Using a proof assistant in an introduction to proof course: first experiment with two proof assistants

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What we did

- Project APPAM (French Agency of Research)
- Pre-experiment for a 1st year undergraduate course
- 2 proof assistants (PAs)
- Observation and survey on students perceptions

Organisation

- Context
- Observations
- Survey
- Questions for further work

Context

The course

- **Title** : Introduction to reasoning and proof
- **Level** : First year undergraduate students
- **Objective**: math vocabulary, reasoning rules, write proofs
- **Content**: naive set theory, functions, relations, natural numbers, induction
- **Organisation** : 24h lectures (and exercises) + 10h using PAs
- **Evaluation** : 2 on paper + 1 on PA

Context

The students

- 200 students of mathematics and computer science
- 6 groups of 30-40 students (2 math + 4 CS)
- Courses taken before:
  - Introduction to programming
  - Introduction to mathematics (algebra and analysis)

The teachers

- Types: lecturers in mathematics and CS, PhD candidates in mathematics, secondary school teachers
- Not experts in PAs
- Lab session conducted by us

Context

Lab Sessions

1. **Introduction to the PA** and to propositional logic using exercises about sets
2. **True or False** exercises about basic arithmetic
3. **Functions**: injectivity, surjectivity, direct and inverse image
4. **Functions**: injectivity, surjectivity, direct and inverse image
5. On computer exam.

The proof assistants

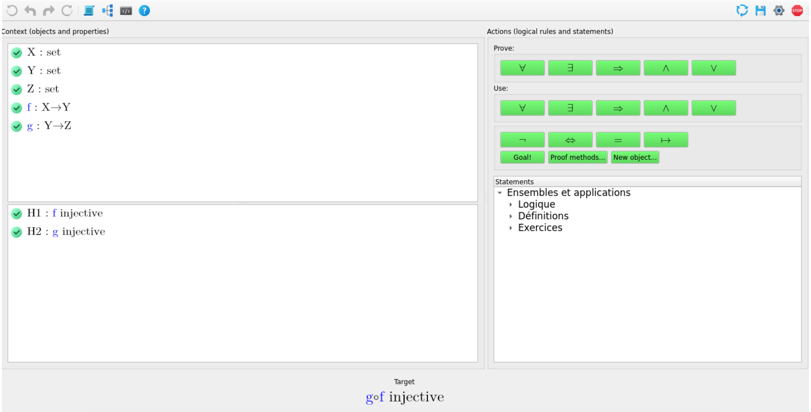
Verbose L∃∀N

- Alternative language for L∃∀N 4
- developed by Patrick Massot
- *help* feature
- global visualization of the proof

D∃∀duction

- Interface on top of L∃∀N 3
- developed by Frédéric Le Roux
- point-and-click user interface
- local visualization of the proof

D∃∀duction



Context (objects and properties)
X : set
Y : set
Z : set
f : X→Y
g : Y→Z
H1 : f injective
H2 : g injective
Actions (logical rules and statements)
Prove:
∀ ∃ ⇒ ∧ ∨
Use:
∀ ∃ ⇒ ∧ ∨
¬ ⇔ = ↦
Goal!
Proof methods...
New object...
Statements
Ensembles et applications
Logique
Définitions
Exercices
Target
g∘f injective

```
variable (X : Type)

Exercise "Tunnel effect"
  Given: (A B C : Set X)
  Assume:
  Conclusion: (A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C))
Proof:
  Let's first prove that A ∪ B ∩ C ⊆ (A ∪ B) ∩ (A ∪ C)
  Fix x
  Assume hx : x ∈ A ∪ B ∩ C
  Let's first prove that x ∈ A ∪ B
  We proceed using hx
  Assume hxA : x ∈ A
  Let's prove that x ∈ A
  We conclude by hxA
  Assume hxIBC : x ∈ B ∩ C
  By hxIBC we get (hxB : x ∈ B) (hxC : x ∈ C)
  Let's prove that x ∈ B
  We conclude by hxB
  Let's now prove that x ∈ A ∪ C
  We proceed using hx
  Assume hxA : x ∈ A
  Let's prove that x ∈ A
  We conclude by hxA
  Assume hxIBC : x ∈ B ∩ C
  By hxIBC we get (hxB : x ∈ B) (hxC : x ∈ C)
  Let's prove that x ∈ C
  We conclude by hxC
  Let's now prove that (A ∪ B) ∩ (A ∪ C) ⊆ A ∪ B ∩ C
```

```
▼ Tactic state
3 goals
▼ case h₁.right.block.prf.left
X : Type
A B C : Set X
x : X
hx : x ∈ A ∪ B ∩ C
⊢ x ∈ A → x ∈ A ∪ C

▼ case h₁.right.block.prf.right
X : Type
A B C : Set X
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hx : x ∈ A ∪ B ∩ C
⊢ x ∈ B ∩ C → x ∈ A ∪ C

▶ case h₂.block

▶ Expected type

▶

▶ All Messages (0)
```

Didactical context

Instrumental approach (Rabardel 1995)

Instrumental genesis: the process of an artifact becoming an instrument

- Instrumentalization
- Instrumentation

Instrumental Genesis

Instrumentalization

The process of adjusting a specific set of characteristics of a PA to direct students' activity towards a specific aspect of proving.

