You are asked to complete the exercise marked with a [★] and to send me your solutions at: nicolas.schabanel@cnrs.fr (or drop it in my mail box on the 4th floor) on Wed. 28/09 before 16:15.

Exercise 1 (Dumb algorithm for Max-SAT). Consider the following algorithm for Max-SAT:
Let \( \tau \) be an arbitrary instantiation of the variables and \( \tau' \) the complementary instantiation (each variable is true in \( \tau' \) if and only if it is false in \( \tau \) ); compute the total weight of the clauses satisfied for \( \tau \) and for by \( \tau' \) and output the best of these two instantiations.

▶ Question 1.1) Show that this algorithm is a \( \frac{1}{2} \)-approximation for Max-SAT, and exhibit a tight instances family.

Exercise 2 (Metric \( k \)-Center). [★] Consider the Metric \( k \)-Center problem: Given a complete undirected graph \( G = (V, E) \) with edge costs \( w \) satisfying the triangle inequality, and \( k \) be a positive integer, let us define for each vertex \( v \in V \) and each vertex set \( S \subseteq V \), connect \( (v; S) \) to be the cost of the cheapest edge from \( v \) to a vertex in \( S \) (connect \( (v; S) = 0 \) if \( v \not\in S \)). The goal is to find a set \( S \subseteq V \), with \( |S| = k \), so as to minimize \( \max_v \{ connect(v, S) \} \), that is to say minimizing the maximum distance from any vertex to \( S \). Let \( OPT \) denote the optimum cost of the \( k \)-center.

Note that if the optimal solution costs \( t \) then one may only consider edges with weight \( \leq t \). The idea is then to sort the edges in non-decreasing order: \( w(e_1) \leq \cdots \leq w(e_m) \) and then to study the graph \( G_i = (V, E_i) \) where \( E_i = \{ e_1, \ldots, e_i \} \).

We say that a set of vertices \( D \subseteq V \) is a dominant of a graph \( H = (V, F) \) if for all \( u \in V \), \( u \not\in D \Rightarrow (\exists v \in D) uv \in F \). We denote by \( \text{dom}(H) \) the minimum size of a dominant set in \( H \): \( \text{dom}(H) = \min \{|D| : D \text{ is a dominant set of } H\} \).

▶ Question 2.1) Let \( i^* = \min \{ i : \text{dom}(G_i) \leq k \} \). Show that the OPT = \( w(e_i^*) \).

Given a graph \( H = (V, F) \), the square graph \( H^2 \) of \( H \) is the graph that contains an edge \( uv \) for all pair of vertices \( u \) and \( v \) connected by a path of at most 2 edges in \( H \). We say that a set of vertices \( I \subseteq V \) is independent if for all \( u, v \in I \), \( uv \not\in F \). An independent set \( I \) is said maximal if for all \( u \in V \setminus I \), \( I \cup \{ u \} \) is not independent.

▶ Question 2.2) Propose an algorithm to compute a maximal independent set of a graph \( H \).

Independent set will allow us to compute a lower bound on OPT as follows:

▶ Question 2.3) Show that for all graph \( H \) and all independent set \( I \) in \( H^2 \), \( |I| \leq \text{dom}(H) \).

▶ Hint. What is the square of a star graph?

Let us now consider Algorithm [I] on the following page.

▶ Question 2.4) Show that Algorithm [I] is a 2-approximation for Metric \( k \)-Center. Exhibit a family of critical instances.
**Algorithm 1** 2-approximation for Metric k-Center

1. Compute $G_1^2, G_2^2, \ldots, G_m^2$
2. Compute a maximal independent set $I_i$ in each $G_i^2$.
3. Let $j = \min\{ i : |I_i| \leq k \}$.
4. Output $I_j$.

**Exercise 3 (SDP approximation for Weighted Max-2SAT).**

A weighted 2SAT formula $\varphi$ over $n$ boolean variables $x_1, \ldots, x_n$ consists in $m$ clauses $C_1, \ldots, C_m$ containing 1 or 2 literals each, together with a non-negative weight $w_1, \ldots, w_m$ for each clause. For instance, 

$$\varphi(x) = \{(C_1 = x_1 \lor x_2, w_1 = 1), (C_2 = \bar{x}_2 \lor x_3, w_2 = 2), (C_3 = x_1, w_3 = 4)\}.$$ 

Given a weighted 2SAT formula $\varphi(x)$, Weighted MAX-2SAT consists in finding an assignment for $x_1, \ldots, x_n$ which maximizes the total weight of the satisfied clauses:

$$\text{weight}(\varphi(x)) = \sum_{j : C_j(x) = 1} w_j.$$ 

In order to obtain a strictly quadratic formulation for this problem (i.e. with terms of degree only 0 or 2), let us introduce $(n + 1)\{-1, 1\}$-variables $y_0, y_1, \ldots, y_n$ such that $x_i = 1$ if and only if $y_i = y_0$.

**Question 3.1** Show that the value of every 1- or 2-clause can be expressed as a positive linear combination of terms of the form $(1 \pm y_i y_j)$ with $i, j \in \{0, 1, \ldots, n\}$. 

▷ **Hint.** $(\pm 1)^2 = 1$.

As a consequence, every instance of Weighted Max-2SAT is equivalent to a strictly quadratic program of the following form for some $(a_{ij})$ and $(b_{ij})$ in $\mathbb{R}^{(n+1)^2}$:

\[
\begin{align*}
\text{(QP)} & \quad \text{Maximize} \quad \sum_{0 \leq i < j \leq n} a_{ij}(1 + y_i y_j) + b_{ij}(1 - y_i y_j) \\
\text{subject to:} & \quad y_i^2 = 1 \quad (\forall 0 \leq i \leq n) \\
& \quad y_i \in \mathbb{Z} \quad (\forall 0 \leq i \leq n)
\end{align*}
\]

We now consider the following vector program:

\[
\begin{align*}
\text{(SDP)} & \quad \text{Maximize} \quad \sum_{0 \leq i < j \leq n} a_{ij}(1 + \langle v_i | v_j \rangle) + b_{ij}(1 - \langle v_i | v_j \rangle) \\
\text{subject to:} & \quad ||v_i||^2 = 1 \quad (\forall 0 \leq i \leq n) \\
& \quad v_i \in \mathbb{R}^{n+1} \quad (\forall 0 \leq i \leq n)
\end{align*}
\]

**Question 3.2** Show that (SDP) is a relaxation of (QP)

**Question 3.3** Propose a randomized rounding scheme for (SDP) to solve Weighted Max-2SAT approximately, and analyze it. Show that you can obtain $\alpha$-approximation for Weighted Max-2SAT where $\alpha = \min_{\theta \in [0,\pi]} \frac{2\theta / \pi}{1 - \cos \theta} = 0.87856 \ldots$

▷ **Hint.** Prove that $\alpha = \min_{\theta \in [0,\pi]} \frac{2(1 - \theta / \pi)}{1 + \cos \theta}$. 

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