On the smallest base in which a number has a unique expansion

Pieter C. Allaart

University of North Texas
(allaart@unt.edu)

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What are univoque bases?

- Fix $q \in (1,2]$. An expression of the form

$$x = \sum_{i=1}^{\infty} \frac{x_i}{q^i}, \qquad x_i \in \{0,1\} \ \forall i$$

  is called a q-expansion of x. We write $x = (x_1 x_2 \dots)_q$.
- Exists iff $x \in I_q := \left[0, \frac{1}{q-1}\right]$, but in general not unique!

Example

Take $q = (1+\sqrt{5})/2 \approx 1.618$, so $q^{-1} + q^{-2} = 1$. Then

$$1 = (11)_q = (1011)_q = (101011)_q = \cdots = ((10)^\infty)_q = (01^\infty)_q.$$

What are univoque bases?

Theorem (Sidorov, 2003)

Let $1 < q < 2$. Then almost every $x \in I_q$ has uncountably many q-expansions.

Definition

- $\mathcal{U}_q := \{x \in I_q : x \text{ has a unique } q\text{-expansion}\}, \quad q \in (1, 2]$.
- $\mathscr{U} := \{q \in (1, 2] : 1 \in \mathcal{U}_q\}$.

Connections between $\mathscr{U}$ and the sets $\mathcal{U}_q$:

- $\mathcal{U}_q$ is closed iff $q \notin \overline{\mathscr{U}}$ (De Vries & Komornik, 2009)
- The description of $\mathcal{U}_q$ is simplest when $q \in \overline{\mathscr{U}} \backslash \mathscr{U}$.

Properties of $\mathscr{U}$

- $\mathscr{U}$ is Lebesgue null (Erdős, Joó & Komornik, 1990)
- $\dim_H \mathscr{U} = 1$ (Daróczy and Kátai, 1995)
- $\min \mathscr{U} = q_{KL}$, where q_{KL} is the Komornik-Loreti constant, i.e. the base q (≈ 1.787) such that

$$1 = (\tau_1 \tau_2 \dots \tau_n \dots)_q,$$

  where $(\tau_i)_{i=0}^{\infty}$ is the Thue-Morse sequence:

$$(\tau_i)_{i=0}^{\infty} = 01\,10\,1001\,10010110\,1001011001101001\dots$$

  (Komornik & Loreti, 1998)
- $\overline{\mathscr{U}}$ is a Cantor set (Komornik & Loreti, 2007), and $\overline{\mathscr{U}} \setminus \mathscr{U}$ is countable.

The set $\mathscr{U}(x)$

Definition

$$\mathscr{U}(x) := \{q \in (1,2] : x \in \mathcal{U}_q\}, \qquad x > 0.$$

Note that $\mathscr{U}(1) = \mathscr{U}$.

- $\mathscr{U}(x)$ is Lebesgue null but of full Hausdorff dimension (Lü, Tan & Wu, 2014)
- The algebraic difference $\mathscr{U}(x) - \mathscr{U}(x)$ contains an interval (Dajani, Komornik, Kong & Li, 2018)
- Local dimension: for $x > 0$ and $q \notin \overline{\mathscr{U}}$,

  $$\lim_{\delta\to 0} \dim_H \big(\mathscr{U}(x) \cap (q-\delta, q+\delta)\big) = \lim_{\delta\to 0} \dim_H \big(\mathcal{U}_q \cap (x-\delta, x+\delta)\big).$$

  (Kong, Li, Lü, Wang & Xu, 2020)
- Kong (2016): initiated a study of the minimum of $\mathscr{U}(x)$.

