Exercise 1 (FPT algorithm for spotting $k$ disjoint triangles).  

Given $G = (V, E)$ an undirected graph ($n = |V|$ and $m = |E|$) and $k$ an integer, we are looking for $k$ vertex-disjoint triangles in $G$. Note that this problem is $NP$-complete when $k$ is part of the input. We are looking for an algorithm of time complexity $O(f(k)n^a m^b)$ where the exponents $a$ and $b$ are constant, independent of $k$. Such an algorithm is called FPT for Fixed Parameter Tractable, which means that the complexity is a fixed-degree polynomial in the size of the input for any fixed value of the parameter $k$. Consider the following randomized algorithm:

**Algorithm 1** FPT randomized algorithm for $k$ disjoint triangles

- Choose independently for each vertex $u$ a color $c_u \in \{1, \ldots, 3k\}$ uniformly at random.
- return "Yes" if there is a colorful solution, i.e. a set of $k$ triangles whose $3k$ vertices use exactly once each color; return "I don’t know" otherwise.

**Question 1.1** Show that if $G$ contains $k$ disjoint triangles, then the probability that the algorithm answer "Yes" is at least $e^{-3k}$.

**Hint.** Use that $k! \geq (k/e)^k$ for all $k$.

**Question 1.2** How many times should you run this algorithm to improve success probability to $1/2$?

In order to check whether a colorful solution exists, we propose to try all permutations $\pi$ on $\{1, \ldots, 3k\}$ and check if there is any triangles of colors $(\pi_1, \pi_2, \pi_3), \ldots, (\pi_{3k-2}, \pi_{3k-1}, \pi_{3k})$.

**Question 1.3** Describe an algorithm that decides if such a collection of triangles exists. (Explicit the exact data structure you are using). What is the overall expected time complexity in $k$, $n$ and $m$, of the algorithm that uses this method to return $k$ disjoint triangles with probability at least $1/2$ if they exists in $G$? What is the time complexity if $k$ is fixed?