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LAMA LABORATOIRE D'ANALYSE ET DE MATHÉMATIQUES APPLIQUÉES – UMR 8050

Badly Approximable Vectors and Littlewood-type Problems

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Outline

A wide–ranging theory

The theory of badly approximable vectors is deeply intertwined with many areas :

- Diophantine Approximation (Analytic, Algebraic and Metric Number Theory) ;
- Ergodic Theory ;
- Dynamical Systems ;
- Classical mechanics (perturbation theory and small divisor problems) ;
- Theory of mathematical quasicrystals (study of aperiodic order) ;
- Physical theory of crystal structures (e.g., structure of graphene) ;
- Fractal Geometry ;
- Convex Geometry ;
- Complexity Theory (computer science) ;
- Theory of Partial Differential Equations ;
- Theory of Signal Processing (Information Theory) ;
- ...

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- ...

A basic result

Theorem (Dirichlet, 1841)

For all $x \in \mathbb{R}$ and all $Q \geq 1$, there exist integers $p, q \in \mathbb{Z}$ such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{qQ} \qquad et \qquad 1 \leq q \leq Q.$$

In particular, if x is irrational, then there exist infinitely many fractions p/q such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}.$$

Figure – Johann Peter Gustav Lejeune Dirichlet (1805 – 1859)

The basic definition

Definition

A real number x is badly approximable is there exists a constant $c > 0$ such that for all p/q,

$$\left| x - \frac{p}{q} \right| > \frac{c}{q^2}.$$

- Equivalently, $x \in \mathbb{R}$ is badly approximable if

$$c := \inf_{\substack{q \geq 1 \\ p \in \mathbb{Z}}} q \cdot |qx - p| > 0 \qquad \text{i.e. si} \qquad c := \inf_{q \geq 1} q \cdot \langle qx \rangle > 0,$$

where

$$\langle \cdot \rangle = \mathrm{dist}\left(\cdot, \mathbb{Z}\right).$$

A characterisation through the theory of continued fractions

- The irrational x can be expanded as a continued fraction :

$$x = [a_0; a_1, a_2, a_3, a_4, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}}$$

  with $a_0 = \lfloor x \rfloor \in \mathbb{Z}$ and $a_i \geq 1$ an integer for for $i \geq 1$.
- The property of being badly approximable admits a very simple formulation with the help of the sequence of partial quotients $(a_i)_{i\geq 1}$:

$$c := \inf_{q\geq 1} q \cdot \langle qx \rangle > 0 \qquad \Longleftrightarrow \qquad M := \sup_{i\geq 1} a_i < \infty.$$

- In this case,

$$M \leq c^{-1} \leq M + 2.$$

A fractal structure

- Given $M \geq 1$, let $F(M)$ be the set of badly approximable numbers $x = [0; a_1, a_2, a_3, a_4, \ldots]$ in the interval $[0, 1)$ such that $\max_{i \geq 1} a_i \leq M$.
- Illustration in the case $M = 2$:

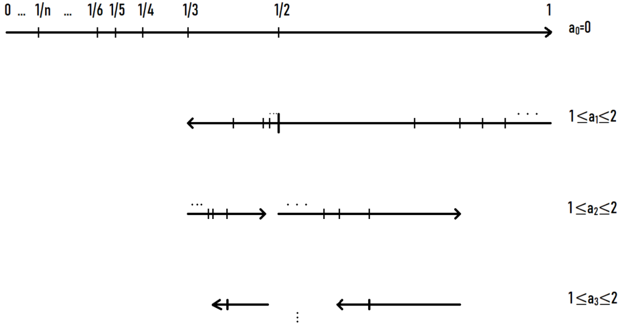


Figure – Fractal construction of the set $F(2)$

A fractal structure (bis)

- The set $F(M)$ admits a fractal structure. The set of badly approximable numbers (in $[0,1)$) is $\bigcup_{M\geq 1} F(M)$.

Theorem (Jarnìk, 1928)

When $M > 8$,

$$1-\frac{4}{M\log(2)} \leq \dim_H(F(M)) \leq 1-\frac{8}{M\log(M)}.$$

In particular, the set of badly approximable numbers has maximal Hausdorff dimension.