The minimal univoque base for x

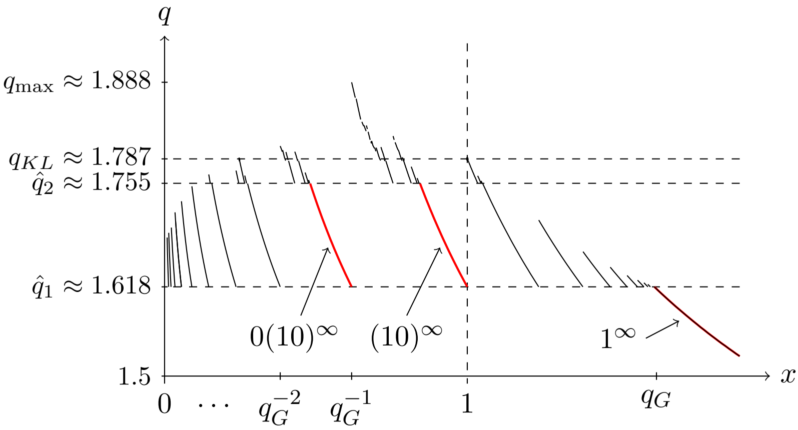
Definition

Let
$$q_{\min}(x) := \inf \mathscr{U}(x), \qquad x > 0.$$

We want to understand the function $x \mapsto q_{\min}(x)$.

- For which x is the "inf" actually a minimum?
- Algorithm for computing $q_{\min}(x)$?
- Continuity properties of $q_{\min}(x)$?
- Structure of level sets of $q_{\min}$? (Finite, countable, uncountable?)

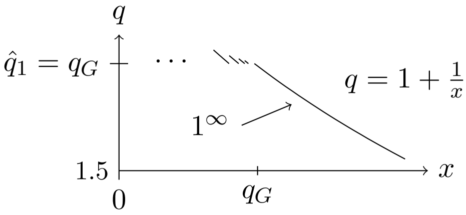
Graph of $q_{\min}$



Graph of $q_{\min}(x)$. Here $q_G := \frac{\sqrt{5}+1}{2}$.

The "tail" of the graph

Easy fact: If $x \geq q_G$, then $q_{\min}(x) = 1 + \frac{1}{x} =: q_x$.



Reason: If $1 < q \leq q_G$, then $\mathcal{U}_q = \{0, \frac{1}{q-1}\}$. (Erdős, Jóo & Komornik, 1990.) Since

$$x = \frac{1}{q-1} \qquad \Longleftrightarrow \qquad q = 1 + \frac{1}{x} = q_x,$$

it follows that $q_x \in \mathscr{U}(x)$ for all $x > 1$. But if $x \geq q_G$ and $q' < q_x$, then $q' < 1 + \frac{1}{q_G} = q_G$, so $\mathcal{U}_{q'} = \{0, \frac{1}{q'-1}\} \not\ni x$.

Some preliminaries

Definition

For $q \in (1,2]$, let $\alpha(q) = \alpha_1(q)\alpha_2(q)\dots$ be the quasi-greedy expansion of 1 in base q; that is, the lexicographically largest expansion not ending in 0^∞. **Ex.** $\alpha(q_G) = (10)^\infty$.

Let $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ denote the shift map.

Lemma

(i) *The map* $q \mapsto \alpha(q)$ *is strictly increasing.*

(ii) $\sigma^n(\alpha(q)) \preceq \alpha(q)$ *for all* $n \geq 1$*;*

(iii) $q \in \overline{\mathscr{U}}$ *if and only if*

$$\sigma^n(\alpha(q)) \succ \overline{\alpha(q)} \quad \forall\, n \geq 1;$$

Here $\overline{\alpha(q)}$ denotes the *reflection* of $\alpha(q)$:

$$\overline{\alpha_1\alpha_2\dots} := (1-\alpha_1)(1-\alpha_2)\dots.$$

An algorithm for determining $q_{\min}(x)$

Set $q_G := (1+\sqrt{5})/2$. For simplicity, we describe the algorithm for $x \in [q_G^{-1}, 1)$, but it extends easily to all $x > 0$.

Proposition

For $x \in [q_G^{-1}, 1)$, let q_1 be the root in $(1,2]$ of $q(q-1) = 1/x$. Then

(i) $q_1 > q_G$, *and* $q_1 \notin \mathscr{U}(x)$;

(ii) $q_{\min}(x) \geq q_1$, *with equality if and only if* $q_1 \in \overline{\mathscr{U}}$.

The point of the choice of q_1 is, that

$$x = (01^\infty)_{q_1} = \left(1\,\overline{\alpha(q_1)}\right)_{q_1}.$$

The number q_1 from the Proposition is our initial lower bound. Write $\alpha(q) = \alpha_1\alpha_2\ldots$.