- using the PA libraries or working in a microworld
- using true or false statements (think semantically)

Instrumentation

The way specific practices of a PA shape the students' process of proving.

- development of automatism about quantifiers
- trial and error strategies

Research questions

General questions

- What are the possible effects of using PAs on students' learning of proof?
- What characteristics of PAs are likely to strengthen or obstruct these effects?

Question for our work

- How do students perceive their use of PAs?
- Does using one of the two PAs help students ameliorate skills in proof and proving? Which skills?
- Do we observe big differences between the groups? What kind of differences?

True or False statements

For every $a \geq 0$, (if for every $\epsilon \geq 0$, $a \geq \epsilon$) then $a = 0$

- **choose the negation**: $\neg(\forall a \geq 0, (\forall \epsilon \geq 0, a \geq \epsilon) \Rightarrow a = 0)$
- **push negation**: $\exists a \geq 0, (\forall \epsilon \geq 0, a \geq \epsilon) \wedge a \neq 0$
- **random value for** a: $\forall \epsilon \geq 0, 1 \geq \epsilon$
- **decide if it is provable or not**

True or False statements

Exploring conjectures

- Many proceed by trial and error
- They used the PA to generate an example of the statement instantiating with an acceptable value
- Same observation for both PAs

Tunnel effect

- D∃∀duction: local visualization of the proof
- Verbose L∃∀N: global visualization of the proof

Tunnel effect: global visualisation

```
variable (X : Type)

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X : Type
A B C : Set X
x : X
hx : x ∈ A ∪ B ∩ C
⊢ x ∈ B ∩ C → x ∈ A ∪ C

▶ case h₂.block

▶ Expected type
▶
▶ All Messages (0)
```
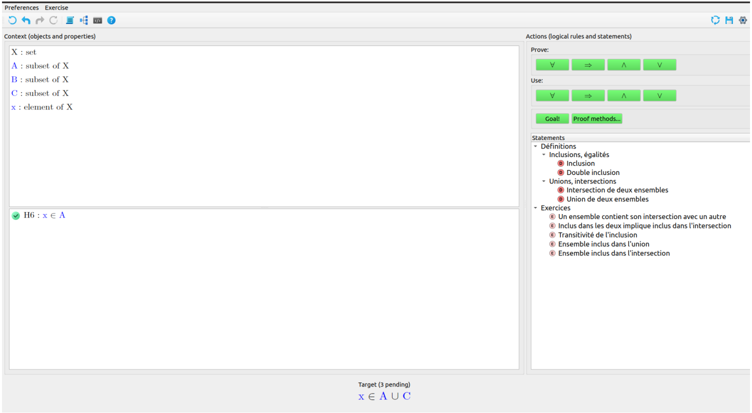

Tunnel effect: local visualisation



Tunnel effect

D∃∀duction

- The interface acts as blinders
- The effect (positive or negative) depends on the proof
- For students the switch between the local and global view of the proof may be difficult

Survey

- Students' perceptions on the use of the PAs
- The survey was optional
- 38 Answers
- Work in progress

Survey

Drag and drop questions

Using the PA allowed me to

- make trial and error
- understand what is to be proven
- verify my proof
- to have an overall view of a proof
- to know how to advance in a proof
- write a proof on paper

Open questions

- What you found easy (/difficult) to do in the PA
- What you liked (/didn't like) about the PA
- What was the reason when I was blocked in the PA?

Survey

Likert scale questions (1 to 5)

- I have an idea of the proof before starting in the PA.
- I've done proofs in the PA without understanding them.
- I anticipate the result I will get before I click/write a line.
- When I had an idea for a proof, I couldn't do it in the PA.
- I need a paper sheet when working with a PA.
- I think I want to use this PA frequently.
- I felt very confident using the PA.

Questions for further work

Questions

- What are the students learning?
  - how to use the PA?
  - how to produce proofs using a PA?
  - how to produce proofs in general?
- What is the impact on understanding the concept of proof?
- What is the impact of a PA on writing proofs?

