Counting balanced words and related problems

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Introduction

$\mathcal{A} = \{0, 1\}$ and define $\mathcal{A}^*$, $\mathcal{A}^{\mathbb{N}}$ as usual. The length of $x \in \mathcal{A}^*$ is $|x|$. We denote $|x|_1$ the cardinality of 1 in $x \in \mathcal{A}^*$. If $y \in \mathcal{A}^*$ is a subword of $x \in \mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}}$, we write $y \prec x$. A word in $x \in \mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}}$ is **balanced** if $||u|_1 - |v|_1| \leq 1$ holds for any $u, v \prec x$ with $|u| = |v|$.

1001010 is balanced

1010001 is not balanced

An infinite word $w \in \mathcal{A}^{\mathbb{N}}$ is **sturmian** if Card$\{u \prec w \mid |u| = n\} = n + 1$ for all $n \in \mathbb{N}$.

Morse and Hedlund showed that if $x \in \mathcal{A}^{\mathbb{N}}$, then followings are equivalent

- x is sturmian.
- x is aperiodic and balanced.
- x is a mechanical word of irrational slope.

A lower mechanical word $(s_n) \in \mathcal{A}^{\mathbb{N}}$ is defined by

$$s_n(\alpha, \rho) = \lfloor \alpha n + \rho \rfloor - \lfloor \alpha(n-1) + \rho \rfloor$$

with a given slope $\alpha \in [0,1]$ and an intercept $\rho \in [0,1)$. An upper mechanical word is similarly defined by replacing $\lfloor \cdot \rfloor$ with $\lceil \cdot \rceil$, which is denoted by $\hat{s}_n(\alpha, \rho)$.

For every balanced word $x = x_1 \ldots x_n$ we can find a slope α and an intercept ρ such that $x_i = s_i(\alpha, \rho) = \hat{s}_i(\alpha, \rho)$. Conversely, $(s_n(\alpha, \rho))_{n=1,\ldots,\ell}$ is balanced.

