Exercise 1 (Let’s try random and derandomize). We consider the $NP$-complete Max-Cut problem: Given an undirected graph $G = (V, E)$, find a cut $C = (S, V \setminus S)$ of maximum size, where:

$$\text{size}(C) = \# \{uv \in E : u \in S \text{ and } v \notin S \}.$$ 

**Question 1.1**  Prove that outputting a random cut is a randomized $\frac{1}{2}$-approximation for the Max-Cut problem. Which upper bound on $OPT$ did you use? Exhibit a family of tight instances.

We now want to derandomize this algorithm by making the “best random choice” for each vertex. We consider each vertex, one after the other in an arbitrary order $u_1, \ldots, u_n$, and place each vertex on the side of the cut which maximizes the expected number of edges in the cut given the choices already made and assuming the next choices are random.

**Question 1.2**  Let $A_i$ and $B_i$ denote the left and right sides of the cut $C_i$ obtained after inserting $u_1, \ldots, u_i$. Show that:

$$\mathbb{E}[\text{size}(C_{i+1}) \mid A_i, B_i, u_{i+1} \in A_{i+1}] - \mathbb{E}[\text{size}(C_{i+1}) \mid A_i, B_i, u_{i+1} \in B_{i+1}] = \deg(u_{i+1}, B_i) - \deg(u_{i+1}, A_i),$$

where $\deg(u, A) = \# \{v \in A : uv \in E \}$.

**Question 1.3**  Conclude that the greedy algorithm that puts each vertex on the side to which it has the less connections is a deterministic $\frac{1}{2}$-approximation, as it is a derandomized version of the random cut algorithm (this derandomization scheme is called the conditional expectation method).

**Question 1.4**  Give a direct analysis for the approximation ratio of this greedy algorithm.

Exercise 2 (Coloring $3$-colorable graphs). Consider the following problem: Given an undirected graph $G = (V, E)$, color its vertices with the minimum number of colors so that the two endpoints of each edge receive distinct colors.

**Question 2.1**  Give a greedy algorithm for coloring $G$ with $\Delta + 1$ colors, where $\Delta$ is the maximum degree of a vertex in $G$.

**Question 2.2**  Give an algorithm for coloring a $3$-colorable graph with $O(\sqrt{n})$ colors.

**Hint.** Prove that for any vertex $v$, the induced subgraph on its neighbors is bipartite, and hence optimally colorable. How many colors do you need to color a vertex of degree $> \sqrt{n}$? How many colors do you need to color all the vertices with degree $\leq \sqrt{n}$?
Exercise 3 (Cost-based approximation for Vertex Cover). [★] Recall the Weighted Vertex Cover problem: Given a undirected graph \( G = (V, E, w) \) with weights \( w : V \to \mathbb{R}_+ \) on its vertices, find a set \( C \subseteq V \) of minimum weight \( w(C) = \sum_{v \in C} w_v \) that covers every edge in \( G \), i.e. such that \((\forall uv \in E) \ u \in C \) or \( v \in C \).

We consider the following algorithm. For each vertex \( v \), a variable \( t_v \) is initialized to its weight, and when \( t_v \) drops to 0, \( v \) is picked in the cover. \( c_e \) is the amount charged to edge \( e \).

Initialization:
- \( C \leftarrow \emptyset \)
- \( \forall v \in V, \ t_v := w_v \)
- \( \forall e \in E, \ c_e := 0 \)

while \( C \) is not a vertex cover do
  Pick an uncovered edge, say \( uv \). Let \( m = \min(t_u, t_v) \).
  \( t_u := t_u - m \)
  \( t_v := t_v - m \)
  \( c_{uv} := m \)
  Include in \( C \) all vertices having \( t_v = 0 \).
Output \( C \).

\[ \textbf{Question 3.1)} \] Show that this is a factor 2 approximation algorithm.
\ (> Hint. Show that the total amount charged to edges is a lower bound on OPT and that the weight of cover \( C \) is at most twice the total amount charged to edges. \)

\[ \textbf{Question 3.2)} \] Exhibit a family of tight instances.