Project 4 : Impossibility of climbing the wall

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Abstract

This report aims at proving that we cannot climb a wall of any height h and reach a target in any cases in a finite number of steps with a finite st of tile for the aTAM model.

1 Definitions

We first introduce some definitions and notation we use in the following.

Definition 1.1 (Tile-Kind). *A Tile-Kind t is a quadruplet $t = (t_N, t_E, t_S, t_W)$ where all t_i are pairs $t_i = (c_{t,i}, s_{t,i})$ where $c_{t,i} \in \mathbb{N}$ and $s_{t,i} \in \mathbb{N}$ are respectively the colour and the strength of the link in position i.*

Definition 1.2 (Tile). *Let $\mathcal{T}$ be a set of tile-kinds. A tile t over $\mathcal{T}$ is a pair $t = (k, p)$ where $k \in \mathcal{T}$ and $p \in \mathbb{Z} \times \mathbb{N}$.*

We define a relation $\rightarrow_{\mathcal{C}}$ over a set of tiles $\mathcal{C}$. We say $(t, (p_0, p_1)) \rightarrow_{\mathcal{C}} (t', (p'_0, p'_1))$ if and only if at least one of those statement is true:

- $c_{t,N} = c_{t',S}$ and $s_{t,N} = s_{t',S}$ and $p_0 = p'_0$ and $p_1 = p'_1 - 1$
- $c_{t,E} = c_{t',W}$ and $s_{t,E} = s_{t',W}$ and $p_0 = p'_0 - 1$ and $p_1 = p'_1$
- $c_{t,S} = c_{t',N}$ and $s_{t,S} = s_{t',N}$ and $p_0 = p'_0$ and $p_1 = p'_1 + 1$
- $c_{t,W} = c_{t',E}$ and $s_{t,W} = s_{t',E}$ and $p_0 = p'_0 + 1$ and $p_1 = p'_1$

One can notice that this relation is symmetric.
We denote $\rightarrow^*_{\mathcal{C}}$ the reflexive-transitive closure of $\rightarrow_{\mathcal{C}}$.

Definition 1.3 (Configuration). *Let $\mathcal{T}$ be a set of tile-kinds. A configuration $\mathcal{C}$ over $\mathcal{T}$ is a subset of $\mathcal{T} \times (\mathbb{Z} \times \mathbb{N})$ such that*

- $\exists t_0 \in \mathcal{T}, (t_0, (0,0)) \in \mathcal{C}$
- $\forall (t,p), (t',p') \in \mathcal{C}, p = p' \Rightarrow t = t'$
- $\forall (t,p) \in \mathcal{C}, \forall t' \in \mathcal{T}, (t', (0,0)) \in \mathcal{C}, (t', (0,0)) \rightarrow^*_{\mathcal{C}} (t,p)$

One can notice that $\exists! t_0 \in \mathcal{T}, (t, (0,0)) \in \mathcal{C}$. It is a direct consequence of the first two points. Let us call it the "seed" of the configuration $\mathcal{C}$. Then the last one says that the seed is always connected to each tile.

Definition 1.4 (Wall). *A τ-wall $\mathcal{W} \subset \mathcal{T} \times (\mathbb{Z} \times \mathbb{N})$ of height h is a set of tiles such that:*

- $\forall (i,j) \in [\![2; +\infty[\![\times [\![0; h]\!], (((0,0), (0,0), (0,0), (0,0)), (i,j)) \in \mathcal{W}$

- $\exists!(c_1,...,c_h) \in \mathbb{N}^h, \exists(s_1,...,s_h) \in \mathbb{N}^k, \forall i \in [\![1,h]\!], s_i < \tau, (((0,0),(0,0),(0,0),(c_i,s_i)),(1,i)) \in \mathcal{W}$

We define a τ-growing family of wall $\mathcal{W}_{\mathcal{F}} = \{\mathcal{W}_i\}_{i\in\mathbb{N}}$ where $\mathcal{W}_i$ is τ-wall of height i such that if $(c_1,...,c_h)$ and $(s_1,...,s_h)$ are the sequences corresponding to W_h then $(c_1,...,c_{h+1})$ and $(s_1,...,s_{h+1})$ are the sequences for W_{h+1}. We say $(c_i)_{i\in\mathbb{N}}$ and $(s_i)_{i\in\mathbb{N}}$ are the sequences for $\mathcal{W}_{\mathcal{F}}$.