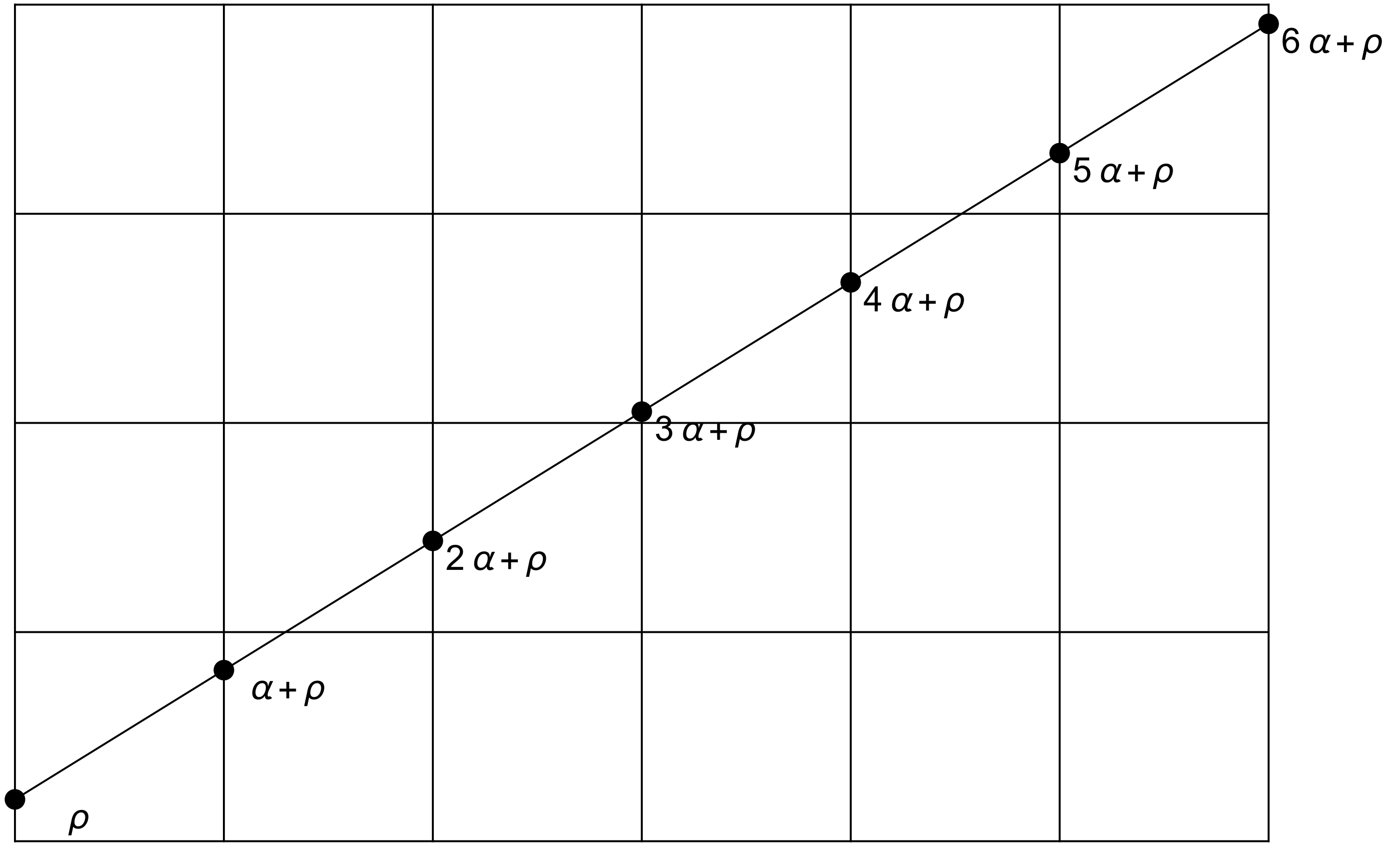

Figure 1: Balanced words

In other words, balanced words are coding of rotation.

