Toward a Certified Encyclopedia of Geometry

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GC 2015 - Nanning

Outline

1. Our formalization of geometry within Coq (joint work with P. Boutry, G. Braun, J.D. Genevaux, P. Schreck)
   - Motivations
   - Overview
     - Axiom system
     - Results
     - Automation
2. Toward a Certified ETC (joint work with D. Braun)
   - Triangles centers
   - Our approach
   - Results

(My) History of Proof

1. Clarify the hypotheses

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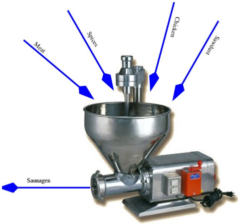

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Hence, in principle we can build proof assistants.

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Hence, in principle we can build proof assistants.

In practice:

Examples

- Coq
- Isabelle
- PVS
- HOL-Light
- ...

What is Coq ?

- A proof assistant
- base on type theory
- that you can download here: `http://coq.inria.fr`.

It allows to :

- define mathematical concepts
- define programs
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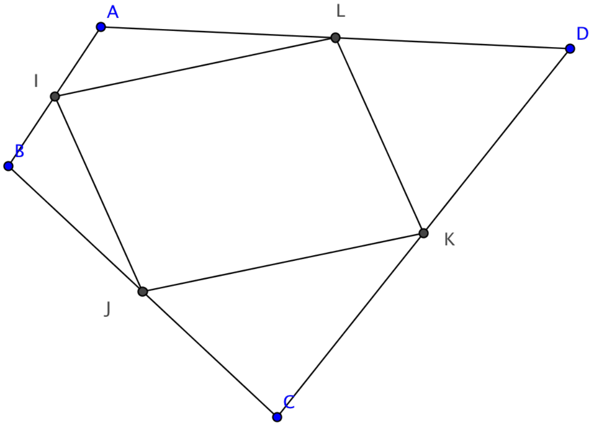
Coq is not

- an automated theorem prover nor
- a tool which help you find proofs.

Demo

Varignon's theorem



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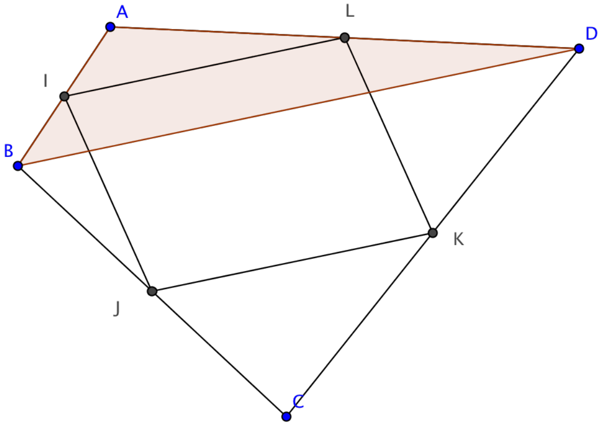

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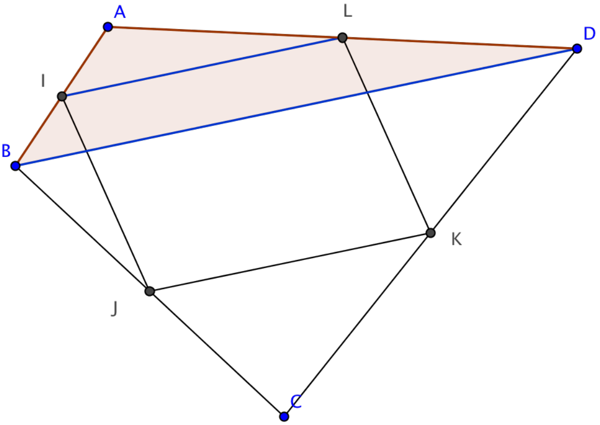

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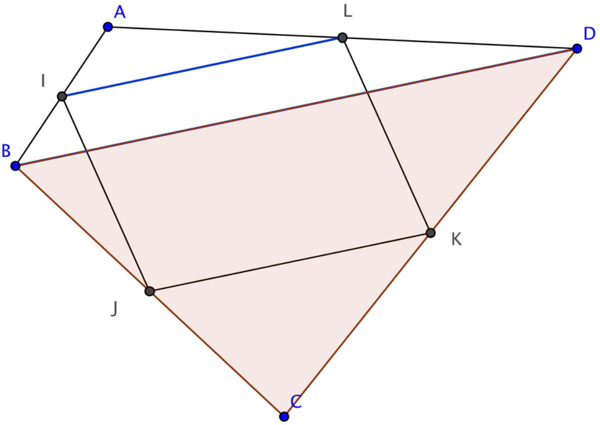