We now give another relation $\perp_\tau$ between a set of tiles $\mathcal{F}$ and a tile (t,p). We say $\mathcal{F}\perp_\tau(t,p)$ if and only if:

- $\forall(t',p') \in \mathcal{F}, (t',p') \rightarrow_{\mathcal{F}\cup\{(t,p)\}} (t,p)$
- $(t,p) \notin \mathcal{F}$
- $\sum_{i\in I} s_{t,i} \geq \tau$ where $N \in I \Leftrightarrow \exists(t',(p'_0,p'_1)) \in \mathcal{F}, p_0 = p'_0 \wedge p_1 = p'_1 - 1$, idem for W E and S.

Definition 1.5 (Execution)**.** *Here we define an $(\mathcal{T},\tau,\mathcal{C},\sigma,\mathcal{W})$-execution $\mathcal{E}$, given a set of tile-kinds $\mathcal{T}$, a configuration $\mathcal{C}$ over $\mathcal{T}$, $\tau \in \mathbb{N}$, $\sigma \in \mathcal{T}$ which is the seed of $\mathcal{C}$ and a τ-wall $\mathcal{W}$ such that $\mathcal{C} \cap \mathcal{W} = \emptyset$.*

An $(\mathcal{T},\tau,\mathcal{C},\sigma,\mathcal{W})$-execution $\mathcal{E}$ is a sequence of length $|\mathcal{E}| \in \mathbb{N}\cup\{\infty\}$, $\mathcal{E} = (\mathcal{E}_i)_{i\in[\![0;|\mathcal{E}|[\![}$ such that:

- $\forall i,j \in [\![0;|\mathcal{E}|[\![, i \neq j \Rightarrow \mathcal{E}_i \cap \mathcal{E}_j = \emptyset$
- $\forall i,j \in [\![0;|\mathcal{E}|[\![, \mathcal{E}_i \cap \mathcal{W} = \emptyset$
- $\cup_{i\in[\![0;|\mathcal{E}|[\![}\mathcal{E}_i = \mathcal{C}$
- $\forall i \in [\![1;|\mathcal{E}|[\![, \forall(t,p) \in \mathcal{E}_i, \exists\mathcal{F} \subset \cup_{j=0}^{i-1}\mathcal{E}_j \cup \mathcal{W}, \mathcal{F}\perp_\tau(t,p)$

Definition 1.6 (Valid execution)**.** *A valid $(\mathcal{T},\tau,\mathcal{C},\sigma,\mathcal{W})$-execution is an $(\mathcal{T},\tau,\mathcal{C},\sigma,\mathcal{W})$-execution $\mathcal{E}$ such that $|\mathcal{E}| < \infty$ and $\exists t \in \mathcal{T}, (t,(10,h)) \in \mathcal{C}$ where h is the height of W*

Definition 1.7 (Ended execution)**.** *An ended $(\mathcal{T},\tau,\mathcal{C},\sigma,\mathcal{W})$-execution is an is an $(\mathcal{T},\tau,\mathcal{C},\sigma,\mathcal{W})$-execution $\mathcal{E}$ such that $\forall(t,p) \in \mathcal{C}, \forall t' \in \mathcal{T}, \forall p' \in \mathbb{N}_- \times \mathbb{N} \cup [\![1;\infty[\![\times[\![h;\infty[\![, (t',p') \in \mathcal{C} \vee \neg((t,p) \rightarrow_{\mathcal{T}} (t',p'))$*

One must notice that this definition is valid for an infinite or finite execution and that we can not add a new tile.

Let $\mathcal{W}_{\mathcal{F},c,s}$ be a family of wall for the sequence $c = (c_i)_{i\in\mathbb{N}}$ and $s = (s_i)_{i\in\mathbb{N}}$.

Now we can precise our goal. In fact we want some (T,τ,σ,c,s) such that for each ended $(\mathcal{T},\tau,\mathcal{C},\sigma,\mathcal{W})$-execution where $\mathcal{W} \in \mathcal{W}_{\mathcal{F},c,s}$ for some $\mathcal{C}$ is finite and valid.

