On the existence of Trott numbers relative to multiple bases

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The basic question

Question:

Is it possible for a number to have its continued fraction expansion agree with its decimal expansion? That is,

$$[0; a_1, a_2, a_3 \dots] = 0.a_1 a_2 a_3 \dots$$

If we require each $a_i \in \{1, 2, \dots, 9\}$, it is easy to see that the answer is no.

Trott's Constant

In 2006, M. Trott published the following example in a blog post:

$$\cfrac{1}{1+\cfrac{1}{0+\cfrac{1}{8+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{0+\cfrac{1}{\dots}}}}}}} = 0.1084101\dots$$

This number is now known as *Trott's constant* - see OEIS sequence A039662.

Trott numbers

The use of the digit 0 is, of course, unnatural from a dynamical point of view. So we slightly tweak the definition:

Definition

Let $b \in \mathbb{N}_{\geq 2}$. A number $x \in (0,1)$ is a Trott number in base b if

$$x = [0; a_1, a_2, a_3, \dots] = (0.\hat{a}_1\hat{a}_2\hat{a}_3\dots)_b,$$

where $\hat{a}_i$ is the string of digits corresponding to the base b representation of a_i.

For example, in base 10,

$$[0; 3, 29, 545, 6, \dots] = 0.3\,29\,545\,6\dots.$$

Question:

For which bases, if any, do Trott numbers exist? And how can we construct them?

Two initial observations

- If x is a Trott number, that is,

$$x = [0; a_1, a_2, a_3, \dots] = (0.\hat{a}_1\hat{a}_2\hat{a}_3\dots)_b,$$

then x is neither rational nor quadratic irrational.

- If $x = [0; a_1, a_2, a_3, \dots]$ is Trott in base b, then $a_1 = \lfloor\sqrt{b}\rfloor$.

Proof. The intervals

$$\left[\frac{a_1}{b}, \frac{a_1+1}{b}\right] \quad \text{and} \quad \left[\frac{1}{a_1+1}, \frac{1}{a_1}\right]$$

must overlap.

This shows also that no Trott numbers exist in base b if b is a perfect square: If $b = k^2$, then $a_1 = k$ and the two intervals only share an endpoint.

The main theorem: good bases

Theorem (A, Jackson, Jones & Lambert, 2021)

There exists a Trott number in base b if and only if

$$b \in \{3\} \cup \bigcup_{k=1}^{\infty} \{k^2+1, k^2+2, \ldots, k^2+k\} =: \Gamma.$$

Furthermore, if T_b is nonempty, then it is uncountable.

So the first few "good" bases are 2,3,5,6,10,11,12,17,18,19,20, ...

The proof of sufficiency is rather involved. See the slides of the talk by T. Jones on October 5, 2021 for the case $b = 10$. The construction is inductive, and the key is to choose a_2 carefully!

Proof of necessity

To prove necessity, suppose $b \notin \Gamma$ and there is a Trott number

$$x = [0; a_1, a_2, a_3, \dots] = (0.\hat{a}_1\hat{a}_2\hat{a}_3\dots)_b.$$

Then there is a unique $k \in \mathbb{N}$, $k \geq 2$ such that

$$k^2 + k < b < (k+1)^2.$$

By our earlier observation, $a_1 = \lfloor \sqrt{b} \rfloor = k$. Suppose $a_2 = j$. Then

$$x \in \left[\frac{1}{k + \frac{1}{j}}, \frac{1}{k + \frac{1}{j+1}} \right].$$

Case 1. If $j > (b-1)/k$, then

$$\frac{1}{k + \frac{1}{j}} > \frac{b-1}{bk} \geq \frac{k+1}{b}$$

so x does not begin with digit k, contradiction.

Proof of necessity (ctd)

Recall that $a_1 = k$, $a_2 = j$ and

$$x \in \left[\frac{1}{k+\frac{1}{j}}, \frac{1}{k+\frac{1}{j+1}}\right].$$

Case 2. If $j \leq (b-1)/k$, then $j < b$ and it can be checked easily that

$$\frac{1}{k+\frac{1}{j}} > \frac{k}{b} + \frac{j+1}{b^2},$$

so the second digit of x is at least $j+1$. Contradiction again!