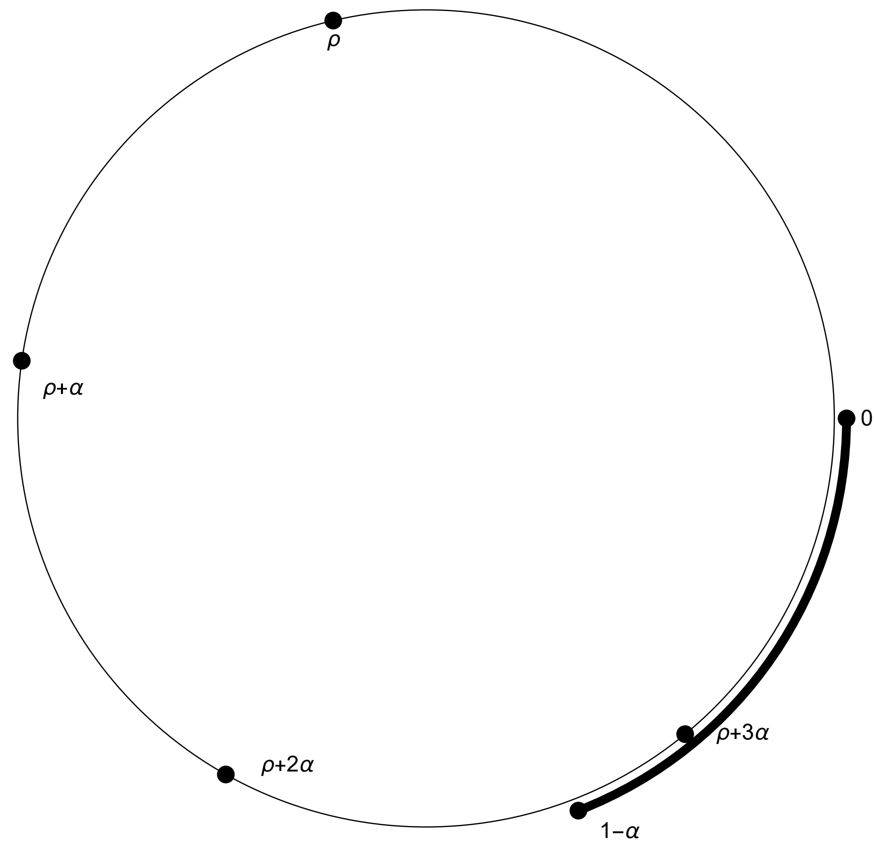

Figure 2: Coding of rotation

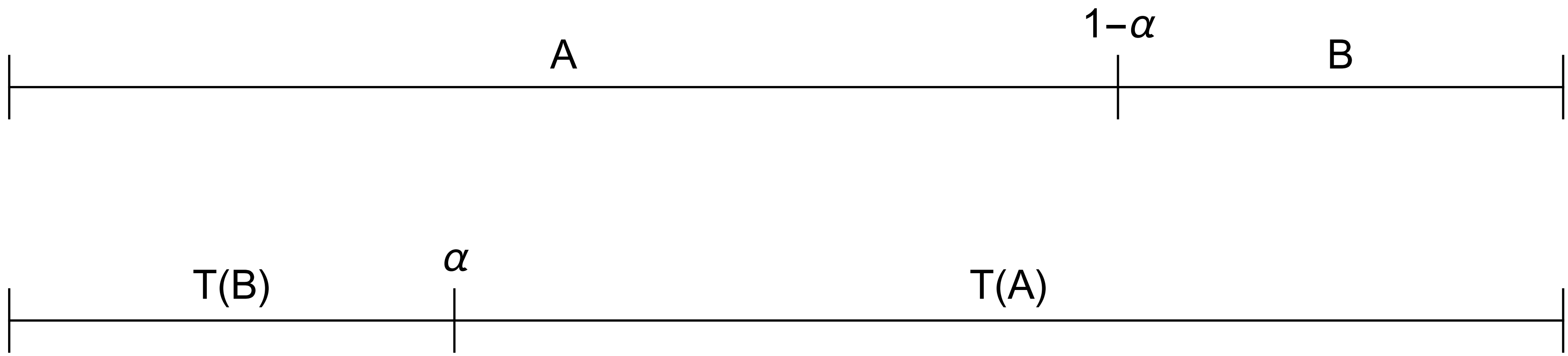

Figure 3: Two interval exchange

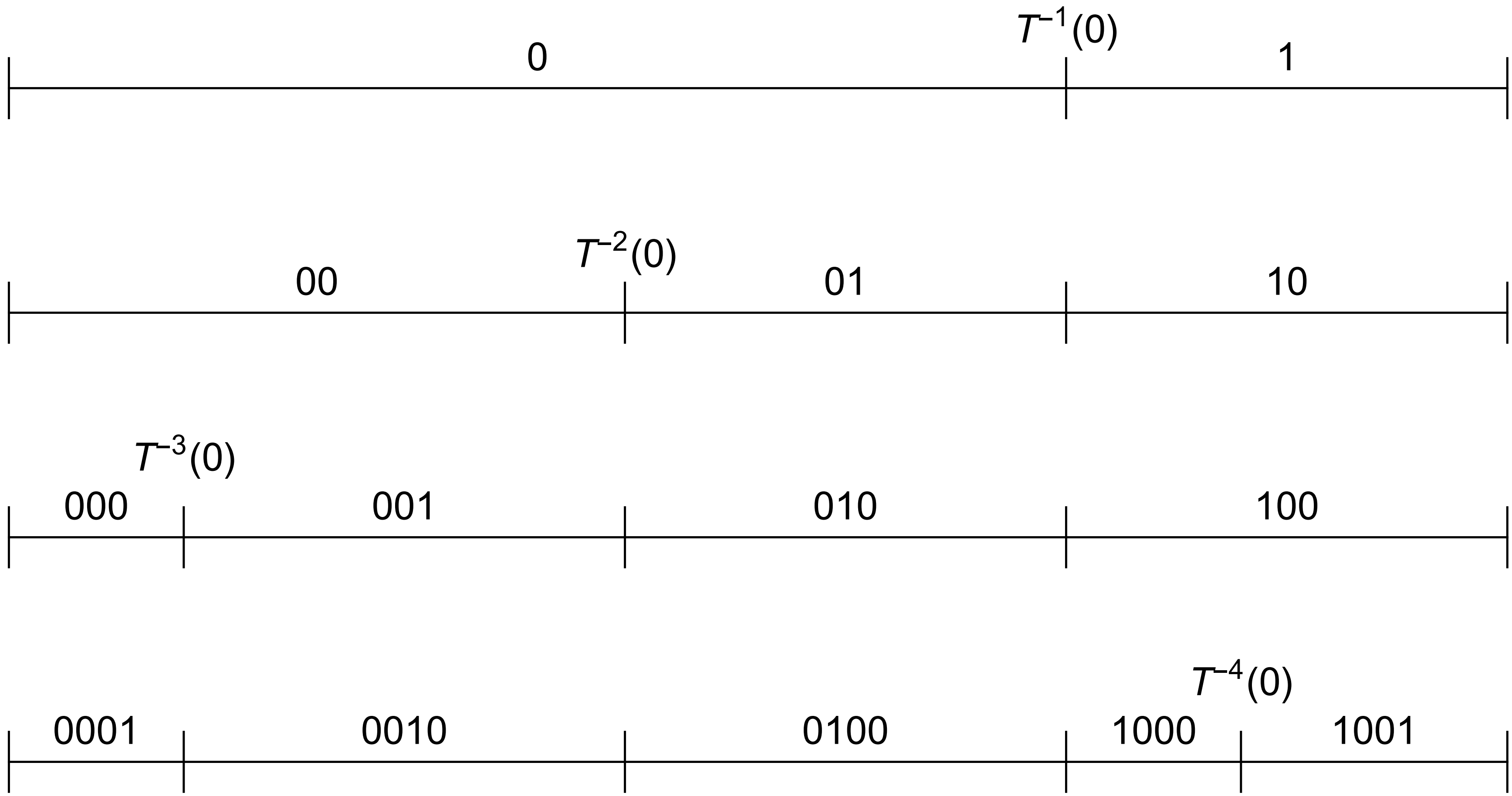

Figure 4: Cylinder sets of rotation

Yasutomi's partition

Let m be a fixed positive integer and put $X := [0,1) \times [0,1]$. The map $\psi : (\rho, \alpha) \to (s_n(1-\alpha, \rho))_{n=1}^m$

$$X := \bigcup_{w \in B(m)} \psi^{-1}(w).$$

gives the partition:

$$Y := X \setminus \{(x,y) \mid x = ny - \ell, n \in \{1, \ldots, m\}, \ell \in \{0, 1, \ldots, n-1\}\}.$$

The slice at $y = 1 - \alpha$ gives the cylinder partition of slope α. Therefore X is the "union" of all rotation. Berstel-Pocchiola [1] and Kamae-Takahashi [7] arrived at the same partition but this way seems to the most direct treatise.