The first lower bound

If $q_1 \in \overline{\mathscr{U}}$, we are done: $q_{\min}(x) = q_1$. Otherwise, there is $n \geq 1$ such that

$$\overline{\alpha_n \alpha_{n+1} \dots} \succeq \alpha(q).$$

Let n_1 be the *smallest* such n, and set $B_1 := 1\overline{\alpha_1 \dots \alpha_{n_1-2}}$. (We have $n_1 \geq 3$ because $\alpha(q)$ begins with 11 for $q > q_G$.)

Example

Let $x = 2/3$. Then $q_1 = (1+\sqrt{7})/2 \approx 1.8228757$. Then

$$\begin{aligned} \alpha(q_1) = & 11011001001011000110 00\dots, \\ & {\color{blue}0010}\ \ {\color{red}11011010011100111\dots} \end{aligned}$$

so $n_1 = 6$ and $B_1 = 10010$.

The first upper bound

Set $y := \left(B_1^\infty\right)_{q_1}$. We can show that $y > x$ and $y \in \mathcal{U}_{q_1}$.
Hence there is a base $r_1 > q_1$ such that $x = \left(B_1^\infty\right)_{r_1}$.
This implies $x \in \mathcal{U}_{r_1}$, and so $q_{\min}(x) \le r_1$.

This is our first upper bound.

Example (ctd.)

$$\frac{2}{3} = \left((10010)^\infty\right)_{r_1} \qquad \Longrightarrow \qquad r_1 \approx 1.8324610.$$

We now know: $1.8228757 \le q_{\min}(2/3) \le 1.8324610.$

The second lower bound

Next, let q_2 be the base such that $(B_1 01^\infty)_{q_2} = x$.

By analog of the Proposition, $q_{\min}(x) \geq q_2$ with equality iff $q_2 \in \overline{\mathscr{U}}$. So if $q_2 \in \overline{\mathscr{U}}$ we are done; otherwise let

$$n_2 := \min\{n \geq 1 : \overline{\alpha_n(q_2)\alpha_{n+1}(q_2)\dots} \succeq \alpha(q_2)\}.$$

Set $B_2 = 1\overline{\alpha_1(q_2)\dots\alpha_{n_2-2}(q_2)}$.

Example (ctd.)

Setting $(10010\,01^\infty)_{q_2} = 2/3$ gives $q_2 \approx 1.83161197$. Then

$$\begin{aligned}\alpha(q_2) =& 110110100110100110101000\,1001\dots,\\ & 0010010110010110010\mathbf{1}\;\; 1110110\dots\end{aligned}$$

so $n_2 = 22$ and $B_2 = 10010010110010110010 1$.
We now know: $1.83161197 \leq q_{\min}(2/3) \leq 1.8324610$.

The second upper bound

Next, let r_2 be the base such that

$$(B_1 B_2^\infty)_{r_2} = x.$$

We can show that $r_2 \le r_1$ and $x \in \mathcal{U}_{r_2}$, so $q_{\min}(x) \le r_2$.

Example (ctd.)

Setting

$$\left(10010(100100101100101100101)^\infty\right)_{r_2} = 2/3$$

gives $r_2 \approx 1.83161199$.

We now know: $1.83161197 \le q_{\min}(2/3) \le 1.83161199$.

(In fact, we'll see later that $q_{\min}(2/3)$ is *exactly* r_2.)

Keep iterating...

At the kth stage, let q_k be the base such that

$$(B_1 \cdots B_{k-1} 0 1^\infty)_{q_k} = x.$$

If $q_k \in \overline{\mathscr{U}}$, we are done and $q_{\min}(x) = q_k$ (but "inf" not attained). Otherwise, let

$$n_k := \min\{n \geq 1 : \overline{\alpha_n(q_k)\alpha_{n+1}(q_k)\dots} \succeq \alpha(q_k)\},$$

and set $B_k := 1\,\overline{\alpha_1(q_k)\dots\alpha_{n_k-2}(q_k)}$. Then let r_k be the base such that

$$(B_1 \dots B_{k-1} B_k^\infty)_{r_k} = x.$$

We can show:

(i) $q_1 < q_2 < \dots$ and $r_1 \geq r_2 \geq \dots$;

(ii) $q_k \leq q_{\min}(x) \leq r_k$ for each k;

(iii) $r_k - q_k \to 0$. So $\lim q_k = \lim r_k = q_{\min}(x)$.