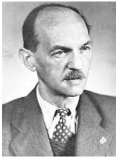


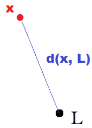
Figure – Vojtech Jarnìk (1897–1970).

Outline

Approximating a vector by a rational point

Let $\boldsymbol{x} = (x_1, \dots, x_n) \in [0,1]^n$ be a vector in dimension $n \geq 2$.

- If one wishes to approximate the vector $\boldsymbol{x}$ by a rational point $L = \boldsymbol{p}/q \in \mathbb{Q}^n$ with $\boldsymbol{p} \in \mathbb{Z}^n$ and $q \geq 1$...



- ... assuming that at least one of its coordinates is irrational, Dirichlet's Theorem (or Minkowski's Convex Body Theorem) guarantees the existence of infinitely many rationals $L = \boldsymbol{p}/q \in \mathbb{Q}^n$ such that

$$\left\| \boldsymbol{x} - \frac{\boldsymbol{p}}{q} \right\|_\infty \leq \frac{1}{q^{1+1/n}};$$

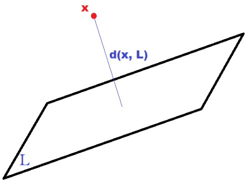
that is,

$$\mathrm{dist}_\infty(\boldsymbol{x}, L) \leq H(L)^{-1-1/n}, \text{ where } H(L) = \|(q, \boldsymbol{p})\|_\infty .$$

Approximating a vector by a rational hyperplane

Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in [0,1]^n$ be a vector in dimension $n \geq 2$.

- If one wishes to approximate the vector vecteur $\boldsymbol{x}$ by a rational hyperplane $L = \{\boldsymbol{y} \in \mathbb{R}^n \ : \ \boldsymbol{q} \cdot \boldsymbol{y} = p\}$ with $\boldsymbol{q} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$ and $p \in \mathbb{Z}$...



- ... assuming that its coordinates are rationally independent, Minkowski's Convex Body Theorem guarantees the existence of a constant $\kappa > 0$ and of infinitely many rational hyperplanes L such that

$$|\boldsymbol{q} \cdot \boldsymbol{x} - p| \cdot \|(\boldsymbol{q}, p)\|_\infty^{-1} \ \leq \ \kappa \cdot \|(\boldsymbol{q}, p)\|_\infty^{-n-1} \, ;$$

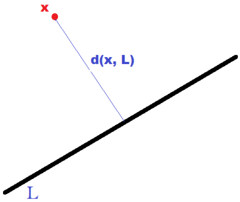
that is,

$$\mathrm{dist}_\infty (\boldsymbol{x}, L) \ \leq \ \kappa \cdot H(L)^{-n-1}, \text{ where } H(L) = \|(\boldsymbol{q}, p)\|_\infty \, .$$

Approximating a vector by a rational line

Let $\boldsymbol{x} = (x_1, \dots, x_n) \in [0,1]^n$ be a vector in dimension $n \geq 2$.

- If one wishes to approximate the vector $\boldsymbol{x}$ by a rational line $L = \{\boldsymbol{q}t + \boldsymbol{p} \in \mathbb{R}^n \ : \ t \in \mathbb{R}\}$ with $\boldsymbol{q} \in \mathbb{Z}^n \backslash \{\boldsymbol{0}\}$ and $p \in \mathbb{Z}^n$...



- ... assuming that at least two of its coordinates are rationally independent, Minkowski's Convex Body Theorem guarantees the existence of a constant $\kappa' > 0$ and of infinitely many rational lines L such that

$$\mathrm{dist}_\infty(\boldsymbol{x}, L) \ \leq \ \kappa' \cdot H(L)^{-1-2/(n-1)}, \text{ where } \ H(L) = \|(\boldsymbol{q}, \boldsymbol{p})\|_\infty .$$