Some other results

Let T_b denote the set of Trott numbers in base b.

Theorem (A, Jackson, Jones & Lambert, 2021)

For each $b \in \Gamma$, T_b is a complete G_δ set. (That is, T_b is G_δ but not F_σ.)

Theorem (A, Jackson, Jones & Lambert, 2021)

Let

$$T := \bigcup_{n=2}^{\infty} T_b$$

be the set of numbers that are Trott in some *base. Then T is nowhere dense, and $\dim_H T < 1$.*

(In fact we have explicit upper bounds for $\dim_H T_b$ for each b, but they are almost surely very bad!)

Conjectures

Conjecture 1

$\dim_H T_b > 0$ for all $b \in \Gamma$.

Conjecture 2

$\dim_H T_b \to 0$ as $b \to \infty$.

Conjecture 3

$T_b \cap T_c = \emptyset$ whenever $b \neq c$.

We'll focus here on Conjecture 3, which says that no number is Trott in more than one base.

Conjecture 3:

$T_b \cap T_c = \emptyset$ whenever $b \neq c$.

Exploration of Conjecture 3

Let $2 \leq b < c$, and suppose

$$x = [0; a_1, a_2, \ldots] \in T_b \cap T_c.$$

Since $a_1 = \lfloor \sqrt{b} \rfloor = \lfloor \sqrt{c} \rfloor$, we see at once that b and c must belong to the same interval $[k^2 + 1, k^2 + k]$.

From here, we tried (with only partial success) to obtain a contradiction by using only information about a_2.

Two lemmas

Lemma 1

Let $b < c$. If a_2 has l digits in base b and m digits in base c, then

$$2 \leq m < l.$$

Lemma 2

Let $k \geq 2$ and $k^2 + 1 \leq b \leq k^2 + k$. Assume $b \geq 6$. Suppose $x = [0; a_1, a_2, \dots] \in T_b$ and a_2 has $l \geq 2$ digits in base b. Then

$$a_2 = \left\lceil \frac{b^l(b - k^2)}{k} - \frac{b}{k(b - k^2)} \right\rceil - 1.$$

Proof. Follows (after some straightforward but cumbersome algebra) from the fact that $a_1 = k$, so these intervals overlap:

$$\left[\frac{k}{b} + \frac{a_2}{b^{l+1}}, \frac{k}{b} + \frac{a_2 + 1}{b^{l+1}}\right] \quad \text{and} \quad \left[\frac{a_2}{ka_2 + 1}, \frac{a_2 + 1}{k(a_2 + 1) + 1}\right].$$

Picking the low-hanging fruit

Corollary (Two special cases)

Let a_2 have l digits in base b, where $l \geq 2$.

(i) *If $b = k^2 + 1$ and $k \geq 3$, then*

$$a_2 = \frac{(k^2+1)^l - (k^2+1)}{k} - 1.$$

(ii) *If $b = k^2 + k$ and $k \geq 2$, then $a_2 = b^l - 2$.*

Proposition

No number is Trott in both base $k^2 + 1$ and $k^2 + k$.

Proof. By the Corollary, we would have

$$\frac{(k^2+1)^l - (k^2+1)}{k} - 1 = (k^2+k)^m - 2.$$

This is impossible!

Specific base pairs: using modular arithmetic

Example. Let $b = 17 = 4^2 + 1$, $c = 19 = 4^2 + 3$ (two "good" bases!). Let a_2 have l digits in base b and m digits in base c. By the Corollary and Lemma 2,

$$a_2 = \frac{17^l - 21}{4} = \begin{cases} \frac{3 \cdot 19^m - 9}{4} & \text{if } m \text{ is odd,} \\ \frac{3 \cdot 19^m - 7}{4} & \text{if } m \text{ is even,} \end{cases}$$

Thus,

$$17^l - 3 \cdot 19^m = \begin{cases} 12 & \text{if } m \text{ is odd,} \\ 14 & \text{if } m \text{ is even.} \end{cases}$$

We can instantly rule out 12. This leaves us with

$$17^l - 3 \cdot 19^m = 14.$$

The example of 17 and 19, ctd.

