Formalization and automation of geometric reasoning using Coq.

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under the supervision of
Hugo Herbelin

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Geometry and proofs

Euclid (−325-−265) The Elements.
The axiomatic method

Hilbert (1862-1943) Die Grundlagen der Geometrie.
Formal mathematics

Tarski (1902-1983) Metamathematische Methoden in der Geometrie.
Automation, axiomatization

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1. clarify what are the assumptions
2. clarify what is a proof
3. make it so precise that one does not need to understand the proof to verify it
4. mechanize proofs

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The problem of correctness of a proof

- The size of the proof (Wiles' theorem)
- The number of theorems (group classification)
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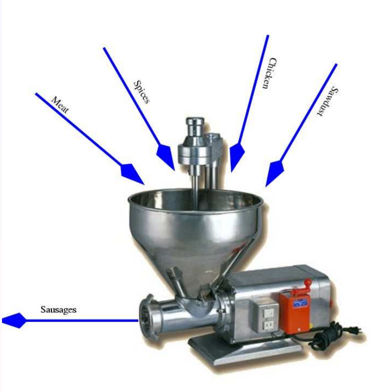
The solution

The use of a proof assistant such as Coq, Isabelle, PVS. . .

- Proofs are objects ⇒ Automation
- Formal proofs should still be convincing proofs



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Outline

1. Formalization
2. Automation
3. GeoProof: A graphical user interface for proofs in geometry
4. Diagrammatic proofs in abstract rewriting

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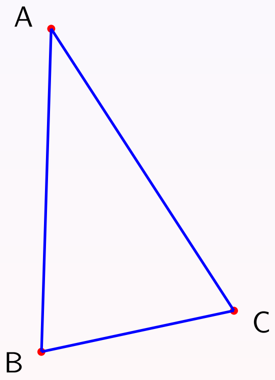
Every triangle is isosceles.

- Let ABC be a triangle.
- Let D be the perpendicular bisector of $[BC]$ and let D' be the bisector of $\angle BAC$.
- Let I be the intersection of D and D'.
- $HI = IG \wedge AH = AG$
- $IB = IC$
- $HB = GC$
- $AB = AC$



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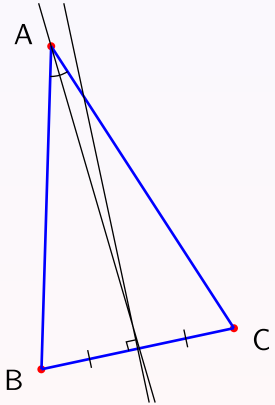

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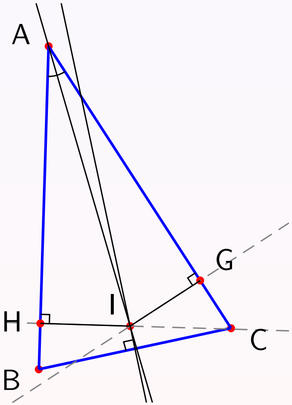

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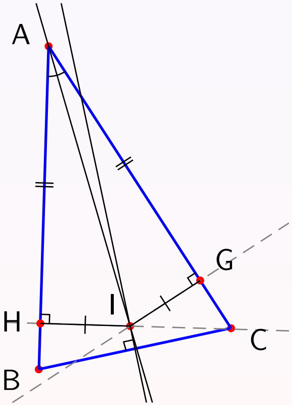

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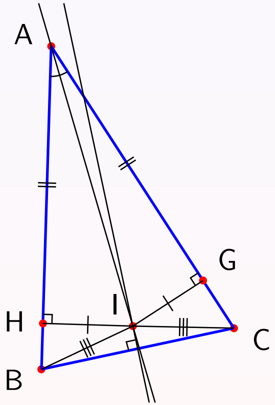