The general case

Proposition

Let $n \geq 2$ (dimension of the Euclidian space) and $0 \leq d \leq n-1$ (dimension of the subspaces). For all $\boldsymbol{x} \in \mathbb{R}^n$ having at least $d+1$ rationally independent coordinates, there exists a constant $\kappa = \kappa(n, d, \boldsymbol{x}) > 0$ and infinitely many affine rational subspaces $L \subset \mathbb{R}^n$ with dimension d such that

$$dist\,(\boldsymbol{x}, L) \;\leq\; \kappa \cdot H(L)^{-1-\omega_{n,d}}, \qquad where \qquad \omega_{n,d} \;=\; \frac{d+1}{n-d}.$$

Definition

Let $n \geq 2$ and $0 \leq d \leq n-1$ be integers. A vector $\boldsymbol{x} \in \mathbb{R}^n$ is (n, d)–badly approximable if there exists a constant $\gamma = \gamma(n, d, \boldsymbol{x}) > 0$ such that for all rational affine subspace $L \subset \mathbb{R}^n$ with dimension d,

$$dist\,(\boldsymbol{x}, L) \;>\; \gamma \cdot H(L)^{-1-\omega_{n,d}}.$$

Unifying the various concepts of multidimensional bad approximability

- Extending works by Khintchine (1925), Dyson (1947), Jarnìk (1938) and Mahler (1939) :

Theorem (Beresnevich, Guan, Marnat, Ramirez, Velani — 2021+)

Let $n \geq 2$ and $0 \leq d < d' \leq n-1$ be integers. An n–dimensional vector est (n, d)–badly approximable if, and only if, it is (n, d')–badly approximable.

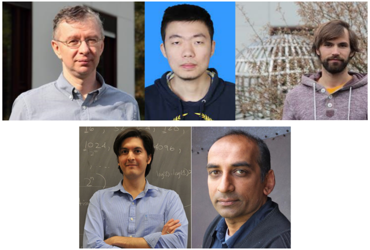


Figure – Victor Beresnevich, Lifan Guan, Antoine Marnat, Felipe Ramirez & Sanju Velani

Hausdorff dimension of the badly approximable vectors

Theorem (Schmidt, 1966)

The set of badly approximable vectors in dimension n has maximal Hausdorff dimension.

Figure – Wolfgang Schmidt (1933–)

Outline

What is a multiplicatively badly approximable vector ?

Let $(\alpha, \beta) \in \mathbb{R}^2$.

- From Dirichlet's theorem applied to the vector (α, β), there exist infinitely many integers $q \geq 1$ such that $\max\{q^{1/2} \cdot \langle q\alpha \rangle, q^{1/2} \cdot \langle q\beta \rangle\} \leq 1$. Consequently,

$$\exists^\infty q \geq 1 \ : \ \langle q\alpha \rangle \cdot \langle q\beta \rangle \ \leq \ \frac{1}{q} \qquad \Longleftrightarrow \qquad \inf_{q \geq 1} q \cdot \langle q\alpha \rangle \cdot \langle q\beta \rangle \ \leq 1.$$

Conjecture (Littlewood, c. 1930)

For all $(\alpha, \beta) \in \mathbb{R}^2$,

$$\inf_{q \geq 1} q \cdot \langle q\alpha \rangle \cdot \langle q\beta \rangle \ = \ 0.$$

Figure – John Edensor Littlewood (1885 – 1977)

What is a multiplicatively badly approximable vector ? (bis)

Conjecture (Badziahin & Velani, 2011)

Given $\lambda \geq 0$, *let*

$$M_\lambda \;=\; \left\{(\alpha, \beta) \in \mathbb{R}^2 \;:\; \inf_{q \geq 1} q \cdot (\log q)^\lambda \cdot \langle q\alpha \rangle \cdot \langle q\beta \rangle \;>\; 0\right\}.$$

Then, $M_\lambda \;=\; \emptyset \textit{ if } \lambda < 1$ *and* $\dim M_\lambda \;=\; 2 \textit{ si } \lambda \geq 1.$

Theorem (Badziahin, 2013)

$$\dim\left\{(\alpha, \beta) \in \mathbb{R}^2 \;:\; \inf_{q \geq 1} q \cdot (\log q) \cdot (\log\log q) \cdot \langle q\alpha \rangle \cdot \langle q\beta \rangle \;>\; 0\right\} \;=\; 2.$$

Figure – Sanju Velani & Dzmitri Badziahin

The p-adic Littlewood Conjecture

Recall : According to the Littlewood conjecture, $\inf_{q\geq 1} q\cdot\langle q\alpha\rangle\cdot\langle q\beta\rangle = 0$ for all $(\alpha,\beta)\in\mathbb{R}^2$.