Does the equation

$$17^l - 3 \cdot 19^m = 14$$

have a solution (l, m) in $(\mathbb{Z}_+ \times \mathbb{Z}_+)$? Consider first the equation modulo 6:

$$(-1)^l - 3 \cdot 1^m \equiv 2 \pmod{6}.$$

This shows l must be odd. Now consider the equation modulo 19:

$$(-2)^l \equiv -5 \pmod{19}.$$

Since l is odd, this becomes $2^l \equiv 5 \pmod{19}$. However, the odd powers of 2 modulo 19 are

$$2, 8, 13, 14, 18, 15, 3, 12, 10, 2, 8, 13, \ldots.$$

Hence, $T_{17} \cap T_{19} = \emptyset$.

What the lemmas tell us

Recall:

Lemma 2

Let $k \geq 2$ and $k^2 + 1 \leq b \leq k^2 + k$. Assume $b \geq 6$. Suppose $x = [0; a_1, a_2, \ldots] \in T_b$ and a_2 has $l \geq 2$ digits in base b. Then

$$a_2 = \left\lceil \frac{b^l (b - k^2)}{k} - \frac{b}{k(b - k^2)} \right\rceil - 1.$$

Now let $k \geq 3$ and $k^2 + 1 \leq b < c \leq k^2 + k$. Suppose a_2 has l digits in base b and m digits in base c. By Lemmas 1 and 2, $2 \leq m < l$ and

$$\left\lceil \frac{b^l (b - k^2)}{k} - \frac{b}{k(b - k^2)} \right\rceil = \left\lceil \frac{c^m (c - k^2)}{k} - \frac{c}{k(c - k^2)} \right\rceil.$$

Two tight inequalities

We just saw that

$$\left\lceil \frac{b^l(b-k^2)}{k} - \frac{b}{k(b-k^2)} \right\rceil = \left\lceil \frac{c^m(c-k^2)}{k} - \frac{c}{k(c-k^2)} \right\rceil.$$

Thus,

$$\left| \frac{b^l(b-k^2)}{k} - \frac{b}{k(b-k^2)} - \left(\frac{c^m(c-k^2)}{k} - \frac{c}{k(c-k^2)} \right) \right| < 1.$$

Some simplification leads to

$$\frac{k^2(c-b)}{(b-k^2)(c-k^2)} - k < b^l(b-k^2) - c^m(c-k^2) < \frac{k^2(c-b)}{(b-k^2)(c-k^2)} + k,$$

and in particular,

$$\left| b^l(b-k^2) - c^m(c-k^2) \right| < k^2.$$

Baker's theorem

Recall $k^2 + 1 \leq b < c \leq k^2 + k$. From the inequality

$$\left| b^l(b - k^2) - c^m(c - k^2) \right| < k^2$$

we can deduce (after minor technicalities) that

$$l > m > k \log k.$$

To make further progress, we need "Baker's theorem", really a collection of theorems (going back to the 1970s) giving lower bounds for expressions of the form

$$\Lambda := \beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 + \cdots + \beta_n \log \alpha_n$$

provided $\Lambda \neq 0$, where $\alpha_1, \ldots, \alpha_n$ are algebraic numbers and $\beta_1, \ldots, \beta_n$ are rational integers.

Baker's theorem for our setting

The version of Baker's theorem that we'll use (stated in simplified form for our setting) is:

Lemma (Matveev, 2000)

Let $\alpha_1, \dots, \alpha_n \in \mathbb{Q}$, *not 0 or 1, and* $\beta_1, \dots, \beta_n$ *be integers. Let*

$$\Lambda := \beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 + \cdots + \beta_n \log \alpha_n \neq 0.$$

Then

$$|\Lambda| \geq \exp\{-C_n h(\alpha_1) \cdots h(\alpha_n) \log(eB)\},$$

where $B := \max\{|\beta_1|, \dots, |\beta_n|\}$,

$$C_n := \min\left\{\frac{e}{2} \cdot 30^{n+3} n^{4.5}, 2^{6n+20}\right\},$$

and for $\alpha = p/q$ *in lowest terms,* $h(\alpha) := \log \max\{|p|, |q|\}$ *is the (logarithmic) Weil height of* α.

