Equipe Logical
LIX, Ecole Polytechnique
Palaiseau, France

Automatizing Geometry using Coq

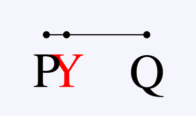
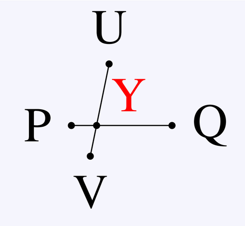
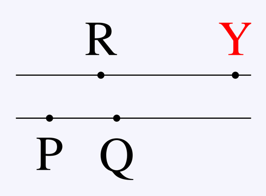
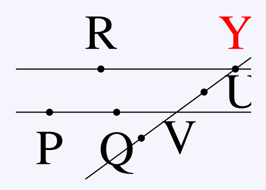
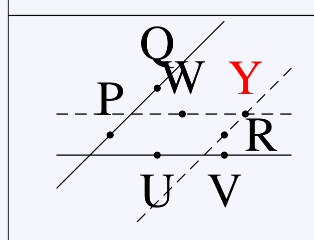
Julien Narboux

Julien.Narboux@inria.fr

The Chou, Gao and Zhang method

1. Find a point which is not used to build any other point.
   - The theorem must be stated *constructively*.
2. *Eliminate* every occurrence of this point from the goal.
   - We need some theorem to *eliminate* the point.
3. Repeat until the goal contains only *free* points.
4. Deal with the *free* points.
5. Check if the remaining goal (an equation on a field) is true.

Elimination lemmas

Construction	Description (Nondegeneracy condition)	Elimination formulas: $\mathcal{S}_{ABY} =$	If $AY \parallel CD$ then $\frac{\overline{AY}}{\overline{CD}} =$
	Take Y on line PQ such that $\frac{\overline{PY}}{\overline{PQ}} = \lambda$. $(P \neq Q)$	$\lambda S_{ABQ} + (1-\lambda)\mathcal{S}_{ABP}$	$\begin{cases} \frac{\frac{\overline{AP}}{\overline{PQ}}+\lambda}{\frac{\overline{CD}}{\overline{PQ}}} & \text{if } A \in PQ \\ \frac{\mathcal{S}_{APQ}}{\mathcal{S}_{CPDQ}} & \text{otherwise}^1. \end{cases}$
	Take Y at the intersection of PQ and UV. $(PQ \nparallel UV)$	$\frac{\mathcal{S}_{PUV}\mathcal{S}_{ABQ}+\mathcal{S}_{QVU}\mathcal{S}_{ABP}}{\mathcal{S}_{PUQV}}$	$\begin{cases} \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{CUDV}} & \text{if } A \notin UV \\ \frac{\mathcal{S}_{APQ}}{\mathcal{S}_{CPDQ}} & \text{otherwise.} \end{cases}$
	Take Y on the parallel to PQ passing through R such that $\frac{\overline{RY}}{\overline{PQ}} = \lambda$. $(P \neq Q)$	$\mathcal{S}_{ABR} + \lambda\mathcal{S}_{APBQ}$	$\begin{cases} \frac{\frac{\overline{AR}}{\overline{PQ}}+\lambda}{\frac{\overline{CD}}{\overline{PQ}}} & \text{if } A \in RY \\ \frac{\mathcal{S}_{APRQ}}{\mathcal{S}_{CPDQ}} & \text{otherwise.} \end{cases}$
	Take Y at the intersection of UV and the parallel to PQ passing through R. $(PQ \nparallel UV)$	$\frac{\mathcal{S}_{PUQR}\mathcal{S}_{ABV}-\mathcal{S}_{PVQR}\mathcal{S}_{ABU}}{\mathcal{S}_{PUQV}}$	$\begin{cases} \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{CUDV}} & \text{if } A \notin UV \\ \frac{\mathcal{S}_{APRQ}}{\mathcal{S}_{CPDQ}} & \text{otherwise.} \end{cases}$
	Take Y at the intersection of the parallel to PQ passing through R and the parallel to UV passing through W. $(PQ \nparallel UV)$	$\frac{\mathcal{S}_{PWQR}}{\mathcal{S}_{PUQV}}.\mathcal{S}_{AUBV} + \mathcal{S}_{ABW}$	$\begin{cases} \frac{\mathcal{S}_{APRQ}}{\mathcal{S}_{CPDQ}} & \text{if } AY \nparallel PQ \\ \frac{\mathcal{S}_{AUWV}}{\mathcal{S}_{CUDV}} & \text{otherwise.} \end{cases}$

The axiomatic

Points Point : Set

Field F is a field
$2 \neq 0$

Signed distance $\overline{\cdot}$: Point $\rightarrow$ Point $\rightarrow F$
$\overline{AB} = 0 \iff A = B$

Signed area $\mathcal{S}$: Point $\rightarrow$ Point $\rightarrow$ Point $\rightarrow F$
$\mathcal{S}_{ABC} = \mathcal{S}_{CAB}$
$\mathcal{S}_{ABC} = -\mathcal{S}_{BAC}$