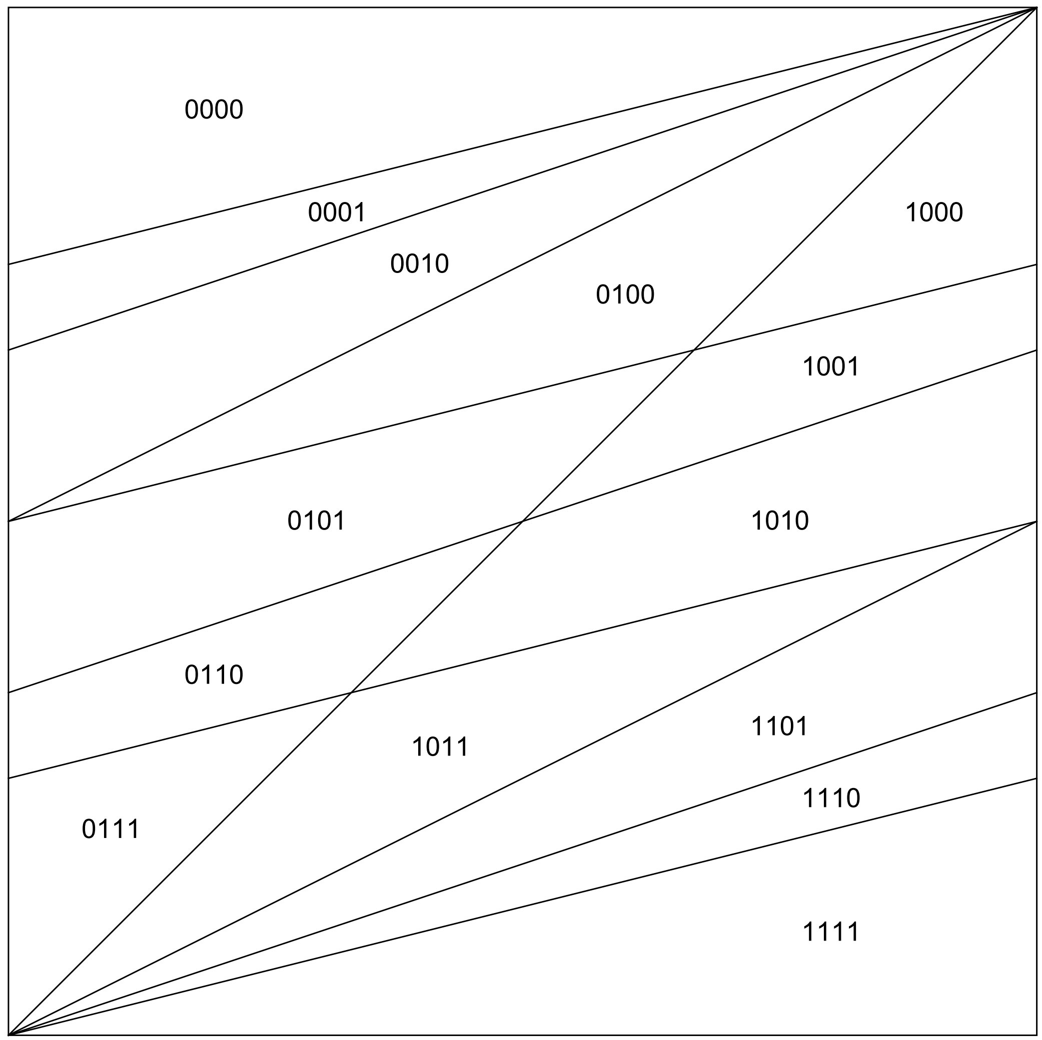

Figure 5: Partition for $m = 4$

Lagrange spectrum appears in

$$\sup \left\{ c > 0 \,\middle|\, \left| x - \frac{p}{q} \right| < \frac{1}{cq^2} \text{ has infinitely many solutions } \frac{p}{q} \right\},$$

for any $x \in \mathbb{R} \setminus \mathbb{Q}$, i.e.,

$$\nu(x) = \frac{1}{\liminf\limits_{n \to \infty} n \| n x \|} \in [\sqrt{5}, \infty].$$

The set $\{\nu(x) \mid x \in \mathbb{R} \setminus \mathbb{Q}\}$ has the minimum accumulation point 3. Below 3, the discrete spectrum is described by balanced words of rational slope and intercept 0 (Christoffel words). Yasutomi [11] gave a characterization of x with $\nu(x) = 3$. Such words may not have a fixed slope and can be understood as a coding of 2-dim dynamics in X.

Denote by $B(n,t,u)$ the cardinality of the set of balanced words of length n whose slope $\alpha \in [1-t,1]$ (**Caution !**) and its intercept $\rho \in [0,u)$.

To enumerate $B(n,t,u)$, we compute $B(m,t,u) - B(m-1,t,u)$, i.e., we count the number of intersections in $[0,u) \times [0,t]$ which appear by adding new segments of slope $1/m$, see Figure 6.

For $t = 0.7$ and $u = 0.59$, we have $(A(m,t,u))_{m=1}^{8} = (1,2,4,4,7,8,10,13)$ and $B(8,t,u) = 50$.