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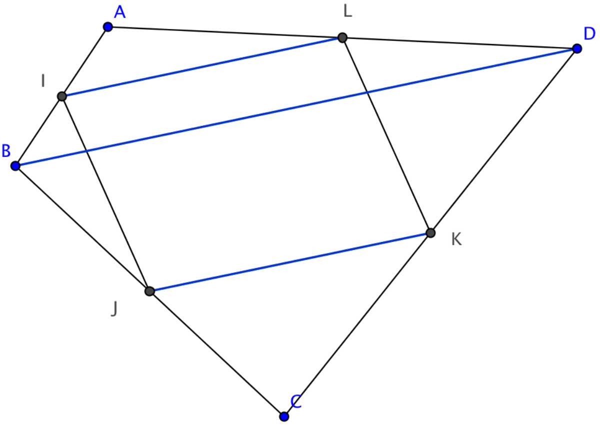

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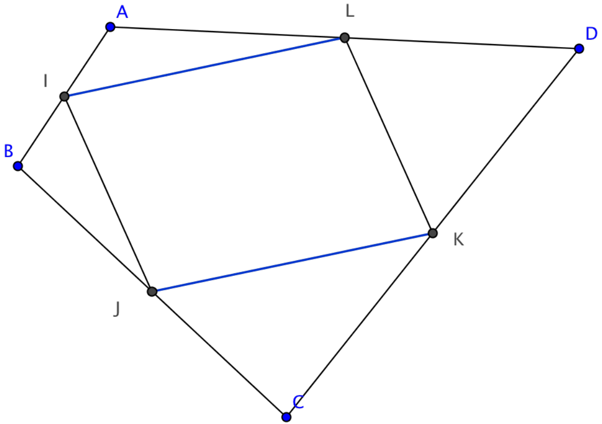

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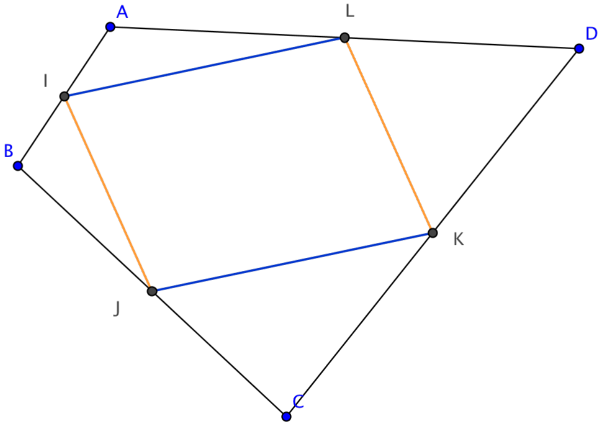

Geometry is central in the history of proofs

Euclid (−325-−265) *The Elements.*
The axiomatic method

Hilbert (1862-1943) *Die Grundlagen der Geometrie.*
Formal mathematics

Tarski (1902-1983)
Metamathematische Methoden in der Geometrie.
Automation, axiomatization

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But also lead to a long history of . . .

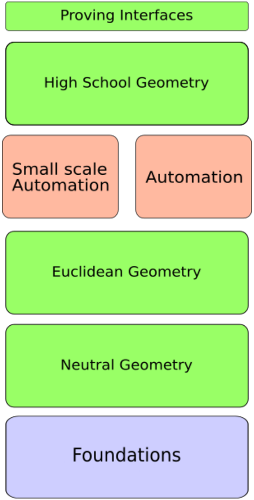
. . . incorrect proofs !

In 1763, in his dissertation Klügel provides a survey of about 30 attempts to "prove" Euclid's parallel postulate" [Klu63].

Examples:

- Ptolemy assumes implicitly Playfair axioms (unicity of parallel).
- Proclus assumes implicitly "If a line intersects one of two parallel lines, both of which are coplanar with the original line, then it must intersect the other also."
- Legendre published several incorrect proofs of Euclid 5 in his best-seller "Éléments de géométrie".

Our project


Proving Interfaces
High School Geometry
Small scale Automation
Automation
Euclidean Geometry
Neutral Geometry
Foundations

Which kind of axiom system ?

Synthetic geometry Start with some geometric objects + axioms about them ...