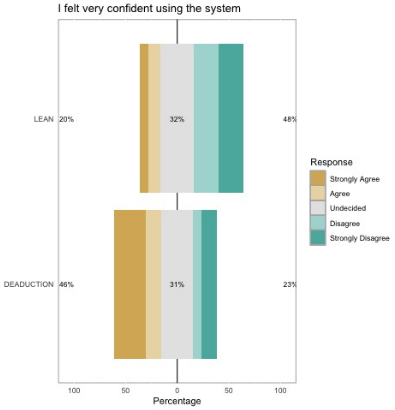
Should take into account different **public**, different **PAs**, different **usage scenarios**, and different **mathematical contents**.

Further work

- Analyze the survey results
- Analyze exam results (qualitative analysis)
- Instrument the PA to collect data
- Repeat the experiment

Thank you for your attention!

Survey



I felt very confident using the system
LEAN
20%
32%
48%
DEADUCTION
46%
31%
23%
100
50
0
50
100
Percentage
Response
Strongly Agree
Agree
Undecided
Disagree
Strongly Disagree

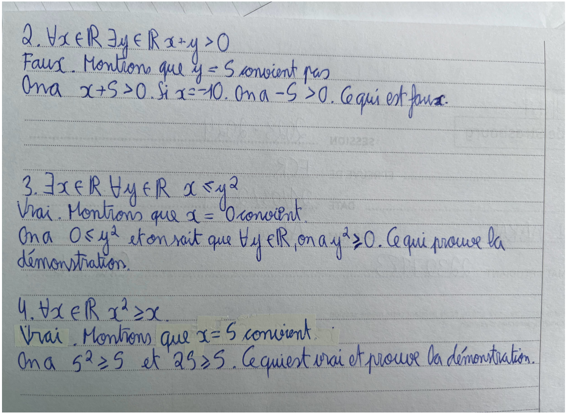
Survey limitations

- Facultative survey – not many answered
- We used the PA as an exerciser – did not work on the translation to paper proofs
- We do not know if they were present during all the lab sessions
- In the meantime they had math courses
- Teachers' different backgrounds (on math, proof, logic, programming) and use of the PA
- Students with different background
- Groups are not homogeneous
- Different proofs on paper and with the PA
- Likert scale

Help

- The assumption hyp is an equality
  One can use it to replace the left-hand-side (namely g (f x1)) by the righ
  (namely g (f x2)) in the goal with:
  We rewrite using hyp
- One can use it to replace the right-hand-side in the goal with:
  We rewrite using ← hyp
- One can also perform such replacements in an assumption hyp_1 with
  We rewrite using hyp dans† hyp_1
- or
  We rewrite using ← hyp dans† hyp_1
- One can also use it in a computation step, or combine it linearly to other
  We combine [hyp, ?_]
  replacing the question mark by one or more terms proving equalities.

Copy 1



2. $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ x+y>0$
Faux. Montrons que $y=5$ convient pas
On a $x+5>0$. Si $x=-10$. On a $-5>0$. Ce qui est faux.

3. $\exists x \in \mathbb{R}\ \forall y \in \mathbb{R}\ x \leq y^2$
Vrai. Montrons que $x=0$ convient.
On a $0 \leq y^2$ et on sait que $\forall y \in \mathbb{R}$, on a $y^2 \geq 0$. Ce qui prouve la démonstration.

4. $\forall x \in \mathbb{R}\ x^2 \geq x$.
Vrai. Montrons que $x=5$ convient.
On a $5^2 \geq 5$ et $25 \geq 5$. Ce qui est vrai et prouve la démonstration.

Copy 2

4. Supposons que $\forall x \in \mathbb{R}\ x^2 \geq x$
Montrons que cette assertation est vraie avec $x = 0,5$
$x^2 = 0,25$ or $0,25 < 0,5$ ↯
Donc cette assertation est fausse.

3. Supposons que $\exists x \in \mathbb{R}\ \forall y \in \mathbb{R}\ x \leq y^2$
Montrons que $x = 0$ convient
$\forall y \in \mathbb{R},\ y^2 \geq 0$
x convient, l'assertation est vraie.

Exercice 4

1. FAUX. Montrons sa négation : $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y \leq 0$.
   Soit $x \in \mathbb{R}$. Montrons que $-x$ convient.
   $x + (-x) \leqslant 0$ est évidemment vrai donc $-x$ convient bien.
   $\Leftrightarrow 0 \leqslant 0$

2. VRAI, et déjà démontré juste au-dessus.

3. Montrons que $x = 0$ convient. Soit $y \in \mathbb{R}$.
   On sait que $y^2 \geqslant 0$. $(\forall y \in \mathbb{R})$.
   Donc 0 convient donc c'est VRAI.