Conjecture (De Mathan & Teulié, 2004)

For all $\alpha\in\mathbb{R}$ and all prime number p,

$$\inf_{q\geq 1} q\cdot|q|_p\cdot\langle q\alpha\rangle = 0,$$

where $|q|_p = p^{-k}$ if $q = p^k n$ with $p^{k+1}\nmid n$. Equivalently,

$$\inf_{n\geq 1, k\geq 0} n\cdot\langle np^k\alpha\rangle = 0.$$

Recall : if $x = [a_0(x); a_1(x), a_2(x), a_3(x), a_4(x),\dots]$, then

$$c(x) := \inf_{q\geq 1} q\cdot\langle qx\rangle > 0 \iff M(x) := \sup_{i\geq 1} a_i(x) < \infty.$$

- Reinterpretation of the p-adic Littlewood conjecture :

$$\forall\alpha\in\mathbb{R},\quad \inf_{k\geq 0} c\left(p^k\alpha\right) = 0 \iff \forall\alpha\in\mathbb{R},\quad \sup_{k\geq 0} M\left(p^k\alpha\right) = \infty.$$

Towards an analogous theory over function fields

- Let α be a real number (e.g., $\alpha = 567.789\,908\,782\ldots$) :

$$\alpha \;=\; \underbrace{b_{-n_0}\cdots b_{-1}b_0}_{\text{integer part}}\,.\,\underbrace{b_1b_2b_3\cdots}_{\text{fractional part}} \;=\; \underbrace{\sum_{k=0}^{n_0} b_{-k}10^k}_{\text{integer part}} + \underbrace{\sum_{k=1}^{\infty}\frac{b_k}{10^k}}_{\text{fractional part}}\;.$$

- Let $\mathbb{K}$ be a finite field and let $A(t) \in \mathbb{K}\left(\left(t^{-1}\right)\right)$ be a Laurent series :

$$A(t) \;=\; \underbrace{\sum_{k=0}^{n_0} b_{-k}t^k}_{\text{polynomial part}} + \underbrace{\sum_{k=1}^{\infty}\frac{b_k}{t^k}}_{\text{fractional part}}\;.$$

Real case	Functional case
$\mathbb{Z}$	$\mathbb{K}[t]$
$\mathbb{Q}$	$\mathbb{K}(t)$
$\mathbb{R}$	$\mathbb{K}\left(\left(t^{-1}\right)\right)$

Littlewood conjectures over function fields

Definition (Norm of a Laurent series)

Given a Laurent series

$$A(t) \;=\; b_{-n_0}t^{n_0} + \cdots + b_{-1}t + b_0 + \frac{b_1}{t} + \cdots + \frac{b_k}{t^k} + \cdots,$$

set $|A| = 2^{-k}$, *where* $k \geq -n_0$ *is the smallest integer such that* $b_k \neq 0$.

- By abuse of notation, denote by $\langle A \rangle$ the fractional part of A and set $\mathrm{dist}\,(A, \mathbb{K}[t]) = |\langle A \rangle|$.