- Hilbert's axiom system: points, lines and planes
- Tarski's axiom system
- ... many others variants (constructive, ...)

Analytic geometry Start with a field. Define geometric objects by equations involving coordinates.

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Birkhoff's axioms
: Start with a field for measuring distances and angles + synthetic axioms

Analytic geometry
: Start with a field. Define geometric objects by equations involving coordinates.

Synthetic geometry approach is appealing because it allows to have results in neutral geometry.
But still we want to obtain the connection with analytic geometry for the efficient automated deductions methods.

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- Work for any dimension without modifying the language.

Congruence axioms

Congruence Pseudo-Transitivity

$$AB \equiv CD \land AB \equiv EF \Rightarrow CD \equiv EF$$

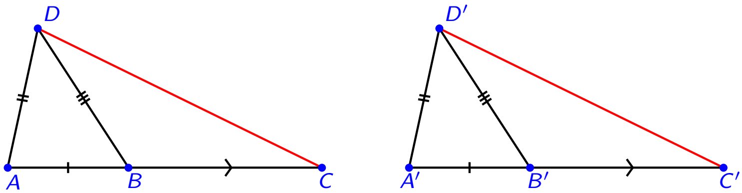
Congruence Symmetry $AB \equiv BA$

Congruence Identity $AB \equiv CC \Rightarrow A = B$

Betweeness axiom

Between identity $\beta\ A\ B\ A \Rightarrow A = B$

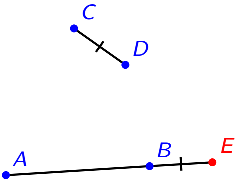
Five segments axiom



$$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$$
$$AD \equiv A'D' \wedge BD \equiv B'D' \wedge$$
$$\beta\ A\ B\ C \wedge \beta\ A'\ B'\ C' \wedge A \neq B \Rightarrow CD \equiv C'D'$$

Some kind of SAS axiom without using angle congruence.

Segment construction axiom

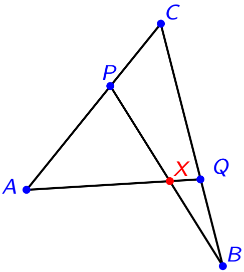


$$\exists E, \beta\ A\ B\ E \land BE \equiv CD$$

Pasch's axiom

Allows to formalize some gaps in Euclid's Elements.
We have the inner form :

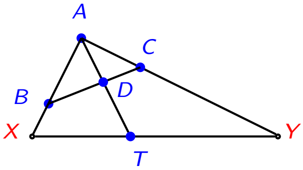
$$\beta\ A\ P\ C \wedge \beta\ B\ Q\ C \Rightarrow \exists X, \beta\ P\ X\ B \wedge \beta\ Q\ X\ A$$



Moritz Pasch
(1843-1930)

Parallel postulate

$$\exists XY, \beta\ A\ D\ T \wedge \beta\ B\ D\ C \wedge A \neq D \Rightarrow$$
$$\beta\ A\ B\ X \wedge \beta\ A\ C\ Y \wedge \beta\ X\ T\ Y$$



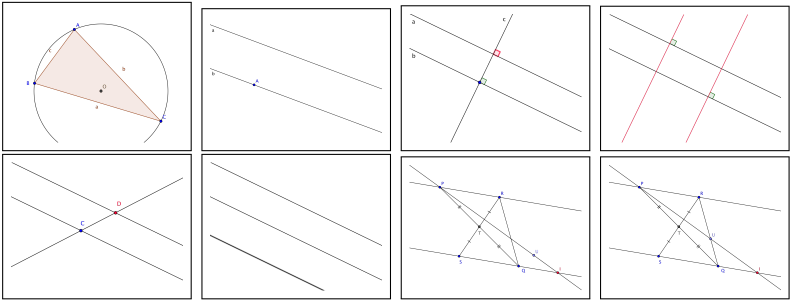
- This statement is equivalent to Euclid 5th postulate.
- Comes from an incorrect proof of Euclid 5th by Legendre.