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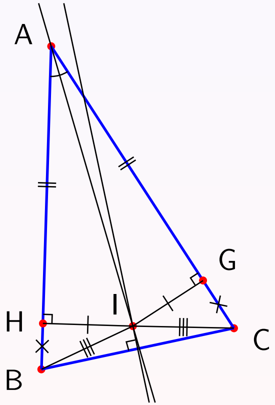

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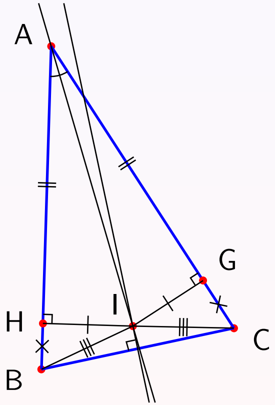

Solution

Formalization of the geometry.

Related work

- Gilles Khan (Coq) [Kah95]
- Christophe Dehlinger, Jean-François Dufourd and Pascal Schreck (Coq) [DDS00]
- Laura Meikle and Jacques Fleuriot (Isabelle) [MF03]
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- They are simple.
  - 11 axioms
  - two predicates ($\beta\ A\,B\,C$, $AB \equiv CD$)
- They have good meta-mathematical properties.
  - coherent
  - complete
  - decidable
  - categorical
  - its axioms are independent (almost)
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History

1940 [Tar67]	1951 [Tar51]	1959 [Tar59]	1965 [Gup65]	1983 [SST83]
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3
4	4	4	4	4
5_1	5_1	$\rightarrow 5$	5	5
6	6	6		6
7_2	7_2	$\rightarrow 7_1$	7_1	$\rightarrow 7$
8(2)	8(2)	8(2)	8(2)	8(2)
$9_1(2)$	$9_1(2)$	$\rightarrow 9(2)$	9(2)	9(2)
10	10	$\rightarrow 10_1$	10_1	$\rightarrow 10$
11	11	11	11	11
12	12			
13				
14	14			
15	15	15	15	
16	16			
17	17			
18	18	18		
19				
20	$\rightarrow 20_1$			
21	21			
20 + 1 schema	18 + 1 schema	12 + 1 schema	10 + 1 schema	10 + 1 schema

Formalization

W. Schwabhäuser
W. Szmielew
A. Tarski

Metamathematische Methoden in der Geometrie

Springer-Verlag 1983

Overview I

About 200 lemmas and 6000 lines of proofs and definitions.

The first chapter contains the axioms.

The second chapter contains some basic properties of equidistance.

The third chapter contains some basic properties of the betweeness predicate (noted `Bet`). In particular, it contains the proofs of the axioms 12, 14 and 16.

The fourth chapters provides properties about `Cong`, `Col` and `Bet`.

The fifth chapter contains the proof of the transitivity of `Bet` and the definition of a length comparison predicate. It contains the proof of the axioms 17 and 18.

The sixth chapter defines the `out` predicate which says that a point is not on a line, it is used to prove transitivity properties for `Col`.

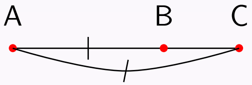
Overview II

The seventh chapter defines the midpoint and the symmetric point and prove some properties.

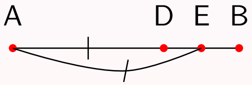
The eighth chapter contains the definition of the predicate “perpendicular”, and finally proves the existence of the midpoint.

Two crucial lemmas

$$\forall ABC, \beta\ A\ C\ B \wedge AC \equiv AB \Rightarrow C = B$$



$$\forall ABDE, \beta\ A\ D\ B \wedge \beta\ A\ E\ B \wedge AD \equiv AE \Rightarrow D = E.$$



($\beta\ A\ B\ C$ means $B \in [AC]$)

About degenerated cases

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Comparison with other formalizations

- ☺ There are fewer degenerated cases than in Hilbert's axiom system.
- ☺ The axiom system is simpler.
- ☺ It has good meta-mathematical properties.
- ☺ Generalization to other dimensions is easy.
- ☹ Lemma scheduling is more complicated.
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Automated deduction in geometry

- Algebraic methods (Wu, Gröbner bases, ...)
- Coordinate free methods (the full-angle method, the area method,...)

