On the recognition of $\frac{1}{2}$-hyperbolic graphs: a subcubic equivalence with the $C_4$-free graph recognition problem

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Understanding Graph Hyperbolicity

- graphs in this study are:
  - undirected;
  - simple;
  - unweighted;
  - and connected.

**distance** = length of a shortest-path  
(an edge = a path of length 1)
A tree-like parameter

- which (maximum) gap between two $u, v$-shortest paths?

- Tree: Unique shortest-path $\rightarrow$ hyperbolicity 0
- Conversely:
  $0$-hyperbolic graph $\rightarrow$ (weighted) tree metric
Definition (4-points Condition, [Gromov, 1987])

Say that graph $G$ is $\delta$-hyperbolic if, for every 4-tuple $a, b, c, d$ of $V$, the two largest sums amongst

$S_1 = d(a, b) + d(c, d)$,
$S_2 = d(a, c) + d(b, d)$, and
$S_3 = d(a, d) + d(b, c)$ differ by at most $2\delta$.

The graph hyperbolicity, denoted $\delta(G)$, is the least $\delta$ such that $G$ is $\delta$-hyperbolic.

- Other "equivalent" definitions
  ($\delta$-slim triangles, Gromov product, . . . ).
Some applications

- network security
- bioinformatics
- routing

→ [Kleinberg, 2007] embedding into a hyperbolic space
The resulting metric has to stay close to the metric of the graph.
Can we compute the hyperbolicity value efficiently?

State of the art:
- naive enumeration of 4-tuples ($\Theta(n^4)$);
  [Fournier et al., 2012]
- reduction to (min, max) matrix product ($O(n^{3.69})$);
  [Coudert et al., 2012]
- sorting of 2-tuples + breaking rule ($O(n^4)$).
Graphs with small hyperbolicity value

- Motivation: complex networks often have a bounded hyperbolicity value.


- [Bandelt et Chepoi, 2003] $\frac{1}{2}$-hyperbolic graphs: Forbidden (isometric) subgraphs

  - no cycle with length $\neq 3, 5 + six\ other\ subgraphs.$
Our main contribution

The complexity of deciding if a graph is $\frac{1}{2}$-hyperbolic is equivalent to the complexity of finding an (induced) $C_4$ in a graph.

- both problems are reducible to fast rectangular matrix multiplication ($O(n^{\omega(1,1,\log n,m)}) = O(n^{3.333953})$).
$C_4$-free detection $\propto \frac{1}{2}$-hyperbolic recognition

- addition of a universal vertex
  $\rightarrow$ linear-time reduction

(using [Bandelt et Chepoi, 2003]) $C_4$ is the only forbidden subgraph in such case.
- Quickly excluding large (isometric) cycles:
  using a constant-factor approximation for hyperbolicity

  - $O(n^{2.69})$ [Fournier et al., 2012];
  - $\tilde{O}(n^{2.37})$ [Chalopin et al., 2013].

Given a $c$-factor approximation, denoted $\delta_c(G)$:

- If $\delta_c(G) > \frac{c}{2}$, then $G$ is not $\frac{1}{2}$-hyperbolic;
- Otherwise, $G$ does not contain isometric cycles with length $\geq \Theta(c)$.
- Excluding the isometric cycles with length bounded by $O(c)$: using graph powers $G = G^1, G^2, G^3, \ldots, G^{O(c)}$.

an edge in $G^i$ = a path of length (at most) $i$ in $G$.

$\rightarrow$ computable in time $\tilde{O}(n^{2.37})$

![Diagram](attachment:image.png)

isometric $C_{4p+2}$ in $G$

induced $C_4$ in $G^{2p}$

- a simple computation shows that every $G^i$ has to be $C_4$-free so that $G$ is $\frac{1}{2}$-hyperbolic.
- Excluding the forbidden (isometric) subgraphs: using modified graph powers $G^{[2]}, \ldots, G^{[O(c)]}$.

We use the characterization of [Bandelt et Chepoi, 2003] for the correctness.
• There is a linear-time reduction from the $C_4$-free graph recognition problem to the $\frac{1}{2}$-hyperbolic graph recognition problem.

• There is a subcubic-time reduction from the $\frac{1}{2}$-hyperbolic graph recognition problem to the $C_4$-free graph recognition problem.
Finding an induced $C_4$

- Basic idea: $G$ is $C_4$-free if, and only if, for every pair $u, v \in V$ at a distance $uv = 2$ we have $N(u) \cap N(v)$ that is a clique.

This can be tested using two matrix multiplications:

- If $A$ is the adjacency matrix of $G$, then $A^2_{u,v} = |N(u) \cap N(v)|$;
- If $T$ is s.t. $T_{u,e} = \delta_{e \in N(u)}$, then $TT^T_{u,v} = |\{e \in E : e \subseteq N(u) \cap N(v)\}|$;
- One has to check for every pair $u, v \in V$ at a distance 2 that we have $TT^T_{u,v} = \frac{A^2_{u,v}(A^2_{u,v} - 1)}{2}$.

The time complexity is $O(n^{\omega(1,1,\log n,m)})$ (using [Huang et al., 1998]).
Conclusion and Perspectives

- Our results:
  - A subcubic equivalence between a *metric* problem (the recognition of a $\frac{1}{2}$-hyperbolic graph) and a *structural* problem (e.g. the detection of a $C_4$)
  - An improved algorithm in time $O(n^{\omega(1,1,\log n, m)}) = O(n^{3.333953}) = o(n^{3.69})$

- Future work: extending our results to 1-hyperbolic graphs
  $\rightarrow$ this would yield a 4-factor approximation for the hyperbolicity
[Bandelt et Chepoi, 2003]

[Bandelt et Mulder, 1986]

[Chalopin et al., 2013]

[Coudert et al., 2012]
[Fournier et al., 2012]  

[Gromov, 1987]  

[Huang et al., 1998]  

[Kleinberg, 2007]  
Questions