DUALITY

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Mijnheer de Rector Magnificus,
Zeer gewaardeerde toehoorders,
Highly esteemed audience,

1. The nature of mathematics

I open with a quote from Sofya Kovaleskaya, the first woman who was appointed to a professorship in mathematics in Europe (this was at Stockholm University in 1889):

> *'You are surprised at my working simultaneously in literature and in mathematics. Many people who have never had occasion to learn what mathematics is confuse it with arithmetic and consider it a dry and arid science. In actual fact it is the science which demands the utmost imagination. One of the foremost mathematicians of our century says very justly that it is impossible to be a mathematician without also being a poet in spirit. It goes without saying that to understand the truth of this statement one must repudiate the old prejudice by which poets are supposed to fabricate what does not exist, and that imagination is the same as 'making things up'. It seems to me that the poet must see what others do not see, and see more deeply than other people. And the mathematician must do the same.'*

My passion, and the subject of my oratie, lies within the field of mathematics. It has been my experience that what mathematics is, is not very well understood by most non-mathematicians. While living in New Mexico, I was once asked by an old cowboy and farmer over a can of Coors Light: "So, what kind of figurin' do you do?". Most people know what 'figuring' or 'rekenen' is, but mathematics or 'wiskunde' is something quite different and before I can attempt to tell you about the mathematical concept of duality, I want to say a few words about mathematics as I see it.

Mathematics is a deeply intellectual and creative endeavour. It might involve some calculation but this is certainly not what it is about. Many mathematicians enter the field through the far end of science, but, to me, mathematics is more of a humanity than a science and I claim that there is another way in than through the sciences. Mathematics is 100% human-made and it draws much of its strength, beauty, and character in general from the fact that it does not talk about anything in particular. The work of a mathematician lies in distilling out the essential features of some aspect of experience, and understanding it as deeply and as fully as possible.

Kovaleskaya drew a parallel between mathematics and poetry. Many have drawn similar parallels between art and mathematics, and I will now walk in their footsteps to try to show you how I see mathematics. Let's start with an artistic phenomenon closely related to nature and to mathematics, namely the appearance of the golden mean, a fundamental constant associated with growth phenomena, in works of artists. In his book, 'The Curves of Life' [3], the Oxford scholar, art critic, and writer Theodore Andrea Cook tries to explain why and how this mathematical constant ends up appearing in the works of artists:

> 'I have described the artist as a man whose brain or temperament, being of more sensitive fibre than the common, is more closely in touch with that world of Nature which he sees and feels more keenly than his fellows. His artistic work is in the strictest sense "creation", for he does not merely copy the natural beauty which he sees around him, he creates a fresh stock of beauty for us all. '

> 'it is, as Goethe elsewhere said, "a revelation working from within; a synthesis of world and mind"; for it is profoundly true that "Art does not exactly imitate that which can be seen by the eyes, but goes back to that element of reason [. . .] of which nature consists and according to which Nature acts." '

So Cook is of the opinion that the artist reaches deep enough down in him- or herself to 'discover', at least on an intuitive plane, the mathematical constants of nature. This view of the work of the artist is interesting to juxtapose with the view of the work of the mathematician given in the famous essay by the Nobel Prize winning physicist and mathematician Eugene Wigner entitled 'The Unreasonable Effectiveness of Mathematics in the Natural Science' [19]. There he argues that only the most basic mathematical concepts are arrived at directly through modelling the world around us, and far the most are obtained by extension via our aesthetic sense and/or their fitness for logical arguments:

> 'the mathematician could formulate only a handful of interesting theorems without defining concepts beyond those contained in the axioms and [. . .] the concepts outside those contained in the axioms are defined with a view of permitting ingenious logical operations which appeal to our aesthetic sense both as operations and also in their results of great generality and simplicity.'

So the mathematician is acting on aesthetic impulse. Why then is mathematics so effective in the sciences? Wigner does not really answer that question but rather poses it as an awe inspiring fact that can make a chill run down our spine from the mystery of it.