The area method

S.C. Chou, X.S. Gao, and J.Z. Zhang.
Machine Proofs in Geometry.
World Scientific, Singapore, 1994.

The elimination method

The elimination method :

1. Find a point which is not used to build any other point.
   - The theorem must be stated constructively.
2. Eliminate every occurrence of this point from the goal.
   - We need some theorem to eliminate the point.
3. Repeat until the goal contains only free points.
4. Deal with the free points.
5. Check if the remaining goal (an equation on a field) is true.

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The goal must be :

- stated constructively (as a sequence of constructions),
- using only two geometric quantities :
  1. the signed area of a triangle ($\mathcal{S}_{ABC} = \mathcal{S}_{BCA} = -\mathcal{S}_{BAC}$)
  2. the ratio of two oriented distances $\frac{\overline{AB}}{\overline{CD}}$ where $AB \parallel CD$
- combined using arithmetic expressions (+,-,*,/).

Using these two quantities :

Geometric notions	Formalization
A,B and C are collinear	$\mathcal{S}_{ABC} = 0$
$AB \parallel CD$	$\mathcal{S}_{ABC} = \mathcal{S}_{ABD}$
I is the midpoint of AB	$\frac{\overline{AB}}{\overline{AI}} = 2 \wedge \mathcal{S}_{ABI} = 0$

We can deal with affine geometry.
The method can be extended to deal with euclidean geometry.

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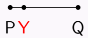
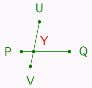
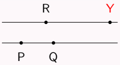
Construction	Elimination formulas	
	$\mathcal{S}_{ABY} =$	If $AY \parallel CD \wedge$ $A \neq Y \wedge$ $C \neq D$ then $\frac{\overline{AY}}{\overline{CD}} =$
	$\lambda \mathcal{S}_{ABQ} + (1-\lambda)\mathcal{S}_{ABP}$	$\begin{cases} \frac{\frac{\overline{AP}}{\overline{PQ}}+\lambda}{\frac{\overline{CD}}{\overline{PQ}}} & \text{if } A \in PQ \\ \frac{\mathcal{S}_{APQ}}{\mathcal{S}_{CPDQ}} & \text{otherwise}^{1}. \end{cases}$
	$\frac{\mathcal{S}_{PUV}\mathcal{S}_{ABQ}+\mathcal{S}_{QVU}\mathcal{S}_{ABP}}{\mathcal{S}_{PUQV}}$	$\begin{cases} \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{CUDV}} & \text{if } A \notin UV \\ \frac{\mathcal{S}_{APQ}}{\mathcal{S}_{CPDQ}} & \text{otherwise.} \end{cases}$
	$\mathcal{S}_{ABR} + \lambda \mathcal{S}_{APBQ}$	$\begin{cases} \frac{\frac{\overline{AR}}{\overline{PQ}}+\lambda}{\frac{\overline{CD}}{\overline{PQ}}} & \text{if } A \in RY \\ \frac{\mathcal{S}_{APRQ}}{\mathcal{S}_{CPDQ}} & \text{otherwise.} \end{cases}$

[1] $\mathcal{S}_{ABCD}$ is a notation for $\mathcal{S}_{ABC} + \mathcal{S}_{ACD}$.

It can not prove automatically:

- Theorems involving a quantification over constructions.
  - The pentagon can be constructed with ruler and compass.
  - The heptagon can not be constructed with ruler and compass.
  - ...
- Theorems stated non constructively.
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The implementation is done :

- using L_{tac} (the tactic language of Coq),
- the reflection mechanism (some sub-tactics are written using Coq itself).