It gets even better!

Recall

$$n_k := \min\{n \geq 1 : \overline{\alpha_n(q_k)\alpha_{n+1}(q_k)\dots} \succeq \alpha(q_k)\}.$$

Now define also

$$m_k := \inf\{m > n_k : \overline{\alpha_{n_k} \dots \alpha_m(q_k)} \succ \alpha_1(q_k) \dots \alpha_{m-n_k+1}(q_k)\}$$

(possibly $m_k = \infty$). If at any point it so happens that

$$\alpha_1(q_k) \dots \alpha_{m_k}(q_k) = \alpha_1(r_k) \dots \alpha_{m_k}(r_k) \quad \text{with} \quad m_k < \infty,$$

then the first m_k digits of $\alpha(q)$ must match this block for all $q \in (q_k, r_k)$, so we have $B_j = B_k$ for all $j \geq k$.

This implies $q_{\min}(x) = r_k = \min \mathscr{U}(x)$, and the smallest unique expansion of x is $B_1 \dots B_{k-1} B_k^\infty$.

Example

Back to our example with $x = 2/3$: we saw

$$\alpha(q_1) = 110110\textcolor{red}{0}100101100011000\ldots,$$
$$\textcolor{blue}{1101101}$$

so $n_1 = 6$ and $m_1 = 12$. With $\left((10010)^\infty\right)_{r_1} = 2/3$, we have

$$\alpha(r_1) = 110110\textcolor{red}{1}0100011011001 01000\ldots.$$

Note $\alpha_7(r_1) \neq \alpha_7(q_1)$. At the next step,

$$\alpha(q_2) = 1101101001101001101 01000\,|1001\ldots,$$

so $n_2 = 22$ and $m_2 = 24$. With $(B_1 B_2^\infty)_{r_2} = 2/3$, we have

$$\alpha(r_2) = 110110100110100110101000\,|1010\ldots.$$

Now $\alpha_1(q_2)\ldots\alpha_{m_2}(q_2) = \alpha_1(r_2)\ldots\alpha_{m_2}(r_2)$, so $q_{\min}(2/3) = r_2$.

Classification of points

The algorithm yields three possibilities for any point x:

- Type I: $q_k \in \overline{\mathscr{U}}$ for some k. Then $q_{\min}(x) = q_k \notin \mathscr{U}(x)$.
- Type II: $q_k \notin \overline{\mathscr{U}}$ for all k, and

$$\alpha_1(q_k) \dots \alpha_{m_k}(q_k) = \alpha_1(r_k) \dots \alpha_{m_k}(r_k) \tag{1}$$

  for some k. Then $q_{\min}(x) = r_k \in \mathscr{U}(x)$.
- Type III: $q_k \notin \overline{\mathscr{U}}$ for all k, and (1) does not hold for any k. Then $q_{\min}(x) = \lim_{k\to\infty} q_k = \lim_{k\to\infty} r_k$. We can show that in this case, too, $q_{\min}(x) \in \mathscr{U}(x)$.

Let X_I, X_{II}, X_{III} be the sets of points of types I, II and III, respectively. We can show:

$$\dim_H X_I \approx .8546, \qquad \lambda(X_{II}) > 0, \qquad \mathrm{card}(X_{III}) \geq \aleph_0.$$

Conjecture: X_{II} has full Lebesgue measure.

Continuity properties of $q_{\min}$

Theorem (A. & Kong, 2020)

The function $x \mapsto q_{\min}(x)$:

(i) *is right-continuous;*

(ii) *has a left-hand limit at every point;*

(iii) *and has no downward jumps.*

As a result, $q_{\min}$ has only countably many discontinuities.

Note

$q_{\min}$ has path properties similar to a spectrally positive Lévy process.

The proof is technical, but relies in part on the Algorithm presented earlier.