Conjectures	$\alpha, \beta \in \mathbb{R}$	$A, B \in \mathbb{K}\left(\left(t^{-1}\right)\right)$
Classical Littlewood	$\inf_{q\in\mathbb{Z}\setminus\{0\}} \lvert q\rvert \cdot \lvert\langle q\alpha\rangle\rvert \cdot \lvert\langle q\beta\rangle\rvert = 0$	$\inf_{Q\in\mathbb{K}[t]\setminus\{0\}} \lvert Q\rvert \cdot \lvert\langle QA\rangle\rvert \cdot \lvert\langle QB\rangle\rvert = 0$
p–adic Littlewood	$\inf_{\substack{n\in\mathbb{Z}\setminus\{0\}\\ k\geq 0}} \lvert n\rvert \cdot \left\lvert\langle p^k n\alpha\rangle\right\rvert = 0$	$\inf_{\substack{N\in\mathbb{K}[t]\setminus\{0\}\\ k\geq 0}} \lvert N\rvert \cdot \left\lvert\langle t^k NA\rangle\right\rvert = 0$

The state of the art

- Hausdorff dimension of the set *not* satisfying the conjecture :

Conjectures	over $\mathbb{R}$	over $\mathbb{K}\left(\left(t^{-1}\right)\right)$
Classical Littlewood	0 (E–Ka–L, 2006)	??? (E–L–M, 2020)
p–adic Littlewood	0 (E–Kl, 2007)	??? (E–L–M, 2020)

E : Einsiedler, Ka : Katok, L : Lindenstrauss, Kl : Kleinbock, M : Mohammadi

Figure – Manfred Einsiedler, Anatol Katok, Dmitri Kleinbock, Elon Lindenstrauss & Amir Mohammadi

The state of the art (bis)

- Hausdorff dimension of the set *not* satisfying the conjecture :

Conjectures	over $\mathbb{R}$	over $\mathbb{K}\left(\left(t^{-1}\right)\right)$
Classical Littlewood	0 (E–Ka–L, 2006)	? ? ? (E–L–M, 2020)
p–adic Littlewood	0 (E–Kl, 2007)	? ? ? (E–L–M, 2020)

E : Einsiedler, Ka : Katok, L : Lindenstrauss, Kl : Kleinbock, M : Mohammadi

- Validity of the conjectures :

Conjectures	over $\mathbb{R}$	over $\mathbb{K}\left(\left(t^{-1}\right)\right)$
Classical Littlewood	? ? ?	? ? ?
p–adic Littlewood	? ? ?	false when char($\mathbb{K}$)=3

Outline

Statement of the main result

Theorem (A., Lunnon, Nesharim, 2021)

Let $(p_n)_{n\geq 1} \in \{0,1\}^{\mathbb{N}}$ be the paper folding sequence. Then, the Laurent series

$$\Pi = \sum_{n=1}^{\infty} \frac{p_n}{t^n} \in \mathbb{F}_3\left(\left(t^{-1}\right)\right)$$

is a counterexample to the t–adic Littlewood conjecture over $\mathbb{K} = \mathbb{F}_3$:

$$\inf_{\substack{N\in\mathbb{K}[t]\setminus\{0\} \\ k\geq 0}} |N| \cdot \left|\langle t^k N\Pi\rangle\right| = 2^{-4} > 0.$$

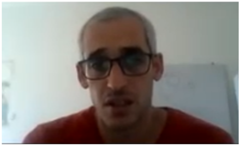
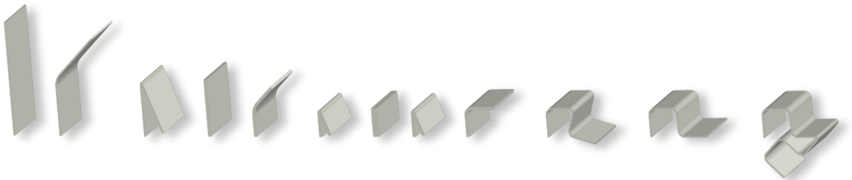


Figure – Erez Nesharim & Fred Lunnon

The paper folding sequence



- The paper folding sequence $(p_n)_{n\geq 1}$ can be obtained as follows : begin with the terms 11 and apply the substitution rule $11 \to 1101, 01 \to 1001, 10 \to 1100, 00 \to 1000$:

$$\begin{aligned} 11 \to 1101 \to 11011001 &\to 1101100111001001 \\ &\to 11011001110010011101100011001001 \\ &\to \dots \end{aligned}$$