We have to :

1. describe the axiomatic,
2. prove the elimination lemmas,
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Some tactics:

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simplification	performs trivial simplifications.
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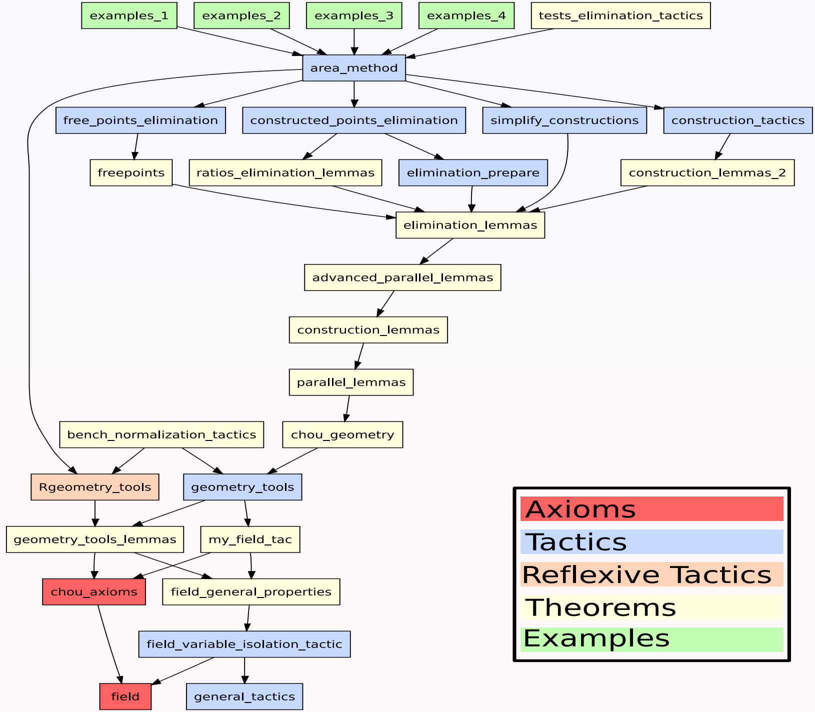
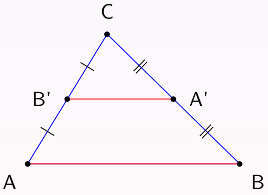


examples_1
examples_2
examples_3
examples_4
tests_elimination_tactics
area_method
free_points_elimination
constructed_points_elimination
simplify_constructions
construction_tactics
freepoints
ratios_elimination_lemmas
elimination_prepare
construction_lemmas_2
elimination_lemmas
advanced_parallel_lemmas
construction_lemmas
parallel_lemmas
bench_normalization_tactics
chou_geometry
Rgeometry_tools
geometry_tools
geometry_tools_lemmas
my_field_tac
chou_axioms
field_general_properties
field_variable_isolation_tactic
field
general_tactics
Axioms
Tactics
Reflexive Tactics
Theorems
Examples

An example

The midpoint theorem

if A' is the midpoint of $[BC]$ and B' is the midpoint of $[AC]$ then $(A'B') \parallel (AB)$.



geoinit.

```
H : on_line_d A' B C (1 / 2)
H0 : on_line_d B' A C (1 / 2)
============================
 S A' A B' + S A' B' B = 0
```

eliminate B'.

```
H : on_line_d A' B C (1 / 2)
============================
 1/2 * S A' A C + (1-1/2) * S A' A A +
 (1/2 * S B A' C + (1-1/2) * S B A' A) = 0
```

basic_simpl.

```
H : on_line_d A' B C (1 / 2)
============================
 1/2 * S A' A C +
     (1/2 * S B A' C + 1/2 * S B A' A) = 0
```

eliminate A'.

```
============================
 1/2*(1/2 * S A C C + (1-1/2) * S A C B) +
(1/2*(1/2 * S C B C + (1-1/2) * S C B B) +
 1/2*(1/2 * S A B C + (1-1/2) * S A B B))= 0
```

basic_simpl.

```
==============================
 1/2*(1/2* S A C B) + 1/2*(1/2* S A B C) = 0
```

unify_signed_areas.

```
==============================
 1/2*(1/2* S A C B)+1/2*(1/2* - S A C B) = 0
```

field_and_conclude.

```
Proof completed.
```

What we learned

- We fixed some details about degenerated conditions.
- We clarified the use of classical logic