Adrien-Marie Legendre (1752-1833) (watercolor caricature by Julien Léopold Boilly)

Some Other Parallel Postulates

with Pierre Boutry

```
Theorem parallel_postulates:
 decidability_of_intersection ->
 ((triangle_circumscription <-> tarski_parallel_postulate) /\
 (playfair                  <-> tarski_parallel_postulate) /\
 (par_perp_perp_property    <-> tarski_parallel_postulate) /\
 (par_perp_2_par_property   <-> tarski_parallel_postulate) /\
 (proclus                   <-> tarski_parallel_postulate) /\
 (transitivity_of_par       <-> tarski_parallel_postulate) /\
 (strong_parallel_postulate <-> tarski_parallel_postulate) /\
 (euclid_5                  <-> tarski_parallel_postulate)).
```



Tarski vs Hilbert

```
Class Tarski := {
 Tpoint : Type;
 Bet : Tpoint -> Tpoint -> Tpoint -> Prop;
 Cong : Tpoint -> Tpoint -> Tpoint -> Tpoint -> Prop;
 between_identity : forall A B, Bet A B A -> A=B;
 cong_pseudo_reflexivity : forall A B : Tpoint, Cong A B B A;
 cong_identity : forall A B C : Tpoint, Cong A B C C -> A = B;
 cong_inner_transitivity : forall A B C D E F : Tpoint,
   Cong A B C D -> Cong A B E F -> Cong C D E F;
 inner_pasch : forall A B C P Q : Tpoint,
      Bet A P C -> Bet B Q C -> exists x, Bet P x B /\ Bet Q x A;
 five_segments : forall A A' B B' C C' D D' : Tpoint,
    Cong A B A' B' -> Cong B C B' C' ->  Cong A D A' D' -> Cong B D B' D' ->
    Bet A B C -> Bet A' B' C' -> A <> B -> Cong C D C' D';
  segment_construction : forall A B C D : Tpoint,
    exists E : Tpoint, Bet A B E /\ Cong B E C D;
  lower_dim : exists A, exists B, exists C, ~ (Bet A B C \/ Bet B C A \/ Bet C
  upper_dim : forall A B C P Q : Tpoint,
    P <> Q -> Cong A P A Q -> Cong B P B Q -> Cong C P C Q ->
    (Bet A B C \/ Bet B C A \/ Bet C A B)
  euclid : forall A B C,    ~ (Bet A B C \/ Bet B C A \/ Bet C A B) ->
    exists CC, Cong A CC B CC /\ Cong A CC C CC
}.
```

Tarski vs Hilbert I

```
Class Hilbert := {
 Point : Type;
 Line  : Type;
 EqL   : Line -> Line -> Prop;
 EqL_Equiv : Equivalence EqL;
 Incid : Point -> Line -> Prop;

 (** Group I Incidence *)
 line_existence : forall A B, A<>B -> exists l, Incid A l /\ Incid B l;
 line_unicity : forall A B l m, A <> B -> Incid A l -> Incid B l -> Incid A m -
 two_points_on_line : forall l, exists A, exists B, Incid B l /\ Incid A l /\ A
 ColH := fun A B C => exists l, Incid A l /\ Incid B l /\ Incid C l;
 plan : exists A, exists B, exists C, ~ ColH A B C;

 (** Group II Order *)
 BetH   : Point -> Point -> Point -> Prop;
 between_col :  forall A B C : Point, BetH A B C -> ColH A B C;
 between_comm : forall A B C : Point, BetH A B C -> BetH C B A;
 between_out :  forall A B   : Point, A <> B -> exists C : Point, BetH A B C;
 between_only_one : forall A B C : Point, BetH A B C -> ~ BetH B C A /\ ~ BetH
 between_one : forall A B C, A<>B -> A<>C -> B<>C -> ColH A B C -> BetH A B C \
 cut := fun l A B => ~ Incid A l /\ ~ Incid B l /\ exists I, Incid I l /\ BetH
```

Tarski vs Hilbert II

```
pasch : forall A B C l, ~ ColH A B C -> ~ Incid C l -> cut l A B -> cut l A C
 (** Group III Parallels *)
Para := fun l m => ~ exists X, Incid X l /\ Incid X m;
euclid_existence : forall l P, ~ Incid P l -> exists m, Para l m;
euclid_unicity :  forall l P m1 m2, ~ Incid P l -> Para l m1 -> Incid P m1-> P

(** Group IV Congruence *)
CongH : Point -> Point -> Point -> Point -> Prop;
cong_pseudo_transitivity : forall A B C D E F, CongH A B C D -> CongH A B E F
cong_refl : forall A B, CongH A B A B;
cong_existence : forall A B l M, A <> B -> Incid M l -> exists A', exists B',
   Incid A' l /\ Incid B' l /\ BetH A' M B' /\ CongH M A' A B /\ CongH M B' A
cong_unicity : forall A B l M A' B' A'' B'', A <> B -> Incid M l ->
 Incid A'  l -> Incid B'  l ->
 Incid A'' l -> Incid B'' l ->
 BetH A' M B' -> CongH M A' A B ->
 CongH M B'  A B -> BetH  A'' M B'' ->
 CongH M A'' A B ->
 CongH M B'' A B ->
 (A' = A'' /\ B' = B'') \/ (A' = B'' /\ B' = A'');
 disjoint := fun A B C D => ~ exists P, BetH A P B /\ BetH C P D;
addition: forall A B C A' B' C', ColH A B C -> ColH A' B' C' ->
                                disjoint A B B C -> disjoint A' B' B' C' ->
```