Chasles'axiom $\mathcal{S}_{ABC} = 0 \rightarrow \overline{AB} + \overline{BC} = \overline{AC}$

Dimension $\exists A, B, C$: Point, $\mathcal{S}_{ABC} \neq 0$
$\mathcal{S}_{ABC} = \mathcal{S}_{DBC} + \mathcal{S}_{ADC} + \mathcal{S}_{ABD}$

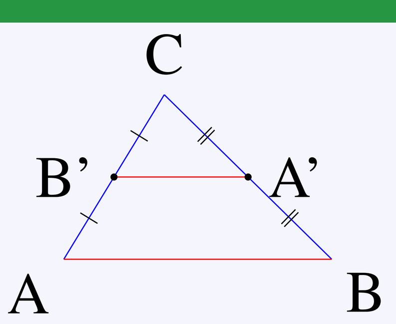
Construction $\forall r : F\ \exists P : Point,\ \mathcal{S}_{ABP} = 0 \wedge \overline{AP} = r\overline{AB}$
$A \neq B \begin{matrix} \wedge\ \mathcal{S}_{ABP} = 0 \wedge \overline{AP} = r\overline{AB} \\ \wedge\ \mathcal{S}_{ABP'} = 0 \wedge \overline{AP'} = r\overline{AB} \end{matrix} \rightarrow P = P'$

Proportions $A \neq C \rightarrow \mathcal{S}_{PAC} \neq 0 \rightarrow \mathcal{S}_{ABC} = 0 \rightarrow \frac{\overline{AB}}{\overline{AC}} = \frac{\mathcal{S}_{PAB}}{\mathcal{S}_{PAC}}$

The formalization of some geometric notions

Geometric notions	Formalization
A,B and C are collinear	$\mathcal{S}_{ABC} = 0$
$AB \parallel CD$	$\mathcal{S}_{ABC} = \mathcal{S}_{ABD}$
I is the midpoint of AB	$\frac{\overline{AB}}{\overline{AI}} = 2 \wedge \mathcal{S}_{ABI} = 0$
$AB \perp BC$	$\mathcal{P}_{ABC} = 0$
$AB \perp CD$	$\mathcal{P}_{ACD} = \mathcal{P}_{BCD}$
$A = B$	$\mathcal{P}_{ABA} = 0$

The midpoint theorem using Coq



Let ABC be a triangle, and let A' and B' be the midpoints of BC and AC respectively. Then the line $A'B'$ is parallel to the base AB.

```
forall A B C A' B' : Point, midpoint A' B C -> midpoint B' A C ->
parallel A' B' A B.
```

At this step it would be enough to type `autogeom` to solve the goal using our decision procedure, but for this presentation we mimic the behavior of the decision procedure using some sub-tactics. For this presentation the fact that A, B, C, D and E are of type `point` has been removed from the context.

```
geoInit.

  H : on_line_d A' B C (1 / 2)
  H0 : on_line_d B' A C (1 / 2)
  ============================
   S A' A B' + S A' B' B = 0

eliminate B'.

  H : on_line_d A' B C (1 / 2)
  ============================
   1 / 2 * S A' A C + (1 - 1 / 2) * S A' A A +
   (1 / 2 * S B A' C + (1 - 1 / 2) * S B A' A) = 0

basic_simpl.

  H : on_line_d A' B C (1 / 2)
  ============================
   1 / 2 * S A' A C + (1 / 2 * S B A' C + 1 / 2 * S B A' A) = 0

eliminate A'.

  ============================
   1 / 2 * (1 / 2 * S A C C + (1 - 1 / 2) * S A C B) +
   (1 / 2 * (1 / 2 * S C B C + (1 - 1 / 2) * S C B B) +
    1 / 2 * (1 / 2 * S A B C + (1 - 1 / 2) * S A B B)) = 0

basic_simpl.

  ============================
   1 / 2 * (1 / 2 * S A C B) + 1 / 2 * (1 / 2 * S A B C) = 0

unify_signed_areas.

  ============================
   1 / 2 * (1 / 2 * S A C B) + 1 / 2 * (1 / 2 * - S A C B) = 0

field_and_conclude.

  Proof completed.
```

References

[1] S.C. Chou, X.S. Gao, and J.Z. Zhang. *Machine Proofs in Geometry*. World Scientific, Singapore, 1994.
[2] Frédérique Guilhot. Formalisation en coq d'un cours de géométrie pour le lycée. In *Journées Francophones des Langages Applicatifs*, Janvier 2004.
[3] Laura I. Meikle and Jacques D. Fleuriot. Formalizing Hilbert's grundlagen in Isabelle/Isar. In *Theorem Proving in Higher Order Logics*, pages 319–334, 2003.
[4] Julien Narboux. A Decision Procedure for Geometry in Coq. In *Theorem Proving in Higher Order Logics*, pages 225–240, 2004.

PROJET LOGICAL

INRIA FUTURS