Example

Let Y on the line PQ such that $\frac{\overline{PY}}{\overline{PQ}} = \lambda$ $(P \neq Q)$.

$$\frac{\overline{AY}}{\overline{CD}} = \begin{cases} \frac{\frac{\overline{AP}}{\overline{PQ}}+\lambda}{\frac{\overline{CD}}{\overline{PQ}}} & \text{if } A \in PQ \\ \frac{S_{APQ}}{S_{CPDQ}} & \text{otherwise.} \end{cases}$$

If $A = Y$ it can happens that $CD \nparallel PQ$.

We need to perform a case distinction using classical logic.

Benchmarks

Some examples

- Ceva
- Menelaus
- Pascal
- Pappus
- Desargues
- Centroïd
- Gauss-Line

$>$ 40 examples

average time : 9 seconds

GeoProof combines these features:

- dynamic geometry
- automatic theorem proving
- interactive theorem proving (using Coq/CoqIDE)

Motivations

- The use of a proof assistant provides a way to combine geometrical proofs with larger proofs (involving induction for instance).
- There are facts than can not be visualized graphically and there are facts that are difficult to understand without being visualized.
- We should have both the ability to make arbitrarily complex proofs and use a base of known lemmas.
- The verification of the proofs by the proof assistant provides a very high level of confidence.

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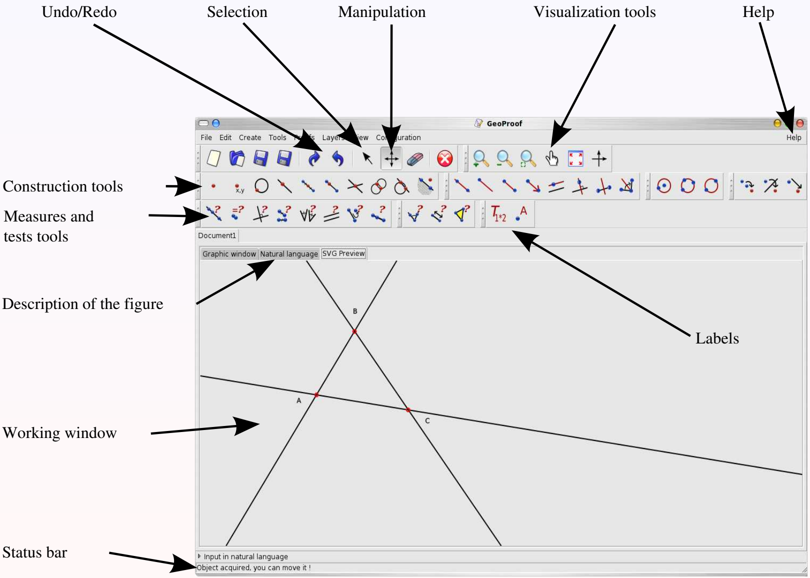
Overview of GeoProof



Dynamic geometry features

- points, lines, circles, vectors, segments, intersections, perpendicular lines, perpendicular bisectors,angle bisectors. . .
- central symmetry, translation and axial symmetry
- traces
- text labels with dynamic parts:
  - measures of angles, distances and areas
  - properties tests (collinearity,orthogonality,. . .)