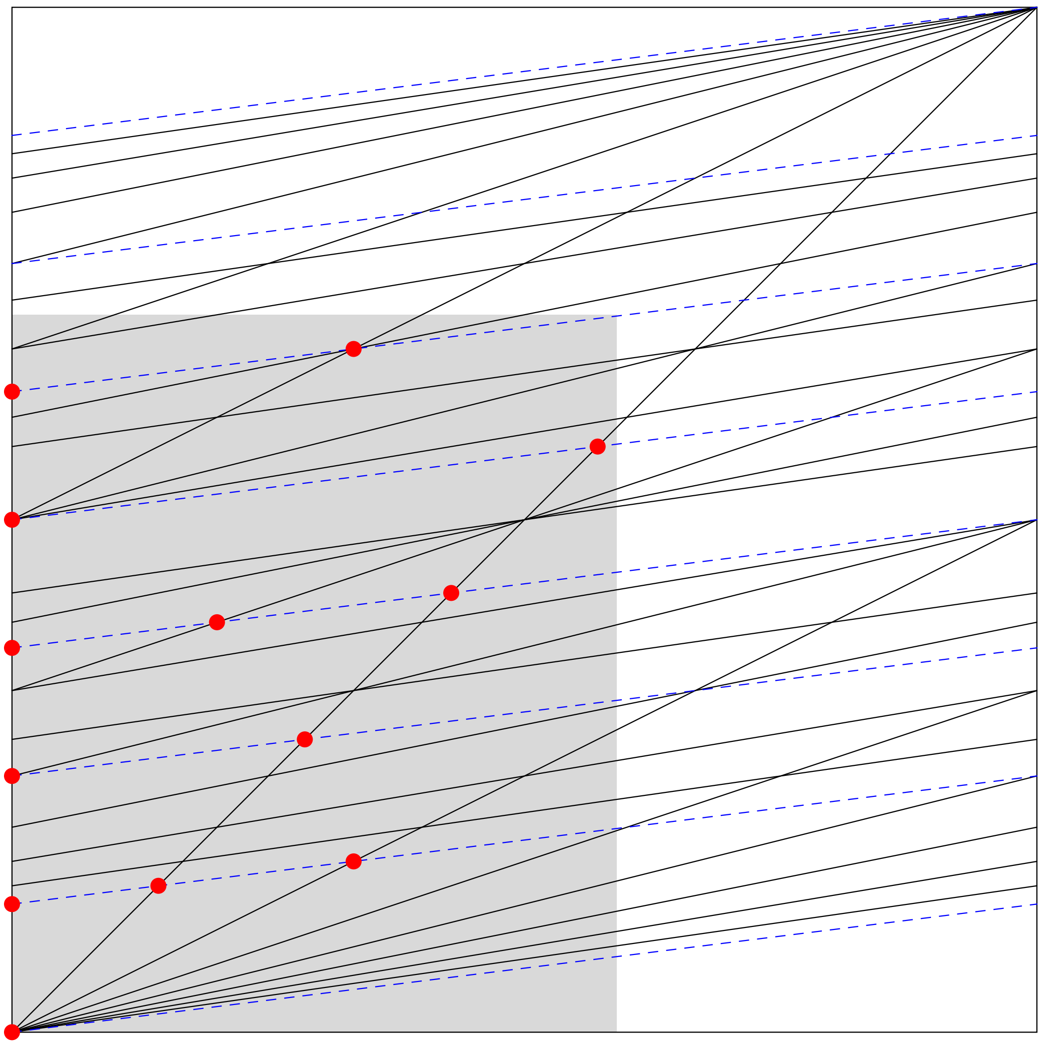

Figure 6: $B(8, 0.7, 0.59) = 50$

Theorem 1 (Yasutomi [11] for $u = 1$)**.**

$$B(n,t,u) = 1 + \sum_{m \le n} A(m,t,u)$$

with

$$A(m,t,u) = \sum_{\substack{0 \le i < j \le m,\ (i,j)=1 \\ i/j \le t,\ \langle mi/j \rangle < u}} 1,$$

where i and j are non negative integers.

- $(i,j) = 1$: Möbius inversion
- $i/j \le t$: Easy
- $\langle mi/j \rangle < u$: Large sieve inequality

Let $\phi(n)$ be the Euler totient function, putting $t = u = 1$,

$$A(m,1,1) = \sum_{\substack{0 \le i < j \le m \\ (i,j)=1}} 1 = \sum_{j=1}^{m} \sum_{\substack{0 \le i < j \\ (i,j)=1}} 1 = \sum_{j=1}^{m} \phi(j) =: \Phi(m).$$

$$\begin{aligned} B(n,1,1) &= 1 + \sum_{m=1}^{n} \Phi(m) \\ &= 1 + \sum_{j=1}^{n} (n+1-j)\phi(j) \end{aligned}$$

Several different proofs are found by Lipatov, Berstel-Pocchiola, A.d.Luca-Mignosi [9, 8, 1, 3, 10].