In the last century art has challenged its form, content and purpose and classical aesthetics is no longer the main concern. This has moved the focus of art much further in the conceptual direction as expressed, in a particularly mathematical form, by British novelist and philosopher Ayn Rand [13, pp. 19–20]and [14, p.10]:

'By a selective recreation, art isolates and integrates those aspects of reality which represent man's fundamental view of himself and existence. Out of the countless numbers of concretes – of single, disorganized and (seemingly) contradictory attributes, actions, and entities – an artist isolates the things which he regards as metaphysically essential and integrates them into a single new concrete that represents an embodied abstraction.'

'A concept is a mental integration of two or more units which are isolated according to a specific characteristic(s) and united by a specific definition.... The act of isolation involved is a process of abstraction: i.e., a selective mental focus that takes out or separates a certain aspect of reality from all others.... The uniting involved is not a mere sum, but an integration, i.e., a blending of the units into a single, new mental entity which is used thereafter as a single unit of thought.'

After the advent of Duchamp's readymades and Warhol's Brillo box, it has become very hard indeed to say what art is. This has lead Canadian philosopher David Davies [4, p.] to conclude that we should

'[give] up the idea that the work [of art] is the product *of the creative process and [say] rather, that the work - what the artist achieves - is the* process *eventuating in that product .'*

This is not a very practical definition of art in many cases, but it does square well with the practice in mathematics that the *proof*, a series of logical steps remapping the creative act of arriving at a mathematical theorem, rather than just the result, is considered to be the mathematics.

One important difference between art and mathematics is that the expression remains more intuitive in art and creative work is encoded in highly subjective and personal forms. The reason it is still of wide interest is of course because deep personal experience often is quite universal and because the avoidance of a symbolic language allows the expression to be more direct and thus does not require realisations to become fully conscious. Mathematics chooses the opposite path, opting for a language far removed from the complexities of everyday in order to allow a language with complete precision. This has its definite drawbacks for communication: it limits what you can say and especially *who* you can say it to, but it also has some advantages for communication: Among mathematicians, the content and nature of a piece of mathematics can be very precisely communicated. As an 18 years old female student of mathematics, I had the experience of stopping by the office of a much older male colleague of my adviser coming from a culture I have little understanding of while on summer holiday, and, within minutes, we could share deep and intricate aspects of a theorem and the beauty of its proof.

Before going on to my subject proper I'd like to talk to you about the similarities of the working process of mathematicians and artists. In an article, 'On the artist and the

creative process', published in the Journal of the American Academy of Psychoanalysis [18], the stages of the creative process are described:

> *'Creativity is experienced as 'giving birth' to new life where none had existed - and most artists experience sudden ecstasy, a 'eureka' sensation, when the phase of creative synthesis is completed. This glorious feeling is sometimes also frightening; it can arouse a fear of exposure, a fear that the work is not good enough to be acceptable. The more original and significant it is, the more the artist is open to scrutiny. A true artist must be brave enough to express his or her inner self. In contrast, repression is detrimental because it can cause the artist to censor valid emotionally laden ideas and intuition, and block insights. [...]*
> *The first phase is that of immersion, the second is the gestation and incubation phase, then a phase of integration takes place, all from the unconscious. A sudden illumination phase occurs when the artist's conscious insight into what is created is revealed, accompanied by feelings of exaltation. The creative artist must have access to the right brain in order to evolve creative ideas, as well as to the left brain in order to store and retrieve personal memories and organize perceptions into gestalten. The artist also needs to be able to slip back and forth easily from one mode of thinking to the other. Much of the anxiety artists may experience derives from discomfort or temporary inability to make this transition without interference from irrelevant or tangential concerns.'*

This description is focused on artists but the anatomy of creativity is apparently quite universal, and even on literary blogs [1] the first modern writings on this topic are attributed to the great French polymath Henri Poincaré [11]. He describes his working schedule as consisting of two two-hour sessions, morning and afternoon, along with mathematical reading in the evening. This is the immersion stage. Poincaré gives several examples of how deep results came to him, fully fledged, after prolonged periods of being taken away from his work. Thus he says:

> *'Arrivés à Coutances, nous montâmes dans un omnibus pour je ne sais quelle promenade; au moment où je mettais le pied sur le marche-pied, l'idée me vint, sans que rien de mes pensées antérieures parut m'y avoir préparé, que les transformations dont j'avais fait usage pour définir les fonctions fuchsiennes sont identiques à celles de la Géométrie non-euclidienne. Je ne fis pas la vérification ; je nen aurais pas eu le temps, puisque, à peine assis dans l'omnibus, je repris la conversation commencée, mais j'eus tout de suite une entière certitude. De retour à Caen, je vérifiai le résultat à tête reposée pour l'acquit de ma conscience.'*

2. The nature of duality

At first glance there are two very different aspects of our reality:

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[1]http://www.onfiction.ca/2008/12/metaphor-inner-voice-and-muse-in.html

mental	physical
verbal	visual
algebraic	geometric

They are however very tightly connected to each other. For example, in education, we focus on the fact that there are visual learners and verbal learners. The point of this is not that we want to teach different things to students of these two categories but rather that there are two different ways of approaching and expressing a single subject matter and that we should provide roads to solutions of both kinds. When we want to solve physical problems in space, such as building a bridge or sending a rocket to the moon — what do we do? We make algebraic calculations to learn what we must do.

The point is that there is really only one thing with two expressions of it. This brings me to 'duality' as a central concept in many Eastern religions. In January last year I gave a course at the Indian Winter School in Logic and went on an excursion to Varanasi and Sarnath, the birthplace of Buddhism. Upon entering the amazing Archaeological Museum at Sarnath, our guide opened with:

> *'Duality underlies the world.'*

This is the kind of sweeping statement that every mathematician, at least secretly, would like to believe about their particular focus, so I liked his statement very much and thought I'd use it for my oratie. However, in most Buddhist writings there is talk not of duality but of *non*-duality. The point being that there are not really two separate parts but that they are deeply connected. This is clearly expressed by such statements in Buddhist teaching as

> *'In seeing, there is just seeing. No seer and nothing seen. In hearing, there is just hearing. No hearer and nothing heard.'* (Bahiya Sutta, Udana 1.10)

This is referred to as non-duality in explanations in English of Buddhism, but in relation to what we mean in mathematics it should be called duality: namely that the two aspects are not separate and unrelated but different aspects of the same thing. Our Buddhist guide in Sarnath did not use the English language in the conventional way and actually meant by duality what mathematicians mean by duality, rather than the separateness and independence otherwise implied by the term. Thus, you may take Buddhist non-duality as an example from outside mathematics that is very similar to what we call duality in mathematics.

The tight connection is not all that makes up the nature of duality — at least in the mathematical sense of the word. If we consider a thing and then the thing itself again, then we wouldn't call it duality but rather equality. In duality there is a sense that, while the things, by some magic, are tightly connected, they are seemingly very different and even somehow opposite. Thus dualities in mathematics are often concerned with relating opposites or turning things around. While examples of duality of one type or another

pervade mathematics and go far back in history, a very general and abstract essence of such relationships was distilled in the twentieth century in category theory. Category theory is a mathematical framework where transformations between mathematical objects are treated as equally important to the objects themselves. Thus a category of algebras consist not only of certain algebras. The idea is that in order to truly identify the nature of the objects, one must also specify what the transformations between them are. Now a duality between a category of algebras and a category of geometrical spaces consists in a correspondence between spaces and algebras and, for each pair of algebras or spaces, a one-to-one correspondence between the transformations between the two algebras in one direction and between the corresponding spaces *in the opposite direction.*

The changing of direction of the transformations is what captures in mathematical terms the difference between equality and duality. This is also what accounts for the power of duality. In total abstraction, whether transformations go one way or the other is almost just a matter of choice and then duality is not much other than equality, but if we somehow have a grounded intuition about the framework then changing directions makes an enormous difference. If you think of a transformation as a chemical reaction, then there are reactions that are reversible but there also are many that aren't. This is the same with transformations. Now the availability of a duality tells us that we can translate a problem about irreversible transformations in algebra or space to a problem in the dual world (of spaces or algebras, respectively) where the direction of the transformations are reversed. This does not make the original transformations reversible, it just translates the original problem into another problem in a completely different setting. I will try to explain this with some examples/thought experiments.