- **layers**
- Computations use **arbitrary precision**
- Input: XML
- Output: XML, natural language, SVG, PNG, BMP, Eukleides (**latex**), **Coq**

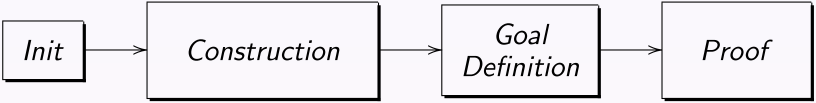
Missing features:

- loci and conics
- macros
- animations

Proof related features

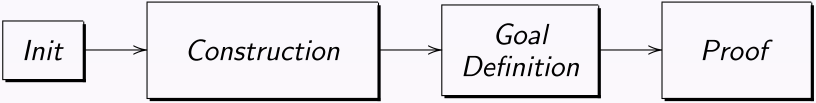
1. Automatic proof using an embedded ATP
2. Automatic proof using Coq
3. Interactive proof using Coq

Interactive proof using Coq



- GeoProof loads the library (Guilhot or Narboux) and updates the interface.
- The user performs the construction.
- It translates each construction as an hypothesis in Coq syntax.
- It translates the conjecture into Coq syntax.
- It translates each construction into the application of a tactic to prove the existence of the newly introduced object.

Interactive proof using Coq



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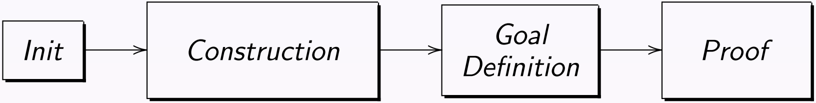

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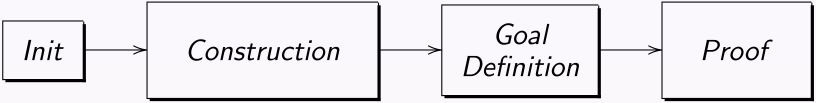

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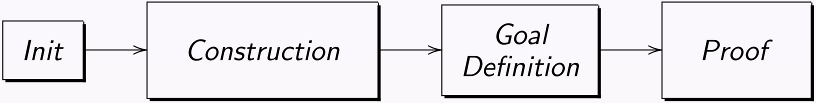

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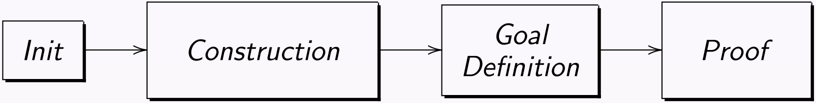

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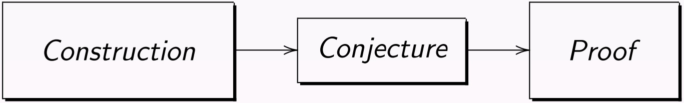
Typical use



- We want to extend GeoProof to perform proof in different domains,
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Running Example

Definition

The composition of two relations $\xrightarrow{a}$ and $\xrightarrow{b}$ is defined by:

$$\forall xy, x \xrightarrow{a.b} y \iff \exists z, x \xrightarrow{a} z \xrightarrow{b} y$$

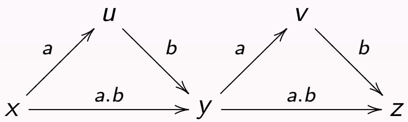
Example

If $\xrightarrow{a}$ and $\xrightarrow{b}$ are transitive and $\xrightarrow{b.a} \subseteq \xrightarrow{a.b}$ then $\xrightarrow{a.b}$ is transitive.