Applying Matveev's lemma

It's not too hard to show that $b^l(b-k^2) - c^m(c-k^2) \neq 0$. Assume WLOG that $c^m(c-k^2) > b^l(b-k^2)$. Let

$$\Lambda := m\log c - l\log b + \log\left(\frac{c-k^2}{b-k^2}\right),$$

so $\Lambda > 0$. Applying Matveev's lemma with $\alpha_1 = c$, $\alpha_2 = b$, $\alpha_3 = (c-k^2)/(b-k^2)$, $\beta_1 = m$, $\beta_2 = -l$ and $\beta_3 = 1$ we obtain

$$\begin{aligned}
\frac{c^m(c-k^2)}{b^l(b-k^2)} - 1 = e^\Lambda - 1 > \Lambda \\
&\geq \exp\{-C_3 \log b \log c \log(c-k^2)\log(el)\} \\
&\geq \exp\{-C_3\big(\log(k^2+k)\big)^2 \log k \log(el)\} \\
&\approx \exp\{-4C_3(\log k)^3 \log l\} \\
&=: A_{k,l}.
\end{aligned}$$

Applying Matveev's lemma (ctd)

Recall that $\left|b^l(b-k^2) - c^m(c-k^2)\right| < k^2$. Hence,

$$\begin{aligned} k^2 > c^m(c-k^2) - b^l(b-k^2) &\gtrsim A_{k,l} b^l (b-k^2) \\ &\geq A_{k,l}(k^2+1)^l > A_{k,l} k^{2l}. \end{aligned}$$

Thus,

$$k^{2l-2} \lesssim A_{k,l}^{-1} = \exp\{4C_3 (\log k)^3 \log l\}.$$

Taking logs,

$$2(l-1)\log k \lesssim 4C_3(\log k)^3 \log l.$$

Since $l > m > k \log k$, we deduce that

$$k(\log k)^2 \lesssim 2C_3(\log k)^3 \log(k \log k),$$

or $k \lesssim 2C_3 \log k \log(k \log k)$. Since $C_3 \approx 1.39 \times 10^{11}$, this yields

$$k \lesssim 3.44 \times 10^{14}, \qquad b, c \leq k^2 + k \lesssim 1.185 \times 10^{29}.$$

This "solves" the conjecture...

The conjecture is now reduced to pairs (b, c) of bases below 1.185×10^{29}, finitely many!

If b and c are below this threshold (and hence $k \leq 3.44 \times 10^{14}$), then the inequality

$$2(l-1)\log k \lesssim 4C_3(\log k)^3 \log l$$

gives an implicit upper bound for l, and since $m < l$, there are only finitely many pairs (l, m) to try.

This reduces the verification of the conjecture to a finite number of operations.

Waiting for quantum computers...

Collecting our partial results

Theorem

Let $b, c \in \Gamma$ with $b < c$. Then $T_b \cap T_c = \emptyset$ if at least one of the following holds:

(i) $\lfloor \sqrt{b} \rfloor \neq \lfloor \sqrt{c} \rfloor$;

(ii) $\gcd(b, c) > 1$;

(iii) $c = b + 1$;

(iv) $c = b + 2$ *and there exists* $k \geq 3$ *such that*

$$k^2 < b \leq k^2 + \sqrt{\frac{2k^2}{k-2} + 1} - 1 \approx k^2 + \sqrt{2k};$$

(v) *There exists* $k \geq 2$ *such that* $b = k^2 + 1$ *and* $c = k^2 + k$;

(vi) $b > 1.185 \times 10^{29}$.

Thank you!