2 Main Theorem

2.1 Preliminaries

Lemma 2.1 (Ending finite executions)**.** *Let $\mathcal{E}$ be a finite $(\mathcal{T},\tau,\mathcal{C},\sigma,\mathcal{W})$-execution $\mathcal{E} = (\mathcal{E}_0,...\mathcal{E}_{|\mathcal{E}|-1})$. Then there is some $(\mathcal{T},\tau,\mathcal{C}',\sigma,\mathcal{W})$-execution $\mathcal{E}' = (\mathcal{E}'_i)_{i\in[\![0;|\mathcal{E}'|[\![}$ such that $\forall i \in [\![0;|\mathcal{E}|[\![, \mathcal{E}_i = \mathcal{E}'_i$ (we say $\mathcal{E} \preceq \mathcal{E}'$) and $\mathcal{E}'$ is ended and $\mathcal{C} \subseteq \mathcal{C}'$.*

This lemma says that given a finite execution, we can continue until we reach an ended execution. It seems evident and the proof is simple. Other things is that $\preceq$ is a (partial) large order over executions.

Proof. Let $\mathcal{E}$ be a finite $(\mathcal{T},\tau,\mathcal{C},\sigma,\mathcal{W})$-execution $\mathcal{E} = (\mathcal{E}_0,...\mathcal{E}_{|\mathcal{E}|-1})$. If $\mathcal{E}$ is ended then there is nothing to do. Otherwise, let $\mathcal{E}^{(0)} = \mathcal{E}$. Suppose $\mathcal{E}^{(n)}$ is built for some $n \in \mathbb{N}$ such that $\mathcal{E}^{(0)} \preceq ... \preceq \mathcal{E}^{(n)}$. We construct $\mathcal{E}^{(n+1)}$. If $\mathcal{E}^{(n)}$ is ended let $\mathcal{E}^{(n+1)} = \mathcal{E}^{(n)}$. Otherwise $\exists(t,p) \in \mathcal{C}^{(n)}, \exists t' \in \mathcal{T}, \exists p' \in \mathbb{N}_- \times \mathbb{N} \cup [\![1;\infty[\![\times[\![h;\infty[\![, (t',p') \notin \mathcal{C}^{(n)} \wedge (t,p) \rightarrow_{\mathcal{T}} (t',p')$. Set $\mathcal{E}_{|\mathcal{E}^{(n)}|} = \{(t',p')\}$ and $\mathcal{E}^{(n+1)} = (\mathcal{E}_0,...,\mathcal{E}_{|\mathcal{E}^{(n)}|})$ to get the result. ☐

Corollary 2.2 (Ending infinite executions)**.** *Let $\mathcal{E}$ be an infinite $(\mathcal{T},\tau,\mathcal{C},\sigma,\mathcal{W})$-execution $\mathcal{E} = (\mathcal{E}_i)_{i\in\mathbb{N}}$. Then there is some $(\mathcal{T},\tau,\mathcal{C}',\sigma,\mathcal{W})$-execution $\mathcal{E}' = (\mathcal{E}'_i)_{i\in\mathbb{N}}$ such that $\forall i \in \mathbb{N}, \mathcal{E}_i \subseteq \mathcal{E}'_i$ (we say $\mathcal{E} \preceq_\infty \mathcal{E}'$) and $\mathcal{E}'$ is ended and $\mathcal{C} \subseteq \mathcal{C}'$.*

The proof is very similar to the previous one. Just say if you can add a tile there is some finite rank i in $\mathcal{E}$ such that you can add this tile, then add it to $\mathcal{E}_{i+1}$. You build an increasing sequence of infinite executions and take the limit to end it.

Lemma 2.3 (Intermediate Tile)**.** *Let $\mathcal{C}$ be a configuration. Let $(t,p),(t',p') \in \mathcal{C}, p <_{lex} p', (t,p) \to_{\mathcal{C}}^{*} (t',p') \wedge \neg((t,p) \to_{\mathcal{C}} (t',p'))$. Then $\exists (t'',p'') \in \mathcal{C}, (t,p) \to_{\mathcal{C}}^{+} (t'',p'') \wedge (t'',p'') \to_{\mathcal{C}}^{+} (t',p') \wedge p <_{lex} p'' <_{lex} p'$.*

Proof. By induction on the length n of the path going from (t,p) to (t',p') with $p = (p_0, p_1)$, $p' = (p'_0, p'_1)$, assuming that $(t,p) \to_{\mathcal{C}}^{n} (t',p')$. We assume without loss of generality that $p_0 \leq p'_0$ and $p_1 \leq p'_1$ by permutation of the coordinates.