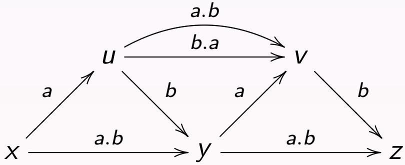
Running example

$$x \xrightarrow{a.b} y \xrightarrow{a.b} z$$

Running example



Running example



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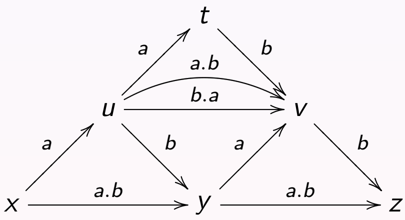

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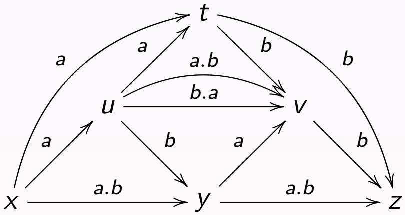

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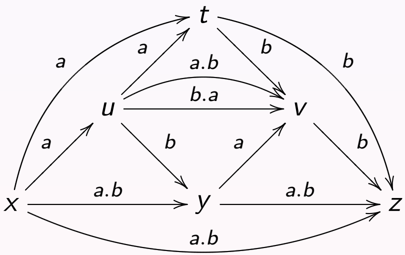

Diagrams as proofs

Diagrams can be seen as proofs hints.

Diagrams as proofs

Diagrams can be seen as proofs ~~hints~~ objects.

Diagrams

Diagrams can be defined by labeled oriented graphs verifying some properties.

Diagrammatic formulas

Formulas which can be represented by a diagram are those of the form:

$$\forall\, \overline{u} \bigwedge_i H_i \Rightarrow \bigvee_i \exists\, \overline{e_i} \bigwedge_j C_{i_j}$$

where H_i and C_{i_j} are predicates of arity two.

This class of formulas is exactly what is called coherent logic by Marc Bezem and Thierry Coquand.

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Inference rules

The system contains five rules of inference:

- `intros` to introduce hypotheses in the context,
- `apply` to use the information contained in a universal diagram to enrich the factual diagram,
- `conclusion` to conclude when the factual diagram contains enough information,
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Correctness and completeness

Intuitionist vs classical logic

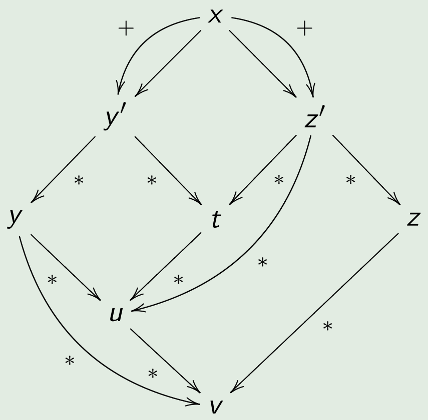
For the class of formulas considered intuitionist and classical provability coincide.

Theorem

The system is correct and complete for the coherent logic (restrained to predicate of arity two).

The system can be extended to deal with well founded induction.

Newman's lemma



A better understanding of diagrammatic reasoning

To have a diagrammatic proof system we need:

1. Visualization by a syntax that mimic the semantic.
2. An inference system which is complete and does not change the conclusion.

intro apply* conclusion

Conclusion

- Foundational work about the formalization of geometry.
- Automation of affine geometry, clarification of the role of classical logic and correction of some proofs.
- A user interface: GeoProof.
- Formalization of diagrammatic proof in abstract rewriting.

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Perspectives

- Formalize other ATP methods (Wu. . .).
- Adapt GeoProof to the education.
- Toward a diagrammatic logic (category theory, projective geometry, . . .).

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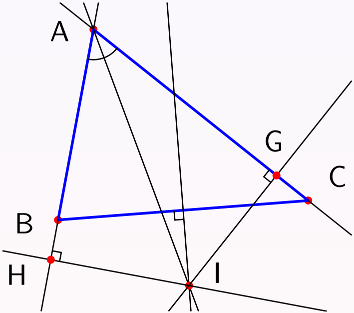
Solution

- Let ABC be a triangle.
- Let D be the perpendicular bisector of $[BC]$ and let D' be the bisector of $\angle BAC$.
- Let I be the intersection of D and D'.
- $HI = IG \wedge AH = AG$
- $IB = IC$
- $HB = GC$
- ~~$AB = AC$~~



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