Let's consider a language, which is something a bit like algebra, and dual to it the interpretation of the language in the world. Let us now transform our language. Suppose in our language we had the element 'horses' and the element 'Friesian horses'. We transform our language to a new one by equating these two elements. That is, we impose that 'all horses are Friesian'. This transforms our original language where 'horses' and 'Friesian horses' were distinct elements to a new language where these two are equal. But 'all horses are Friesian' does not hold in general. So, if we want a new world or space that corresponds to our new language where 'horses' and 'Friesian horses' have been equated, we must remove all the parts of the old world where 'all horses are Friesian' is violated. In other words, the transformation of *equating* two elements of the language has the effect of *removing* part of the space. Transformations that are functions allow us to equate, this is in fact irreversible as a function; two distinct things have become one, and in trying to make an assignment of the new thing backwards, we would have to give two values and this is not a function assignment. So our act of equating is a functional transformation from the old algebra to the new algebra but not in the other direction. On the spatial side, what happens? We delete or remove points. This is not a transformation that is expressible as a function as all the deleted points don't have an assignment in the new world. However, from the new world to the old world with more points, this is an inclusion, i.e. each point in the new world can just be assigned to itself and this is a transformation that is expressible by a function. So on the language or algebra side our transformation is described by a function that equates in the forward direction, and on the space side by a function that includes in

the *backwards* direction.

Let us try a second thought experiment in the same setting but with a different transformation. Suppose we delete part of our language or algebra. This is a functional transformation from the new language to the old, the new language being included in the old. What effect does that have on the spatial interpretation of this language? A bunch of the points in our old world can no longer be distinguished from each other as our new language is too poor to tell those points apart. That is, some points of the old space have been equated in our new world. In this second thought experiment, the transformation from the new language back to the old is the one that is describable by a function, the inclusion function. On the spatial side however, the transformation is a function in the forward direction as points of the old world get glued together in the new.

Both these examples attempt to illustrate the nature of duality by showing how the direction of the simple transformations of equating and including are dual to each other but work in opposing directions. These examples deal only with these simple cases, but, for those of you who know some undergraduate algebra, the fundamental 'Isomorphism Theorem' tell us that these two cases in combination actually are sufficient to account for all possible algebraic transformations that can be described by functions.

3. Categorical dualities in mathematics

I will now give a selective view of some categorical dualities in mathematics. This part will require a little more familiarity with mathematics than what I have said so far, but I will still try to keep it as accessible as possible.

Let me start with a very classical subject: algebraic geometry. As the name already indicates, it does involve both algebra and geometry. Fundamentally, it is built around a relationship of the type I described above where an algebraic setting speaks about a space and the space speaks about the algebra. Let k be a number field, something like the real numbers or preferably the complex numbers. The basic idea of space here is a Cartesian coordinate system: n-tuples of numbers that we can think of as describing points in an n-dimensional space. The algebra or language is the ring of polynomials in n variables, $k[x_1, \ldots, x_n]$. Something like $y - x^2$ or $x^2 + y^2$ or xy or even just 3 which is a polynomial with no variables in it. These speak about the space, in the sense that at each point they have a value. We can now equate two of these and it gives us a subset of the space: for example $y - x^2 = 3$, which is the same as $y = x^2 + 3$. This is the parabola $y = x^2 + 3$ in the Cartesian plane. This is an algebraic curve, a conic section as you have probably learned about in school. In general these are called *algebraic sets*. I did not stress it above, but we can also think of the space as talking about the algebra: Once we have a subset of the space, such as a parabola or a circle, we see that relative to this subset many polynomials will have the same effect: For example, the polynomials xy and x^3 will be equal for every point on the parabola $y = x^2$ since xy is the same as xx^2 which is

x^3. This yields what is called a quotient ring: it is obtained from the polynomial ring by equating elements according to their action on the restricted space. This gives a duality between algebraic sets, which are certain kinds of spaces, and certain quotient rings of the polynomial ring. Hilbert's famous Nullstellensatz states that these rings are exactly the (n-generated) reduced rings (corresponding to radical ideals). I am leaving out some very important details: what are the correct notions of morphism or transformation on either side of the duality, but even though the whole power lies in this, it is unfortunately too technical for this talk.