Reduction of the problem to the existence of singular matrices

- Let $A = \sum_{n=1}^{\infty} b_n t^{-n}$ be a Laurent series and let $N = \theta^h t^h + \cdots + \theta_1 t + \theta_0$ be a polynomial with degree $h \geq 1$ (i.e. $|N| = 2^h$).
- Given $l \geq 1$, asking that $|N| \cdot \left|\langle t^k NA \rangle\right| < 2^{-l}$, i.e. $\left|\langle t^k NA \rangle\right| < 2^{-(l+h)}$, amounts to imposing that a certain number of the coefficients of the Laurent series $t^k NA$ should vanish. Explicitly, one requires that $H_A \boldsymbol{\theta} = \mathbf{0}$, where $H_A = H_A(k, l, h)$ is a $(h + l) \times (h + 1)$ Hankel matrix formed from the coefficients of A, and where $\boldsymbol{\theta} = (\theta_h, \ldots, \theta_1, \theta_0)^T \in \mathbb{K}^{h+1} \backslash \{\mathbf{0}\}$.

Reduction of the problem to the existence of singular matrices (bis)

- From properties of Hankel matrices, the condition $|N| \cdot \left|\langle t^k NA \rangle\right| < 2^{-l}$ amounts to asking that there should exist l nested singular submatrices inside the infinite Hankel matrix of the sequence $(b_n)_{n\geq 1}$.

$$
\begin{array}{ccccccccccc}
b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & b_{10} & \rightarrow \\
b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & b_{10} & b_{11} & \iddots \\
b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & b_{10} & b_{11} & b_{12} & \iddots \\
b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & b_{10} & b_{11} & b_{12} & b_{13} & \iddots \\
b_5 & b_6 & b_7 & b_8 & b_9 & b_{10} & b_{11} & b_{12} & b_{13} & b_{14} & \iddots \\
b_6 & b_7 & b_8 & b_9 & b_{10} & b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & \iddots \\
b_7 & b_8 & b_9 & b_{10} & b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} & \iddots \\
b_8 & b_9 & b_{10} & b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} & b_{17} & \iddots \\
b_9 & b_{10} & b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} & b_{17} & b_{18} & \iddots \\
b_{10} & b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} & b_{17} & b_{18} & b_{19} & \iddots \\
\downarrow & \iddots & \iddots & \iddots & \iddots & \iddots & \iddots & \iddots & \iddots & \iddots & \searrow
\end{array}
$$

The Number Wall of a sequence

Definition

The Number Wall of the sequence $\mathcal{B} = (b_n)_{n\geq 1} \in \mathbb{K}^{\mathbb{N}}$ is an infinite matrix $W(\mathcal{B}) = (w_{mn}(\mathcal{B}))_{m\geq 0, n\in\mathbb{Z}}$ whose $(m, n)^{th}$ coefficient is (up to the sign) the Hankel determinant

$$w_{mn}(\mathcal{B}) \;=\; (-1)^{m(m-1)/2} \cdot \begin{vmatrix} b_{n-m} & \cdots & b_k & \cdots & b_n \\ \vdots & \iddots & & \iddots & \vdots \\ b_k & & b_n & & b_l \\ \vdots & \iddots & & \iddots & \vdots \\ b_n & \cdots & b_l & \cdots & b_{n+m} \end{vmatrix}.$$

(In particular, when $m = 0$, $w_{0,n}(\mathcal{B}) = b_n$ for all $n \in \mathbb{Z}$.)

The Number Wall of the paper folding sequence over $\mathbb{F}_3$

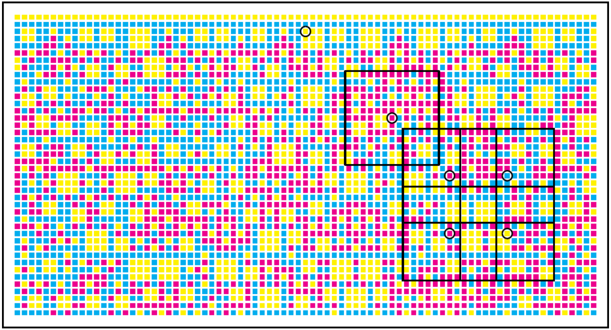


Figure – The Number Wall of the paper folding sequence over $\mathbb{F}_3$. In yellow (resp. in blue, in pink) the determinants equal to 0 (resp. to 1, to 2).