Theorem 2.

$$B(n,t,u) = \frac{tu}{\pi^2} n^3 + O\left(n^2 (\log n)^{15/2}\right).$$

Moreover, we have

$$B(n,1,1) = \frac{n^3 + 3n^2}{\pi^2} + O\left(n^2 \exp\left(-c\left((\log n)^{3/5} (\log\log x)^{-1/5}\right)\right)\right)$$

and

$$B(n,t,1) = \frac{tn^3}{\pi^2} + O(n^2).$$

Let $\mathcal{B}(n)$ be the set of balanced words of length n.

$$^{\#}\{x \in \mathcal{B}(n) \mid (\rho, \alpha) \in (a, b] \times [c, d)\} = \frac{(b-a)(d-c)}{\pi^2} n^3 + O\left(n^2 (\log n)^{\frac{15}{2}}\right).$$

For a Jordan measurable region W in the unit square, we have
Corollary 3.

$$^{\#}\{x \in \mathcal{B}(n) \mid (\rho, \alpha) \in W\} = \frac{\mathrm{Area}(W)}{\pi^2} n^3 + O\left(n^2 (\log n)^{15/2}\right)$$

where Area is the 2-dimensional Lebesgue measure.

Farey series $(f_m(i))$ of order m is the finite increasing sequence composed of irreducible fractions in $[0, 1)$ whose denominators are not larger than m:

$$0 = f_m(1) < f_m(2) < \cdots < f_m(\Phi(m)) < 1.$$

For $m = 6$, we have

$$\frac{0}{1} < \frac{1}{6} < \frac{1}{5} < \frac{1}{4} < \frac{1}{3} < \frac{2}{5} < \frac{1}{2} < \frac{3}{5} < \frac{2}{3} < \frac{3}{4} < \frac{4}{5} < \frac{5}{6} < 1$$

Clearly $A(m, t, 1) = \max \{j \mid f_m(j) \leq t\}$. For example

$$A\left(6, \frac{1}{2}, 1\right) = 7.$$

The function $A(m,t,u)$ counts special Farey fractions. An easy estimate

$$A(m,t,1) - t\Phi(m) = O(m)$$

gives uniform distribution property. Farey series is expected to be "highly" uniform in $[0,1]$. J. Franel showed

$$\text{Riemann Hypothesis} \Longleftrightarrow \int_0^1 \left(A(m,t,1) - t\Phi(m)\right)^2 dt = O\left(m^{1+\varepsilon}\right).$$

We add another one directly related to the number of balanced words:

Corollary 4.

$$\textit{Riemann Hypothesis} \Longleftrightarrow B(n,1,1) = \frac{n^3 + 3n^2}{\pi^2} + O\left(n^{3/2+\varepsilon}\right). \qquad (1)$$

Counting technique in number theory

An arithmetic functions is the map $\mathbb{N} \to \mathbb{C}$. With binary operations

$$(f+g)(n) = f(n) + g(n),$$

and

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right), \qquad \text{(Dirichlet convolution)}$$

they form a commutative ring $\mathcal{R}$ with the multiplicative identity $\mathbf{e}$:

$$\mathbf{e}(n) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$$

The invertible elements are $\mathcal{R}^* = \{f \in \mathcal{R} \mid f(1) \neq 0\}$. Denote by $\mathbf{1}$ the function $\mathbf{1}(n) = 1$. Then we have

$$\mathbf{1} * \mu = \mathbf{e}$$

where

$$\mu(n) = \begin{cases} 1 & n = 1 \\ (-1)^k & n = p_1 \dots p_k \ (p_i : \text{ distinct primes}) \\ 0 & n \text{ is not square free} \end{cases}$$

Möbius inversion formula is well-known.

$$f = g * \mathbf{1} \Longleftrightarrow f * \mu = g$$

i.e.

$$f(n) = \sum_{d|n} g(d) \Longleftrightarrow g(n) = \sum_{d|n} f(d)\mu\left(\frac{n}{d}\right)$$

We often study its generating Dirichlet series

$$D(f,s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

Then we have formally

$$D(f * g, s) = D(f,s)D(g,s).$$

Many important arithmetic functions are connected with the Riemann zeta function.

$$D(\mathbf{1}, s) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}.$$

For e.g.

$$D(\mu, s) = \sum_{n=1} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}, \quad D(\phi, s) = \sum_{n=1} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}$$

We use properties of Riemann zeta function as an analytic function in $\mathbb{C}$ to deduce asymptotic formula of summatory function, by Perron's formula:

$$\sum_{n<x} f(n) = \lim_{T\to\infty} \frac{1}{2\pi\sqrt{-1}} \int_{c-T\sqrt{-1}}^{c+T\sqrt{-1}} \left(\sum_{n=1}^{\infty} \frac{f(n)}{n^s} \right) \frac{x^s}{s} ds$$

or its weighted variants. The most famous one is the Prime number theorem:

$$\sum_{p<x} 1 = \int_1^x \frac{dx}{\log x} + O(x \exp(-c\sqrt{\log x})).$$

Case $t = u = 1$.