Using algebra to speak about space is as old as algebra, just about, but, if we are given an algebra that is not born with an associated space, it has not been obvious to mathematicians that one should go look for a space or that having one would be an advantage for algebra. This is the greatest insight of Marshall Stone, my foremost mathematical hero. He said ([17] and [10, p.xiv]:

> 'A cardinal principle of modern mathematical research may be stated as a maxim: "One must always topologize"'.

Here 'topologize' means introduce a topology, and topology is a very general mathematical framework for spaces. It makes a set into a space-like object by defining on it a notion of neighborhood (the idea is that in space points are too hard to locate, only neighborhoods are observable). Stone worked in functional analysis, an area of mathematics with many dualities and a close interaction between algebra and topology. He realised that, for some kinds of algebras, having only an algebra, you can actually manufacture a space: points, neighborhoods and all that will yield an algebra-space duality. We will speak directly about Stone's duality shortly, but remaining on the topic of algebraic geometry for a minute, let us see how this fits in Stone's set-up.

The algebraic sets, which are the spaces, are equipped with a topology à la Stone: the neighborhoods in an algebraic set are the complements of its algebraic subsets. This is known as the Zarisky topology [20] and is central in algebraic geometry. But even more interesting, while the topology is right, looking at the duality in algebraic geometry with the knowledge of Stone's ideas, one realises that while Zariski's neighborhoods are the right ones, the *points* of the space, namely the tuples in the Cartesian space, are *not* the right points, or more precisely, some are missing. In order for the space to be the intrinsic space associated with the topology it has to be made what was later called *sober*, that is, certain points have to be added. One needs to have a point, not just for each coordinate tuple in the Cartesian space, but a point for each irreducible algebraic set. An algebraic set is irreducible if it cannot be written as the union of two strictly smaller algebraic sets, that is, it acts like a single point. Adding these points one obtains what is known as the spectrum of the dual ring (first introduced by Jacobson in 1945 [9]). This idea was independently introduced into algebraic geometry by Grothendieck [7] who, in addition realised that this had revolutionary consequences: once you have the right set of points, this allows you to represent the ring as the global sections of a sheaf over the spectrum. A sheaf representation is something like a representation by continuously varying functions over a space, except that the function values fall in different structures at each point.

Getting the setting right, finding the intrinsic content of a piece of mathematics, often allows one to generalise it greatly. The generalised version may then suddenly apply to other things that mathematicians have already been working on and create surprising and fruitful connections. Thus getting the setting right in algebraic geometry made connections to many other areas, most spectacularly maybe, to algebraic number theory where topo-geometric methods contributed to the recent proof of Fermat's Last Theorem. [2]

Centrally related to Stone's insight is the representation of algebras as algebras of continuous functions on a structure that is simultaneously an algebra and a topological space. This idea in functional analysis predates Stone, e.g., in the work of Pontryagin [12], but found a culmination later in the Gelfand-Naimark duality [6] which provides a duality between compact Hausdorff spaces and certain topological algebras, called commutative unital C^*-algebras. This duality may be seen as a far reaching generalisation of such central engineering tools as Fourier transform, and it is the basis of so called non-commutative geometry. The latter is possibly the ultimate in reliance on duality: In physics, classical mechanics is described on phase space, which is generally a (locally) compact Hausdorff space. By the Gelfand-Naimark duality the theory of these phase spaces can be translated to the theory of the corresponding commutative C^*-algebras. You may or you may not want to do that, but once you move to quantum mechanics there is *no phase space* any more, no particle with a trajectory. There is, however, still a C^*-algebra, a non-commutative one. The audacity then, is simply, in the absence of spaces, and in the absence even of a duality, to say that the 'geometry', the phase space, 'is' the formal contravariant dual of the algebra.