If $n = 2$, then

- If $p_0 = p'_0$, the only path is (t'',p'') with $p''_0 = p_0 + 1$, $p''_1 = p_1$.
- If $p_0 = p'_0 - 1$, $p_1 < p'_1$, and two only paths are either $(t'',(p_0,p'_1))$ and $(t'',(p'_0,p_1))$; both verifying the theorem.
- If $p_0 = p'_0 - 2$, the only path is (t'',p'') with $p''_0 = p_0$, $p''_1 = p_1 + 1$.
- Else, there cannot be any path of length 2 from t to t'.

Let us assume that for all $(t,p) \to_{\mathcal{C}}^{n} (t',p')$, there exists such (t',p''). Let $(t^{(i)},p^{(i)})_{i \leq n+1}$ verify $(t^{(0)},p^{(0)}) = (t,p)$, $(t^{(n+1)},p^{(n+1)}) = (t',p')$ and $\forall i \in [\![0,n]\!], (t^{(i)},p^{(i)}) \to_{\mathcal{C}} (t^{(i+1)},p^{(i+1)})$. Let us consider $(t^{(n)},p^{(n)})$.

- If $p_0^{(n)} < p'_0$ or $p_0^{(n)} = p'_0$ and $p_1^{(n)} < p'_1$, then $p^{(n)} <_{lex} p'$ and by hypothesis, there exists n_0 such that $p <_{leq} p^{(n_0)} <_{leq} p^{(n)} <_{leq} p$
- If $p_0^{(n)} = p'_0$ and $p_1^{(n)} > p'_1$, then, by contradiction, there exist such (t'',p''): Assuming that $\forall k, p^{(k)} >_{lex} p^{(n)}$, then either $p_0^{(0)} = p_0^{(n)}$ and $p_1 > p_1^{(0)}$ or $p_0^{(0)} > p_0^{(n)}$ which contradicts our hypothesis.
- If $p_0^{(n)} > p'_0$, the same reasoning holds.

□

Corollary 2.4 (Path through each abscissa)**.** *Let $\mathcal{C}$ be a configuration over $\mathcal{T}$. Let $(t,p),(t',p') \in \mathcal{C}, p_0 + 1 < p'_0, (t,p) \to_{\mathcal{C}}^{*} (t',p') \wedge \neg((t,p) \to_{\mathcal{C}} (t',p'))$ Then $\forall p''_0 \in]\!]p_0; p'_0[\![\exists p''_1 \in \mathbb{N} \exists t'' \in \mathcal{T}, (t'',(p''_0,p''_1)) \in \mathcal{C} \wedge (t,p) \to_{\mathcal{C}}^{+} (t'',(p''_0,p''_1)) \wedge (t'',(p''_0,p''_1)) \to_{\mathcal{C}}^{+} (t',p')$.*

Proof. The proofs directly comes from the previous lemma 2.3 by supposing it is false and induct it on the number of columns that are in between the two considered tile. □

2.2 Main Theorem

Theorem 2.5 (Main theorem)**.** *There is no (T,τ,σ,c,s) such that for each ended $(\mathcal{T},\tau,\mathcal{C},\sigma,\mathcal{W})$-execution where $\mathcal{W} \in \mathcal{W}_{\mathcal{F},c,s}$ for some $\mathcal{C}$ is finite and valid.*

Proof. We proceed by contradiction. Suppose given such a quintuplet (T,τ,σ,c,s). Let $h_0 \in \mathbb{N}$ greater than (or equal to) 1 and $\mathcal{W}_{h_0} \in \mathcal{W}_{\mathcal{F},c,s}$ the wall of height h_0. $(\{\sigma\})$ is an execution (see Figure 1). Using lemma 2.1 we can complete it in a ended execution $\mathcal{E}^{(0)}$. By our hypothesis, $\mathcal{E}^{(0)} = (\mathcal{E}_0^{(0)}, ..., \mathcal{E}_{|\mathcal{E}^{(0)}|-1}^{(0)})$ is finite and valid. Let $\mathcal{C}^{(0)}$ be the configuration of $\mathcal{E}^{(0)}$ as a $(\mathcal{T},\tau,\mathcal{C}^{(0)},\sigma,\mathcal{W}_{h_0})$-execution. Then $\exists t \in \mathcal{T}, (\sigma,(0,0)) \to_{\mathcal{C}}^{*} (t,(10,h))$. We use corollary 2.4 to say that there some tile in $\mathcal{C}^{(0)}$ such that its abscissa is 1. We now define i_0:

$$i_0 = \min\{i | \exists t \in \mathcal{T}, \exists p_1 \in \mathbb{N}(t,(1,p_1)) \in \mathcal{E}_i\}$$