Tarski vs Hilbert III

```
                                   CongH A B A' B' -> CongH B C B' C' -> CongH A
Angle := @Triple Point;
angle := build_triple Point;
CongaH : Angle -> Angle -> Prop;
cong_5 : forall A B C A' B' C', forall H1 : (B<>A /\ C<>A), forall H2: B' <> A
 forall H3 : (A<>B /\ C<>B), forall H4: A' <> B' /\ C' <> B',
 CongH A B A' B' -> CongH A C A' C' -> CongaH (angle B A C H1) (angle B' A' C'
  CongaH (angle A B C H3) (angle A' B' C' H4);

same_side := fun A B l => exists P, cut l A P /\ cut l B P;
outH := fun P A B => BetH P A B \/ BetH P B A \/ (P <> A /\ A = B);

InAngleH := fun a P =>
(exists M, BetH (V1 a) M (V2 a) /\ ((outH (V a) M P) \/ M = (V a))) \/
                                   outH (V a) (V1 a) P \/ outH (V a) (V2 a) P;

Hline := @Couple Point;
line_of_hline : Hline -> Line;
hline_construction := fun a (h: Hline)  P (hc:Hline) H =>
(P1 h) = (P1 hc) /\
CongaH a (angle (P2 h) (P1 h) (P2 hc) (conj (sym_not_equal (Cond h)) H)) /\
 (forall M, InAngleH (angle (P2 h) (P1 h) (P2 hc) (conj (sym_not_equal (Cond h
  same_side P M  (line_of_hline h));
```

Tarski vs Hilbert IV

```
 aux : forall (h h1 : Hline), P1 h = P1 h1 -> P2 h1 <> P1 h;
  hcong_4_existence: forall a h P,
 ~Incid P (line_of_hline h) -> ~ BetH (V1 a)(V a)(V2 a) ->
 exists h1, (P1 h) = (P1 h1) /\ (forall CondAux : P1 h = P1 h1,
                                  CongaH a (angle (P2 h) (P1 h) (P2 h1) (conj
                                  /\ (forall M, ~ Incid M (line_of_hline h) /
                                                 -> same_side P M  (line_of_


hEq : relation Hline := fun h1 h2 => (P1 h1) = (P1 h2) /\
                        ((P2 h1) = (P2 h2) \/ BetH (P1 h1) (P2 h2) (P2 h1) \/
                                               BetH (P1 h1) (P2 h1) (P2 h2))
hcong_4_unicity :
 forall a h P h1 h2 HH1 HH2,
 ~Incid P (line_of_hline h) -> ~ BetH (V1 a)(V a)(V2 a) ->
 hline_construction a h P h1 HH1 -> hline_construction a h P h2 HH2 ->
 hEq h1 h2

}.
```

Results

- Formalization of the first 14 chapters of SST, this includes:
  - A big library in neutral dimensionless geometry: projection, axial symetry, angles, midpoint ...
  - Geometric proof of Pappus and Desargues by Gabriel Braun
  - Construction of the field of coordinates by Gabriel Braun
- Integration of automated deduction methods
- Connection with other axiom systems
- Some "high-level" theorems: quadrilaterals, midpoints, Varignon, Euler line, well known triangle centers, ...