Now I would like, finally, to focus on Stone's duality specifically. It was arrived at through functional analysis and precursors within Stone's work dealt with issues of representation of algebras of functions and compactifications of spaces. Nevertheless, the actual Stone duality is for Boolean algebras [15] and more generally for distributive lattices [16]. These are topics to come out of logic, not geometry and physics. In 1854 George Boole published his work entitled *An Investigation of the Laws of Thought* [2] in which he argued that logic is part of mathematics rather than philosophy, and that in fact, the laws of thought yield a kind of algebraic system much like the arithmetic of numbers. His set-up is similar to the one I used above to explain the basic nature of duality. He takes fragments of language to describe classes or sets of individuals. Thus he argues [2, p.20] that if x stands for 'white things', and y for 'sheep', then xy stands for 'white sheep'. He then argues that one may consider this as a kind of multiplication which obeys certain arithmetic laws, for example,

$$xy = yx \qquad\qquad x^2 = x.$$

Boole's main point was that one can treat logic as an algebraic system, however, he introduced his algebra via a spatial counterpart though this was not his focus. Stone duality, on the other hand, focuses on this relationship and provides an actual duality. In particular, it shows that every Boolean algebra is representable as a collection of neighbourhoods, or classes, over some space. For Boolean algebras, Stone's theorem has a clear meaning: It says that the axioms of Boolean algebras exactly capture the algebra of 'classes'. For logic in general, it has done much more as Stone duality is the common denominator of most

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[2]A conjecture that remained unproved for more than 350 years.

dualities between syntax and semantics for logical frameworks.

What are syntax and semantics? To describe a logic we first identify some kind of language of symbols or signs to encode the logic in, and then, in order to specify the logic, we have two fundamental approaches: through deductive rules and axioms or through meaning or interpretation of the language. Giving deductive rules and axioms is algebraic in nature whereas specifying what the language means in some model of the world is generally spatial in nature. Since a fully satisfactory logic should tell us how to make deductions *and* what meaning it carries, having algebra-geometry type dualities is a central issue for logic. Note that in algebraic geometry or non-commutative geometry, as discussed above, dualities play a central role simply because they allow us to get at the information that we want. Here, in the logic setting, the duality of syntax and semantics is fundamental to the nature of logic in itself.

In mathematical logic viewed as a foundation of mathematics there are just a few logics, namely classical logic, based on Boole's laws, and a few variants of the intuitionistic logic developed by the famous Dutch mathematician Luitzen Egbertus Jan Brouwer, and the focus is on a metamathematical study of mathematics in the form of such topics as set theory, model theory, or proof theory. The advent of computer science has shifted this focus somewhat and has intensified and inspired the development of logic. I quote from the abstract of a recent talk by the eminent theoretical computer scientist, Moshe Vardi, entitled 'And Logic Begat Computer Science: When Giants Roamed the Earth': [3]

> *'During the past fifty years there has been extensive, continuous, and growing interaction between logic and computer science. In fact, logic has been called "the calculus of computer science". The argument is that logic plays a fundamental role in computer science, similar to that played by calculus in the physical sciences and traditional engineering disciplines. Indeed, logic plays an important role in areas of computer science as disparate as architecture (logic gates), software engineering (specification and verification), programming languages (semantics, logic programming), databases (relational algebra and SQL), artificial intelligence (automated theorem proving), algorithms (complexity and expressiveness), and theory of computation (general notions of computability). [We] provide an overview of the unusual effectiveness of logic in computer science by surveying the history of logic in computer science, going back all the way to Aristotle and Euclid, and [by] showing how logic actually gave rise to computer science.'*

Now you might want to hear Vardi's talk rather than the rest of mine! Then I can highly recommend the article written by Vardi and four other computer scientists entitled 'On the unusual effectiveness of logic in computer science'[8].