We have $1 \leq i_0 \leq |\mathcal{E}^{(0)}|$. Now consider $\mathcal{E}^{(0)}_{|i_0-1} = (\mathcal{E}^{(0)}_0, ..., \mathcal{E}^{(0)}_{i_0-1})$. $\mathcal{E}^{(0)}_{|i_0-1}$ is a non-ended $(\mathcal{T}, \tau, \cup_{j=0}^{i_0-1} \mathcal{E}^{(0)}_j, \sigma, \mathcal{W}_{h_0})$-execution (see Figure 2). Let h_1 such that:

$$h_1 = \max\{h \in \mathbb{N} | h \geq h_0, \exists t \in \mathcal{T}, (t, (0, h)) \in \mathcal{C}^{(0)}\}$$

Because for all $t \in \mathcal{T}$ and $h \geq h_0$ such that $(t, (0, h) \in \mathcal{C}$ has been added with out any wall, and because mismatches are allowed, $\mathcal{E}^{(0)}_{|i_0-1}$ is a finite non-ended $(\mathcal{T}, \tau, \sigma, \cup_{j=0}^{i_0-1} \mathcal{E}^{(0)}_j, \sigma, \mathcal{W}_{h_1})$-execution. Using lemma 2.1 we can complete it in an $(\mathcal{T}, \tau, \mathcal{C}^{(1)}, \sigma, \mathcal{W}_{h_1})$-execution $\mathcal{E}^{(1)} = (\mathcal{E}^{(1)}_0, ..., \mathcal{E}^{(1)}_{|\mathcal{E}^{(1)}|-1})$. More-over suppose that we cannot get one without using the blocks of the wall, then there is some tile is $\mathcal{E}^{(1)}_i$ for some $i \geq i_0$ that has height lower than h_1 and abscissa 0. Then as well as corollary 2.4, one can show that the path can go trough each ordinate. Then because we work in $\mathbb{N}^2$ each path starting from one of those new tiles going to target cuts a path the seed to the tile achieving previous h_1. The tiles are then "useless", we mean that the cannot imply any nex path going above the wall (all possible neighbours are already there). We can assume there is no new tile in any position $(0, h)$ with $h < h_1$ in $\mathcal{E}^{(1)}$. Then no glue from $\mathcal{W}_{h_1}$ which is not in $\mathcal{W}_{h_0}$ is used to get further. It says that is we use the same notations as for $\mathcal{E}^{(0)}$, $\mathcal{E}^{(1)}_{|i_1-1}$ is an $(\mathcal{T}, \tau, \mathcal{C}^{(1)}, \sigma, \mathcal{W}_{h_0})$-execution (see Figure 3).

By recurrence, replace any 0 by n in the previous reasoning and any 1 by $n+1$ to get a family of $(\mathcal{T}, \tau, \_, \sigma, \mathcal{W}_{h_0})$-executions $\{\mathcal{E}^{(n)}\}_{n \in \mathbb{N}}$ and $(i_n)_{n \in \mathbb{N}}$ an increasing sequence such that $\forall n \in \mathbb{N}, \forall i \in [\![0; i_n[\![, \mathcal{E}^{(n)}_i = \mathcal{E}^{(n+1)}_i$.

Then define $\mathcal{E}^{(\infty)} = (\mathcal{E}^{(\infty)}_i)_{i \in \mathbb{N}}$ with $\mathcal{E}^{(\infty)}_i = \lim_{n \to +\infty} \mathcal{E}^{(n)}_i$ which is in fact just the limit of a stationary sequence (well defined). Then one can check that its satisfies the definition of a $(\mathcal{T}, \tau, \mathcal{C}^{(\infty)}, \sigma, \mathcal{W}_{h_0})$-execution.

