspond bijectively with the completely meet irreducible elements of the lattice of Galois closed sets given by the polarity; whenever $x \in X$ and $y \in Y$ and xRy, then there needs to exist a Galois closed pair (C,D) which are both closed in the respective topologies with $x \in C$ and $y \in D$; finally the complement of R in the product space $X \times Y$ must be compact in the topological sense.

Remark 7. Given a standard topological context, the corresponding lattice consists of the 'triply closed sets', that is, of those Galois closed sets C in X so that C is topologically closed and the corresponding Galois closed set D in Y is also topologically closed in Y.

Given a bounded lattice L it is not difficult to see that the Galois closed sets of (X_L,Y_L,R_L) is an isomorphic copy of L^δ (by Theorem 3) and $a \mapsto \hat{a}$ embeds L in the lattice of triply closed sets of (X_L,Y_L,R_L). To show that this embedding is surjective, one needs a non-trivial argument which is given in the proof of Proposition 2.1.7 of [40] and which uses the Axiom of Choice in the form of Rado's Selection Theorem [38].

Before Hartung, Urquhart [55] already provided a topological duality for lattices (albeit only treating surjective morphisms). In this incarnation, the dual of a lattice is a doubly ordered topological space $(Z_L, \tau_L, \leq_1, \leq_2)$ whose points are filter-ideal pairs (F,I) that are maximal with respect to being disjoint from each other. Using Theorem 16, it is not difficult to see that Z_L corresponds to the points of

$$P = (X_L \times Y_L) - R_L$$

that are maximal with respect to the order given by

$$(x,y) \preceq (x',y') \text{ if and only if } x \geq_{L^\delta} x' \text{ and } y \leq_{L^\delta} y'.$$

The topology τ_L defined by Urquhart is the same as the subspace topology inherited from $X_L \times Y_L$ with the product topology $\tau_X^L \times \tau_Y^L$, and the orders $\leq_1$ and $\leq_2$ are the orders $\geq_{L^\delta}$ and $\leq_{L^\delta}$ applied to the first and second coordinates of the points in Z_L, respectively.

Recently, alternative topological dual structures for lattices have been developed. In [50, 51] Jipsen and Moshier develop a purely topological duality theory (without the use of additional relational structure such as the orders in the Urquhart and Hartung dualities) for lattices (and semilattices) and use it to obtain a characterisation of lattice expansions. The aim in this work is to obtain a purely topological category consisting of topologically nice spaces. Their spaces are as nice as spaces get: sober, even spectral, with the open filters forming a base. They call these Hoffman-Mislove-Stralka (HMS) spaces: they are the spaces that arise as algebraic lattices with the Scott topology [50, Theorem 3.7]. These spaces are characterised among HMS spaces as those in which the open filters form a sublattice of the complete lattice, $\mathrm{FSat}(X)$, generated by closing the lattice of open filters of X by arbitrary intersections.

For a lattice L, its dual space in the Jipsen-Moshier duality is simply the space $Filt(L)$ of filters of L equipped with the Scott topology. Another way to understand this space is as the Stone dual of the free finite join semilattice over the underlying meet semilattice (forgetting its join structure completely). The drawback if any of this approach is that the original lattice actually sits inside its dual space (e.g. for a finite Boolean algebra, the Jipsen-Moshier dual space is based on the algebra itself while the Stone dual is logarithmically smaller). Also, on the topological side, all spaces are HMS spaces, a class of spaces less rich than that of general spectral spaces. A lattice L is recovered from its dual space X as the compact open filters of the space. Interestingly, the canonical extension also has a natural incarnation in this setting being isomorphic to the complete lattice $\mathrm{FSat}(X)$ [50, Theorem 4.1]. In recent work González and Jansana [37] have generalised this approach to obtain a duality for posets. This work is also related to [32] where canonical extension is treated with domain theoretic methods.

Another recently developed duality for lattices [17] takes a similar point of departure: wanting spaces that are nice from a topological point of view. However, a second criteria is that when restricted to distributive lattices one should get the usual dual space of Stone/Priestley duality. This is made possible by an algebraic fact. The inclusion of distributive lattices in the category of lattices (and even of meet-semilattices) with morphisms preserving distributive meets also has a left adjoint, that is, there is a *free distributive envelope* over any (semi)lattice which preserves all existing distributive joins (i.e. ones that do not contradict distributivity). This construction is a finitary version of the classical construction of Bruns and Lakser of the injective hull of a meet semilattice [3], which it turns out is a frame (in the sense of pointfree topology). The advantage of the free distributive envelope construction over the fully free join-semilattice as used by Jipsen and Moshier is that a distributive lattice is its own envelope and there isn't any unnecessary blow-up in size. The choice of keeping meet and extending by joins rather than the other way around is of course arbitrary. In the approach in [17], both the join- and the meet-distributive envelope are used and the original lattice may then be recaptured by a Galois connection between these two distributive lattices. The dual space of a lattice is then taken to be the duals of these two distributive lattices and the relation between them dual to the Galois connection under the usual duality for distributive lattices. This topological polarity extends the one from Hartung's duality but the spaces involved are spectral. An important point of this duality is that it isolates precisely which morphisms between lattices lift to continuous functions (rather than relations as is the case for general homomorphisms between lattices in the Hartung duality).

6 Concluding remarks

Canonical extensions provide an algebraic approach to duality theory which allows formulations and proofs as simple as the ones in the distributive setting. An additional advantage is that canonical extensions is just one of a whole zoo of Δ_1-completions so that one has the opportunity to consider the possibility that, for particular varieties (and corresponding proof theories or more generally logics), other parametric choices of Δ_1-completions may be more suitable. This gives the possibility of broadening the notion and scope of 'duality theory' in this algebraic form considerably. This may be of interest in various settings with binding of variables, but also in various settings coming from proof theory and abstract algebraic logic [22, 49].

Here we have surveyed the results available, mainly for one particular choice of Δ_1-completion, namely the canonical extension. This completion encodes Stone's topological duality and its generalisations to non-distributive lattices. Some results are available also for MacNeille completions [34]. For other potential choices of interest, one would first have to develop the theory and see to which extent the methods and results available for canonical extension and MacNeille completion carry over. This is of course most interesting to do in test cases where open problems have a chance of being settled by such an approach. We hope this chapter may be an inspiration to algebraically minded proof theorists in considering this possibility.

Even in the case of canonical extensions of lattices, a large number of open questions remains such as the topological treatment of extensions of maps and the right categorical setting for understanding canonical extension as a free construction, and even more about the generalisation from lattices to posets, such as identifying the 'right' choice of ideals and filters for the notion of canonical extension. Finally, it would be nice to see examples worked out and applied of how canonical extension can be used for unbounded but otherwise well-behaved structures, e.g. for ℓ-groups. It is clearly possible but the details need to be worked out.

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