For the 'applied logic' topics that have emerged from interaction with computer science, duality theory is often central. This goes for example for the many extensions of classical

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[3]http://www.newton.ac.uk/programmes/LAA/seminars/060611001.html

Boolean logic by so-called modalities. These are logical mode modifiers that may model temporal aspects, distributed agent knowledge, and many other things. The logics are often specified algebraically but problems are resolved dually using Kripke semantics. Another example comes from the theory of computation, where classes of finite state automata and their corresponding formal languages have been studied using semigroup invariants. Interestingly, in this situation, the semigroups - which one would think were algebras - are actually the spaces! (Recent work of mine with Jean-Eric Pin and Serge Grigorieff [5] identified this theory as a case of extended Stone duality). Another role of duality theory in the fast developing foundations of computer science is to realise connections between otherwise separately evolving subfields. Samson Abramsky's celebrated work 'Domains in logical form' [1] is possibly the most spectacular example of this kind. Quoting from the introduction to Abramsky's paper:

> *The mathematical framework of Stone duality is used to synthesize a number of hitherto separate developments in Theoretical Computer Science:*
> - *Domain Theory, the mathematical theory of computation introduced by Scott as a foundation for denotational semantics.*
> - *The theory of concurrency and systems behaviour developed by Milner, Hennessy et al. based on operational semantics.*
> - *Logics of programs.*
>
> *Stone duality provides a junction between semantics (spaces of points = denotations of computational processes) and logics (lattices of properties of processes).'*

The interaction of duality theory with the foundations of computer science has influenced the mathematical field in at least two ways in my opinion. One point of influence comes from the 'modeling perspective' inherent in any kind of applied mathematics: One needs to consider and understand a whole hierarchy of possible logics relative to each other, either in order to understand the relative complexity and nature of various problems, or in order to have a battery of possible models to fit to a given problem. Examples of this include the theories of modal, intermediate, and of substructural logics, all of which treat hierarchic families of logics. It also includes domain theory and in particular methods for solving domain equations. The other effect of the interaction with computer science is a further focus on duality itself: its nature, content, mechanisms, and limits. This focus stems partially from the equal importance of the two sides of duality and partially from the need to generalise its principles and tools to wider settings in order to provide new models. Examples of this include the theory of canonical extensions, in which I have been quite active, the idea of geometric theories in pointfree topology, and coalgebraic logic.

I have attempted to give you an idea of the nature of duality and its role in mathematics and computer science. Now I would love to really get started and tell you all about the many plans I have for the future! The work on semigroups and automata that I have done recently with Pin and Grigorieff has opened a whole Pandora's box and we cannot wait to explore it. Secondly, my work on spatial semantics for substructural logics reveals a duality betweens 'worlds' and 'information quanta' much like the wave-particle duality in physics. Is there a connection? And what is the power of these semantics in linguistic

applications? and how do they tie up with current methods in proof theory? Thirdly, Bart Jacobs and I are about to embark on an exciting project to understand the dualities for rings and C^*-algbras and those for logics in a common framework as a foundational contribution to current work on introducing stochastics in possibilistic models of computation. However, my time is drawing to a close and I think I have to be content if I have managed to give you a general flavour of my mathematical world. I hope you can at least guess at the beauty and excitement that mathematics contains.

With this lecture, I symbolically take my place among the mathematics professors in the Netherlands, a country with a distinguished history in logic and its interaction with other fields. This is my pleasure but certainly by no means just my doing and I am deeply indebted to many, both near and far: employers, colleagues, friends, family, teachers, and students. Being a mathematician is an absorbing task and being a female mathematician involves yet another level of complication. My family and friends have received far less from me and have had to put up with all sorts of shortcomings for the sake of my involvement with mathematics. My colleagues and students have had to weather my insecurities and excitements. In return, they have given me love and support. Rather than engaging in the fabrication of long lists of thanks, I offer in this oratie my attempt at an explanation of why I dare to feel that mathematics is worth it.

I thank you for your attention,
ik dank u voor uw aandacht.

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