$\mathcal{E}^{(\infty)}$ may be not ended. Up to renaming, consider it is (applying corollary 2.2). However it is not finite (and may be not valid), that is a contradiction. $\square$

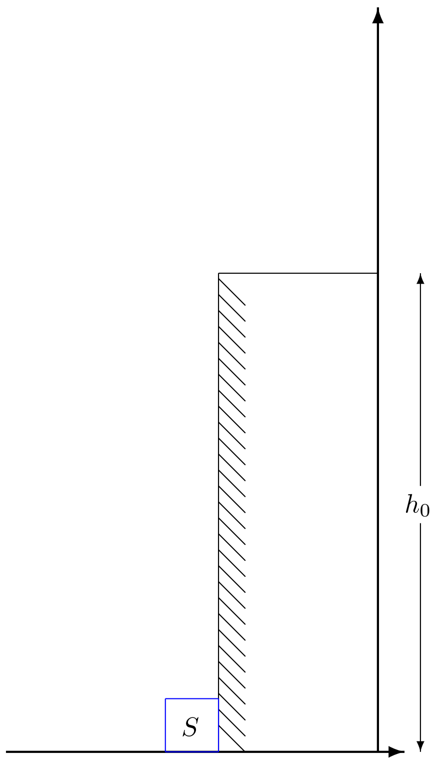

Figure 1: Initial state

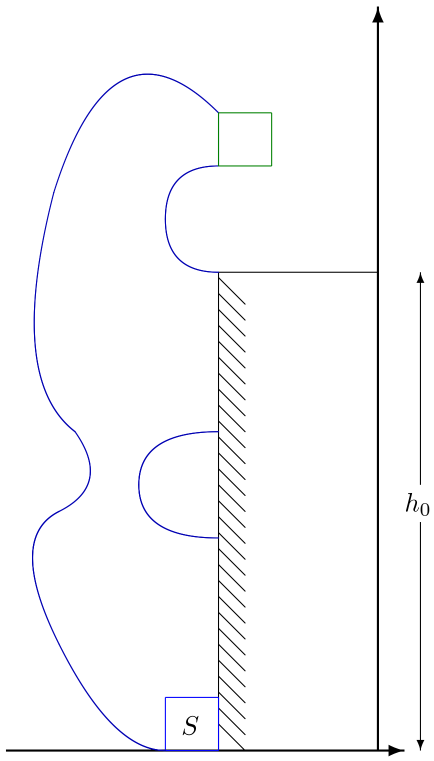

Figure 2: First growth of the tile algorithm

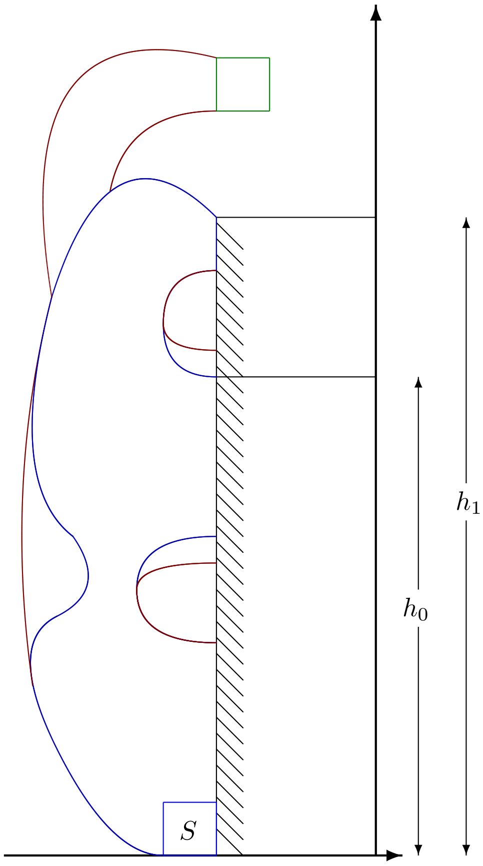

Figure 3: Second growth of the tile algorithm

2.3 Remarks

In fact we prove that it's only sufficient that there some wall $\mathcal{W}_0$ of height grater than (or equal to) 1, such that each wall of height greater than $\mathcal{W}_0$'s one, $\mathcal{W}_0$ is the base of this wall. We also need the fact that for each height, there is a wall for this height in the family.

More over we also prove that is a family of walls contains such a sub-family, it is impossible to find the correct $(\mathcal{T}, \tau, \sigma)$.