Automation

Big scale automation

Tools to prove a theorem completly:

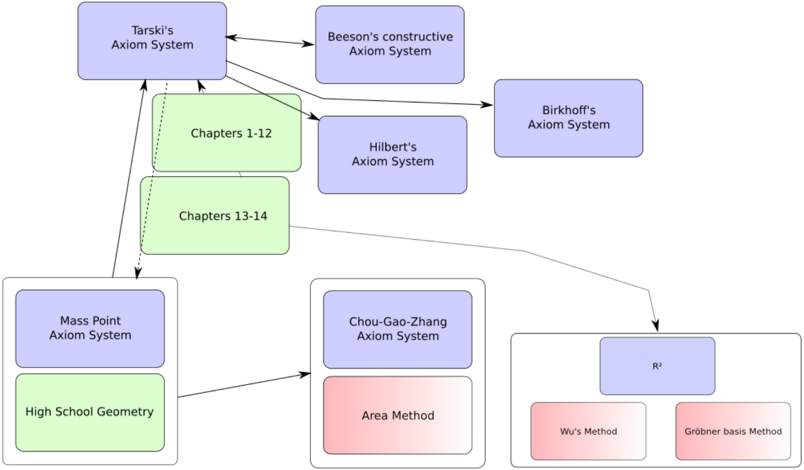
- Simple version of Wu's method (with Jean-David Genevaux) [GNS11].
- Area Method of Chou, Gao and Zhang [Nar04, JNQ12].

Small scale automation [BNSB14] (with Boutry and Schreck)

Tools to simplify interactive proofs:

- Tactics to deal with ndgs: $A \neq B$, $\neg Col(A, B, C)$
- Tactics to deal with permutations: $AB \parallel CD \equiv DC \parallel BA$
- Tactics to deal with pseudo transitivity of Col, *etc.*

Overview of the formalization in Coq



Statistics

Definitions	356
Lemmas (manual)	2300
Proofs	104 kloc

We need proof search !

Toward a Certified ETC (joint work with D. Braun)

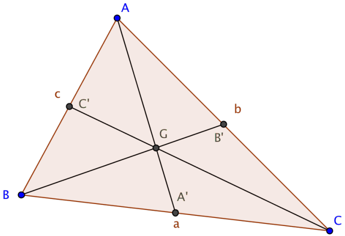
Triangle centers

Since centuries geometers have studied some special points of triangles.

1. **Center of gravity**
2. Circumcenter
3. Orthocenter
4. Incenter
5. ...

These points have some properties, for example :

1. H,G and O are collinear: Euler line
2. ...



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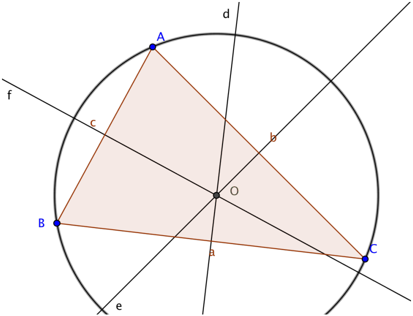

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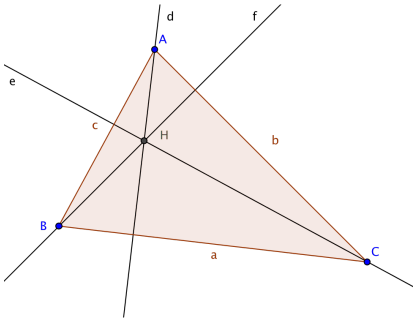

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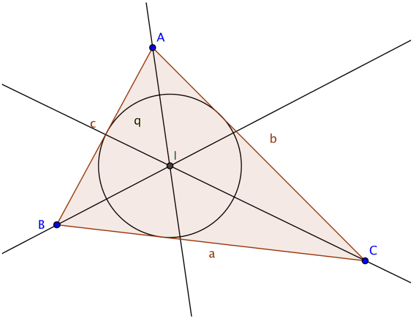

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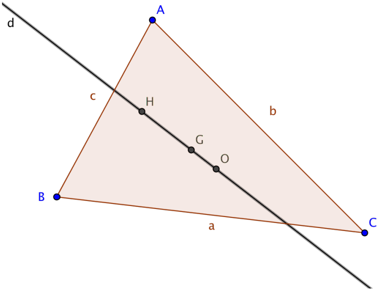
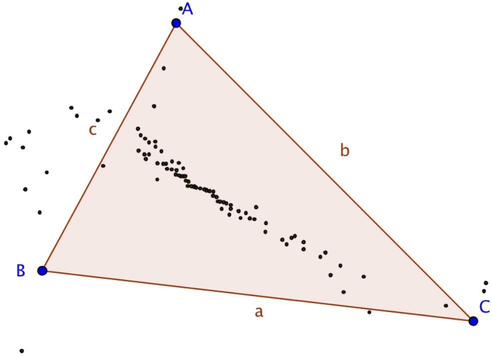


Clark Kimberling's Encyclopedia of Triangle Centers

Clark Kimberling's encyclopedia contains:

- more than **6000 centers**

,



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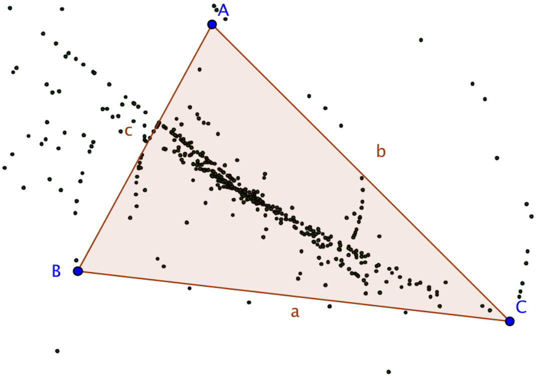

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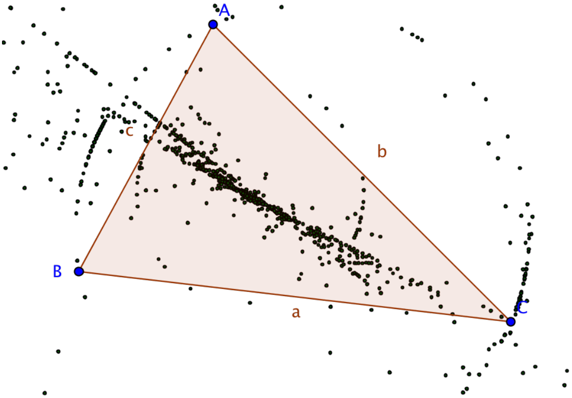

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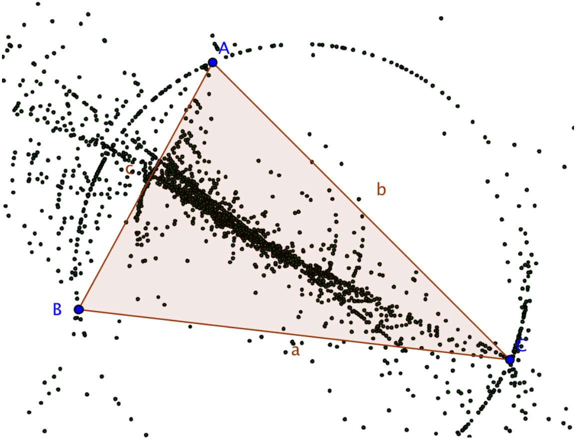

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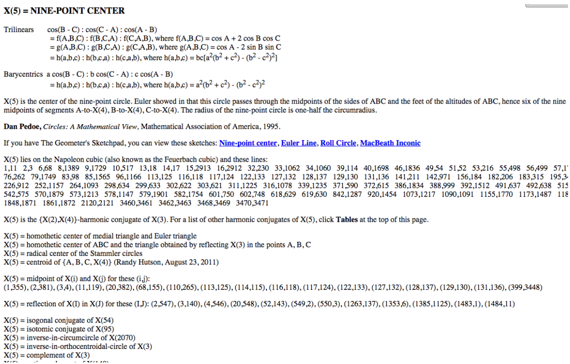
- and **thousands** of properties,



X(5) = NINE-POINT CENTER

Trilinears cos(B - C) : cos(C - A) : cos(A - B)
= f(A,B,C) : f(B,C,A) : f(C,A,B), where f(A,B,C) = cos A + 2 cos B cos C
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Clark Kimberling's Encyclopedia of Triangle Centers

Clark Kimberling's encyclopedia contains:

- more than **6000 centers**
- and **thousands** of properties,
- presented **without proofs**.

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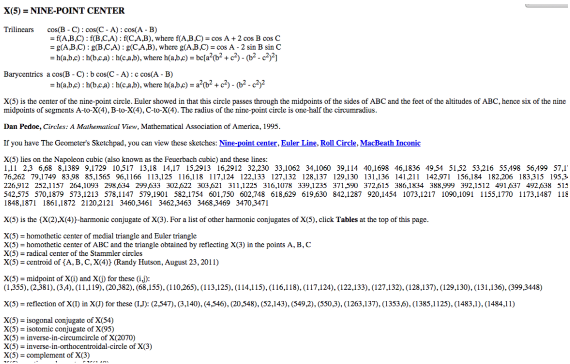

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How to trust these results ?

To use one of these result you need to assume:

1. The definitions corresponds to our intention.
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In this project we want to reduce the trusted code base to the one of Coq.