$$B(n,1,1) = 1 + \sum_{j=1}^{n} (n+1-j)\phi(j).$$

We use

$$\sum_{j \le x} (x-j)\phi(j) = \lim_{T \to \infty} \frac{1}{2\pi\sqrt{-1}} \int_{a-\sqrt{-1}T}^{a+\sqrt{-1}T} \frac{\zeta(s-1)x^{s+k}}{\zeta(s)s(s+1)} ds$$

for $a > 2$. For Corollary 4, if Riemann Hypothesis is valid,

$$\lim_{T \to \infty} \int_{a_0-\sqrt{-1}T}^{a_0+\sqrt{-1}T} \left| \frac{\zeta(s-1)}{\zeta(s)s(s+1)} \right| ds < \infty,$$

with $a_0 = 1/2 + \varepsilon$, we get the estimate (1). Conversely inverse Mellin transformation shows

$$\int_1^\infty \left(\sum_{n \le x} (x-n)\phi(n) - \frac{x^3}{\pi^2} \right) x^{-s-2} dx = \frac{\zeta(s-1)}{\zeta(s)s(s+1)} - \frac{1}{\pi^2(s-2)}$$

for $\sigma > 2$. If (1) is valid, then the parenthesis in the integrand is $O(x^{3/2+\varepsilon})$. This gives the holomorphic continuation of the right side to $\sigma > 1/2 + \varepsilon$, which finishes the proof. Without RH, we use the current best ”zero free” region by shifting the path.

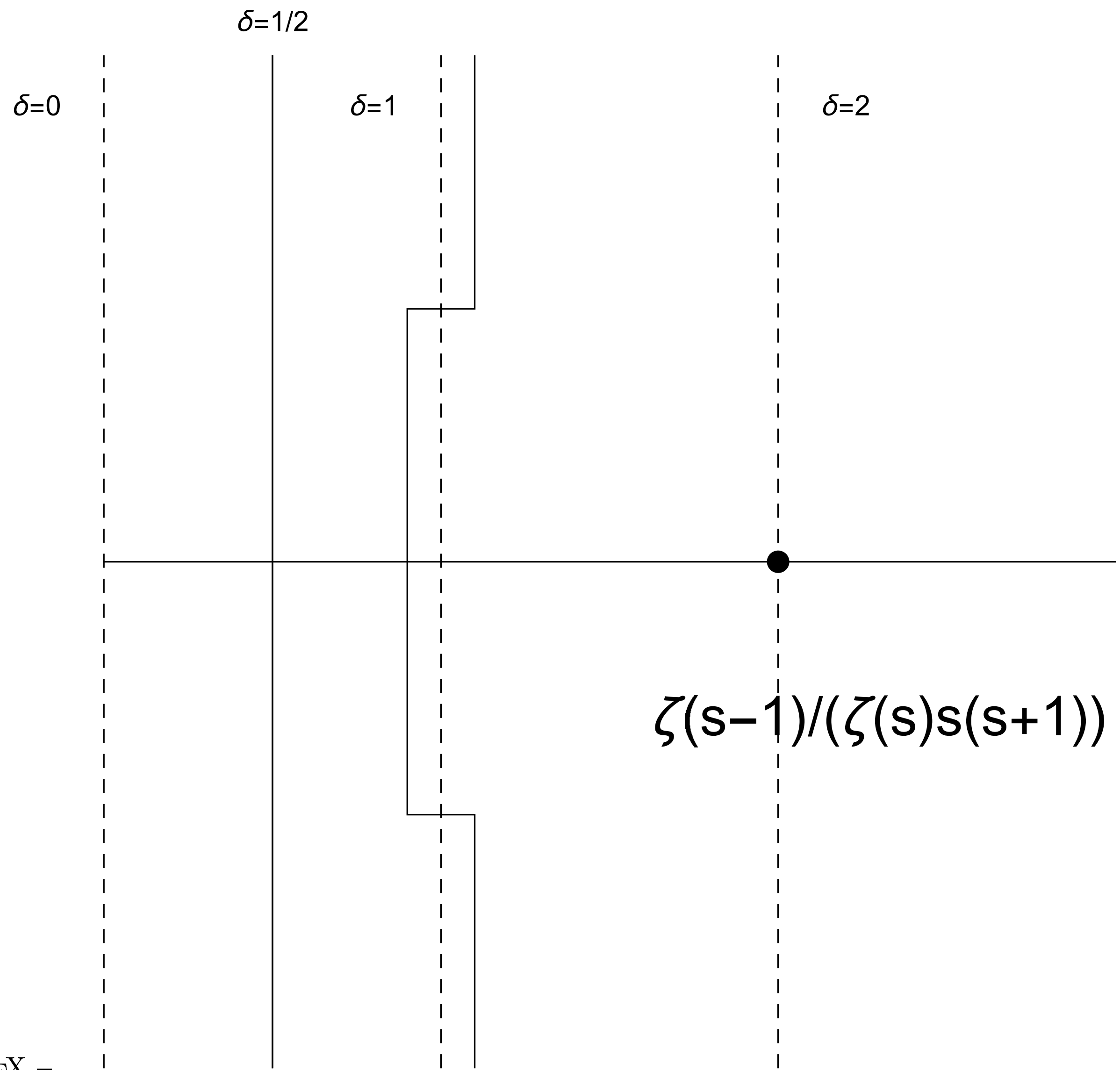

Figure 7: Shifting the integration path

to obtain

$$B(n,1,1) = \frac{n^3+3n^2}{\pi^2} + O\left(n^2 \exp\left(-c\left((\log n)^{3/5}(\log\log x)^{-1/5}\right)\right)\right).$$

Case $u = 1$.

$$B(n,t,1) = 1 + \sum_{m \le n} A(m,t,1), \qquad A(m,t,u) = \sum_{\substack{i<j\le m,\ (i,j)=1 \\ i/j \le t}} 1.$$

The key equation is

$$B(n,t,1) = 1 - t + n + tB(n,1,1) - \sum_{kb \le m \le n} \mu(k) \langle bt \rangle$$

for $t < 1$.