Definition

Let

$$L(q) := \{x > 0 : q_{\min}(x) = q\}, \qquad q \in (1, 2]$$

Theorem (A. & Kong, 2020)

(i) $L(q)$ *is finite whenever* $q \notin \overline{\mathscr{U}}$. *Thus,* $L(q)$ *is finite for Lebesgue-almost every* q.

(ii) $L(q)$ *is infinite for infinitely many* $q \in \overline{\mathscr{U}}$, *including* $q = q_{KL}$.

(iii) $L(q_{KL})$ *contains infinitely many left- and infinitely many right accumulation points.*

The level set $L(q_{KL})$

We can describe an infinite subset of $L(q_{KL})$ as follows. Recall

$$\alpha(q_{KL}) = (\tau_i)_{i=1}^{\infty} = 11010011001011010010110011010011\ldots$$

Set

$$x_n := (\tau_n \tau_{n+1} \ldots)_{q_{KL}}.$$

So $x_1 = 1$, $x_2 = q_{KL} - 1$, etc. Define a set $S \subset \mathbb{N}$ recursively by

(a) $2, 3, 4 \in S$;

(b) For each $k \geq 2$ and $2^k < n \leq 2^{k+1}$, $n \in S$ if and only if

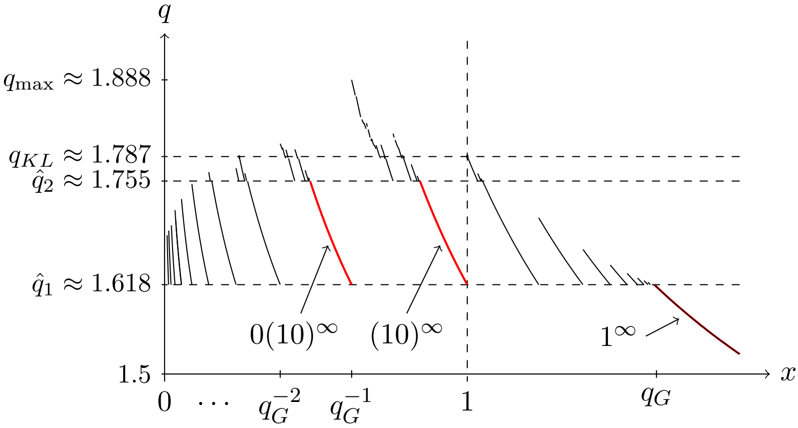
$$n > 3 \cdot 2^{k-1} \qquad \text{and} \qquad n - 3 \cdot 2^{k-1} \in S.$$

Thus, $S = \{2, 3, 4, 8, 14, 15, 16, 26, 27, 28, 32, \ldots\}$.

Theorem (A. & Kong, 2020)

$q_{\min}(x_n) = q_{KL}$ *if and only if* $n \in S \cup \{1\}$.

Graph of $q_{\min}$



Graph of $q_{\min}(x)$. Here $q_G := \frac{\sqrt{5}+1}{2}$.

The maximum of $q_{\min}(x)$

Theorem (A. & Kong, 2020)

The maximum value of $q_{\min}$ *is attained at* $x = 1/q_G$, *and*

$$\max_{x>0} q_{\min}(x) = q_{\min}(1/q_G) \approx 1.888453328$$

is the unique base q *such that*

$$\left(1(0010001100011)^\infty\right)_q = \frac{1}{q_G} = \frac{\sqrt{5}-1}{2}.$$

(It is an algebraic integer of degree 26.)

Open problems (selection)

Question 1

Does $L(q_{KL})$ have any *two*-sided accumulation points?
(If the answer is no, then it follows that $L(q_{KL})$ is countable.)
More generally, are all level sets of $q_{\min}$ countable?

Question 2

Is $L(q_{KL})$ closed? More generally, are all level sets of $q_{\min}$ closed?
(We do know each level set has a smallest and a largest element.)

Question 3

Is $q_{\min}(x)$ algebraic over $\mathbb{Q}(x)$ for every $x > 0$?
(This is true for all points of type I or II, and also for the points $x_n = (\tau_n \tau_{n+1} \dots)_{q_{KL}}$, which are of type III. We don't know if it's true for *all* points of type III.)

Thank you!