The points in the encyclopedia are defined by

geometric constructions: midpoint, reflection, intersection of lines, center of circle, ..., and many advanced geometric constructions

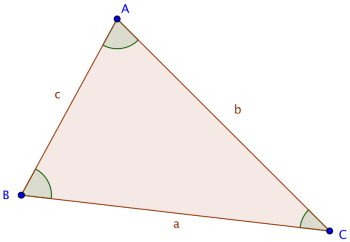
physical properties: X(5626) = Center of eletrostatic potential

coordinates: X(1092) = trilinear cube of X(3)

Coordinates of the points

Can be expressed using homogeneous coordinates as:

- Polynomials in a, b, c
- Radical expressions in a, b, c
- Trigonometric expressions involving A, B, C



The kind of problems

Given a set P of points:

1. Find all triples $(A, B, C) \in P^3$ such that $Col(A, B, C)$.
2. For some function of arity n, find all $(P_1, \ldots, P_n, Q) \in P^{n+1}$ such that $f(P_1, \ldots P_n) = Q$

Our Approach

1. Start with the list of homogeneous coordinates
2. Find properties using symbolic numeric computations on few triangles

3. Check properties using a CAS (Maple)
   - Could not check all properties
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   - Heuristic to pseudo-normalize some expressions involving radicals

Preliminary Results

Statistics for 6000 points

Type of property	**CAS**	**Coq**
Col	68820	67454
Isotomic conjugate	616	604
Complement	932	906
Cyclocevian Conjugate	17	16
Isogonal Conjugate	1954	1905
Hirst inverse	20769	9130
Ceva conjugate	12131	7592
...		
Total	*271568*	*193042* (71 %)

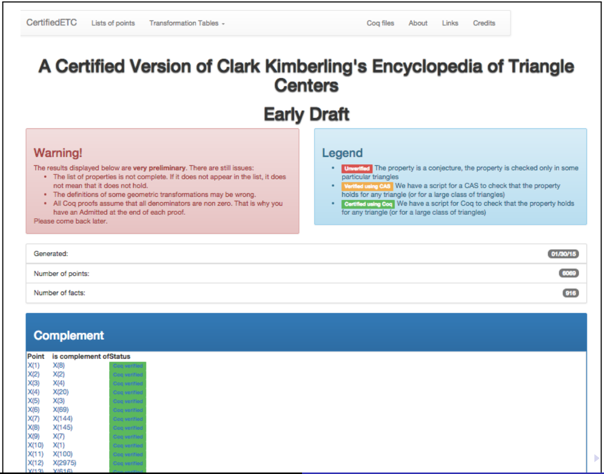
Computation time

About 50 core/day(s) of computation.

Preliminary Results

Draft available here:

`http://dpt-info.u-strasbg.fr/~narboux/draftCETC/`



CertifiedETC Lists of points Transformation Tables Coq files About Links Credits

A Certified Version of Clark Kimberling's Encyclopedia of Triangle Centers

Early Draft

Warning!

The results displayed below are **very preliminary**. There are still issues:

- The list of properties is not complete. If it does not appear in the list, it does not mean that it does not hold.
- The definitions of some geometric transformations may be wrong.
- All Coq proofs assume that all denominators are non zero. That is why you have an Admitted at the end of each proof.

Please come back later.

Legend

- Unverified The property is a conjecture, the property is checked only in some particular triangles
- Verified using CAS We have a script for a CAS to check that the property holds for any triangle (or for a large class of triangles)
- Certified using Coq We have a script for Coq to check that the property holds for any triangle (or for a large class of triangles)

Generated:	01/30/15
Number of points:	6089
Number of facts:	916

Complement

Point	is complement of	Status
X(1)	X(8)	Coq verified
X(2)	X(2)	Coq verified
X(3)	X(4)	Coq verified
X(4)	X(20)	Coq verified
X(5)	X(3)	Coq verified
X(6)	X(69)	Coq verified
X(7)	X(144)	Coq verified
X(8)	X(145)	Coq verified
X(9)	X(7)	Coq verified
X(10)	X(1)	Coq verified
X(11)	X(100)	Coq verified
X(12)	X(2975)	Coq verified
X(13)	X(616)	Coq verified

Conclusion

- We have a large library of formal geometry.
- Still needs to be completed to integrate all differents parts.

Perspective

- How to certify more properties ?
- How to search in this database ? using sketches ?
- How to find new properties ?

Thank you.

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