Our problem is the summatory function:

$$\sum_{kb \le m} \mu(k)\langle bt\rangle = \sum_{\ell=1}^{m} \sum_{k|\ell} \mu(k) \left\langle \frac{\ell}{k} t \right\rangle$$

Therefore the analytic property of

$$Z_t(s)/\zeta(s)$$

plays a key role where

$$Z_t(s) := \sum_{b=1}^{\infty} \frac{\langle bt\rangle - \frac{1}{2}}{b^s}.$$

is the Hecke's Dirichlet series.

Fujii [4] studied Hecke's Dirichlet series [5]: The analytic property of $Z_t(s)$ heavily depends on the Diophantine approximation property of t by rationals. [4, Theorem 1 and 2] imply

$$B(n,t,1) = \frac{t(n^3+3n^2)}{\pi^2} + O\left(n^2 \exp\left(-c\left(\log n \cdot \log\log n\right)^{1/3}\right)\right)$$

for almost all t, including all algebraic numbers.

However for general t, the error term is pretty big for now.

$$B(n,t,1) = \frac{tn^3}{\pi^2} + O(n^2).$$

General case.

$$B(n,t,u) = 1 + \sum_{m \le n} A(m,t,u), \quad A(m,t,u) = \sum_{\substack{i<j\le m,\ (i,j)=1 \\ i/j \le t,\ \langle mi/j \rangle < u}} 1.$$

- Fourier expansion of a smooth approximation of $\chi_{[0,u] \pmod{\mathbb{Z}}}$.
- Remove other constraints by Fourier technique.
- Apply large sieve inequality to obtain cancellation.

Fourier expansion

Choose $\chi_{[0,u] \pmod{\mathbb{Z}}} \approx \sum_{h\in\mathbb{Z}} a_h \exp\left(\frac{2\pi m i h\sqrt{-1}}{j}\right)$ to obtain

$$\sum_{\substack{i<j\leq m,\ (i,j)=1 \\ i/j\leq t,\ \langle mi/j\rangle<u}} 1 \approx \sum_{\substack{i<j\leq m,\ (i,j)=1 \\ i/j\leq t}} \sum_{h\in\mathbb{Z}} a_h \exp\left(\frac{2\pi m i h\sqrt{-1}}{j}\right)$$

$$\approx \sum_{\substack{i<j\leq m,\ (i,j)=1 \\ i/j\leq t}} \sum_{h=-H}^{H} a_h \exp\left(\frac{2\pi m i h\sqrt{-1}}{j}\right)$$

Constant term $h = 0$ gives the main term.

Remove other constraints

Use Möbius inversion to delete $(i,j)=1$. Remove $i/j \le t$ by

$$\frac{2}{\pi}\int_0^T \frac{\sin x}{x}dx = 1 + O\left(\min\left(1, \frac{1}{T}\right)\right),$$

The target becomes

$$\sum_{i<j\le m} \sum_{h=1}^{H} a_h \exp\left(\frac{2\pi mih\sqrt{-1}}{j}\right)$$

Apply large sieve inequality

Large sieve stands for a certain variants of Bessel inequality. Letting $S(\alpha) = \sum_{n=1}^{N} a_n \mathbf{e}(n\alpha)$, and $\alpha_1, \ldots \alpha_R \in \mathbb{T}$ with $\mathbf{d}(\alpha_i, \alpha_j) \geq \delta$ for $i \neq j$, we have

$$\sum_{r=1}^{R} |S(\alpha_r)|^2 \leq \left(N + \frac{1}{\delta} - 1 \right) \sum_{n=1}^{N} |a_n|^2$$

is a well-known one by Halberstam-Davenport-Selberg.

There are many many variants but I had a Hiroshi Mikawa's advice.

We shall use a large sieve inequality [6, Theorem 7.2],[2, Lemma 2.4]:

Lemma 5. *For any real numbers* x_m, y_m *with* $|x_m| \leq X$ *and* $|y_m| \leq Y$ *and* $\alpha_m, \beta_m \in \mathbb{C}$*, we have*

$$\begin{aligned} &\left| \sum_m \sum_n \alpha_m \beta_n \exp(2\pi x_m y_n \sqrt{-1}) \right| \\ &\leq \quad 5\sqrt{1+XY} \left(\sum_{|x_i - x_j| < 1/Y} |\alpha_i \alpha_j| \sum_{|y_i - y_j| < 1/X} |\beta_i \beta_j| \right)^{1/2}. \end{aligned}$$

We obtain

$$B(n,t,u) = \frac{tu}{\pi^2} n^3 + O\left(n^2 (\log n)^{15/2}\right)$$

by the choice

$$x_n = mi, \quad y_m = \frac{h}{j}.$$

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