- From the so–called Desnanot–Jacobi determinental identity, the zero coefficients in this Number Wall can only appear inside squares.

The Number Wall of the paper folding sequence over $\mathbb{F}_3$ (bis)

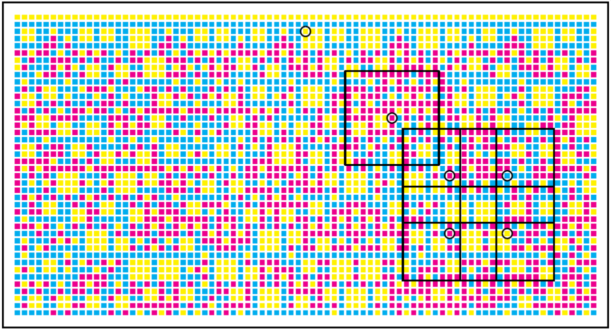


Figure – The Number Wall of the paper folding sequence over $\mathbb{F}_3$. In yellow (resp. in blue, in pink) the determinants equal to 0 (resp. to 1, to 2).

- Reformulation of the problem :

$$\inf_{\substack{N\in\mathbb{K}[t]\setminus\{0\}\\ k\geq 0}} |N|\cdot\left|\langle t^k N\Pi\rangle\right| = 2^{-4} \iff \begin{cases}\text{there exists no } 4\times 4 \text{ zero square}\\ \text{and there exists one such } 3\times 3 \text{ square.}\end{cases}$$

Strategy 1/2

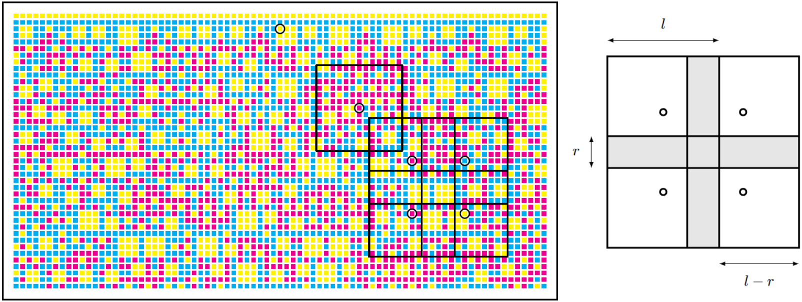
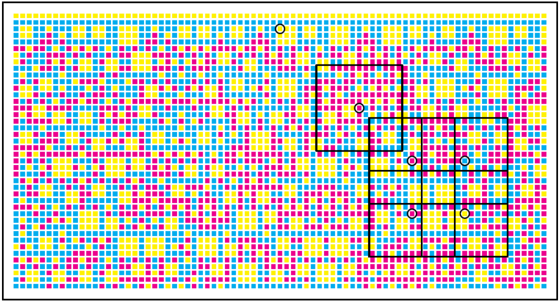


Figure – Left : the Number Wall of the paper folding sequence over $\mathbb{F}_3$. Right : schematic illustration of the coding of the Number Wall.

Step 1 : To show that a sufficiently large portion of the Number Wall can be obtained as a the image under a coding and a substitution rule of a 2-dimensional tiling.

Strategy 2/2



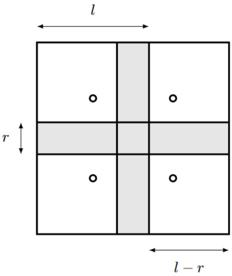


Figure – Left : the Number Wall of the paper folding sequence over $\mathbb{F}_3$. Right : schematic illustration of the coding of the Number Wall.

Step 2 : To consider the (infinite) tiling of the plane obtained from the above coding and substitution rule. Working locally, to show that :

2.a. it is the Number Wall of a sequence (the tiling must satisfy the Desnanots–Jacobi determinental rules) and that the coding and the substitution rules do not generate 4×4 zero–squares.

2.b. the restriction of the coding and the substitution rule to the first row generates the paper folding sequence.

The end