Diophantine Approximation for Systems of Linear Forms

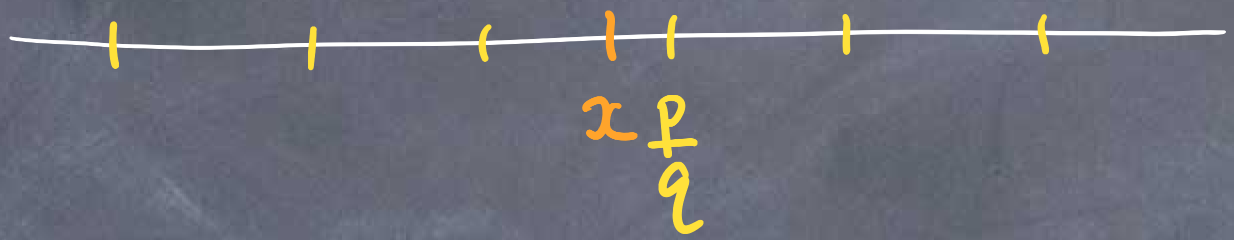
Joint work with Felipe Ramírez (Wesleyan, US)

One World Numeration Seminar

21st March 2023

Diophantine Approximation

Example



Given any $x \in \mathbb{R}$ and $q \in \mathbb{N}$, $\exists\ p \in \mathbb{Z}$ s.t.

$$\left| x - \frac{p}{q} \right| < \frac{1}{q}.$$

Theorem (Dirichlet, 1842) For any $x \in \mathbb{R}$, $\exists$ i.m. $q \in \mathbb{N}$ s.t.

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}$$

for some $p \in \mathbb{Z}$.

Given $\psi : \mathbb{N} \to [0,\infty)$, define

$$\mathcal{A}(\psi) = \left\{ x \in [0,1] : \left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q} \text{ for i.m. } (p,q) \in \mathbb{Z} \times \mathbb{N} \right\}.$$

Khintchine's Theorem (1924) For any $\psi : \mathbb{N} \to [0,\infty)$,

$$\mathcal{L}(\mathcal{A}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \psi(q) < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} \psi(q) = \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$

First Borel–Cantelli Lemma Let (X,μ) be a measure space and let $(A_q)_{q \in \mathbb{N}}$ be a sequence of measurable subsets of X. If $\sum_{q=1}^{\infty} \mu(A_q) < \infty$, then

$$\mu\left(\limsup_{q \to \infty} A_q \right) = 0.$$

$$\limsup_{q \to \infty} A_q = \{ x \in X : x \in A_q \text{ for i.m. } q \in \mathbb{N} \}$$

Proof of Convergence Part of Khintchine's Theorem

For each $q \in \mathbb{N}$, let $A_q = \bigcup_{p=0}^{q} B\left(\frac{p}{q}, \frac{\psi(q)}{q}\right) \cap [0,1]$.

Then $\mathcal{A}(\psi) = \limsup_{q \to \infty} A_q$.

Note that $\mathcal{L}(A_q) \leq q \times \frac{2\psi(q)}{q} = 2\psi(q)$.

So, if $\sum_{q=1}^{\infty} \psi(q) < \infty$, then $\sum_{q=1}^{\infty} \mathcal{L}(A_q) < \infty$ and $\mathcal{L}(\mathcal{A}(\psi)) = 0$.

Question: Do we really need monotonicity of ψ in Khintchine's Theorem?

Theorem (Duffin + Schaeffer, 1941) Khintchine's Theorem is not true in general if ψ is not monotonic.

- Let $\mathcal{A}'(\psi) = \{x \in [0,1] : |x - \frac{p}{q}| < \frac{\psi(q)}{q}$ for i.m. $(p,q) \in \mathbb{Z} \times \mathbb{N}$ with $\gcd(p,q) = 1\}$.

Duffin-Schaeffer Conjecture (1941) For any $\psi : \mathbb{N} \to [0,\infty)$,

$$\mathcal{L}(\mathcal{A}'(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \varphi(q) \frac{\psi(q)}{q} < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} \varphi(q) \frac{\psi(q)}{q} = \infty, \end{cases}$$

where $\varphi(n) = \#\{1 \leq k \leq n : \gcd(k,n) = 1\}$ for $n \in \mathbb{N}$.

Theorem (Koukoulopoulos + Maynard, 2020) The DSC is true.

Diophantine Approximation for Systems of Linear Forms

Let $n, m \geq 1$ be integers.

Let $\psi : \mathbb{N} \to [0, \infty)$.

Write $\|\cdot\|$ for the supremum norm.

Define

$n \times m$ matrix

$$A_{n,m}(\psi) = \left\{ \underline{x} \in [0,1]^{nm} : \|\underline{q}\,\underline{x} - \underline{p}\| < \psi(\|\underline{q}\|) \text{ for i.m. } (\underline{p}, \underline{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \right\}.$$

n-dim. row $\qquad$ m-dim. row

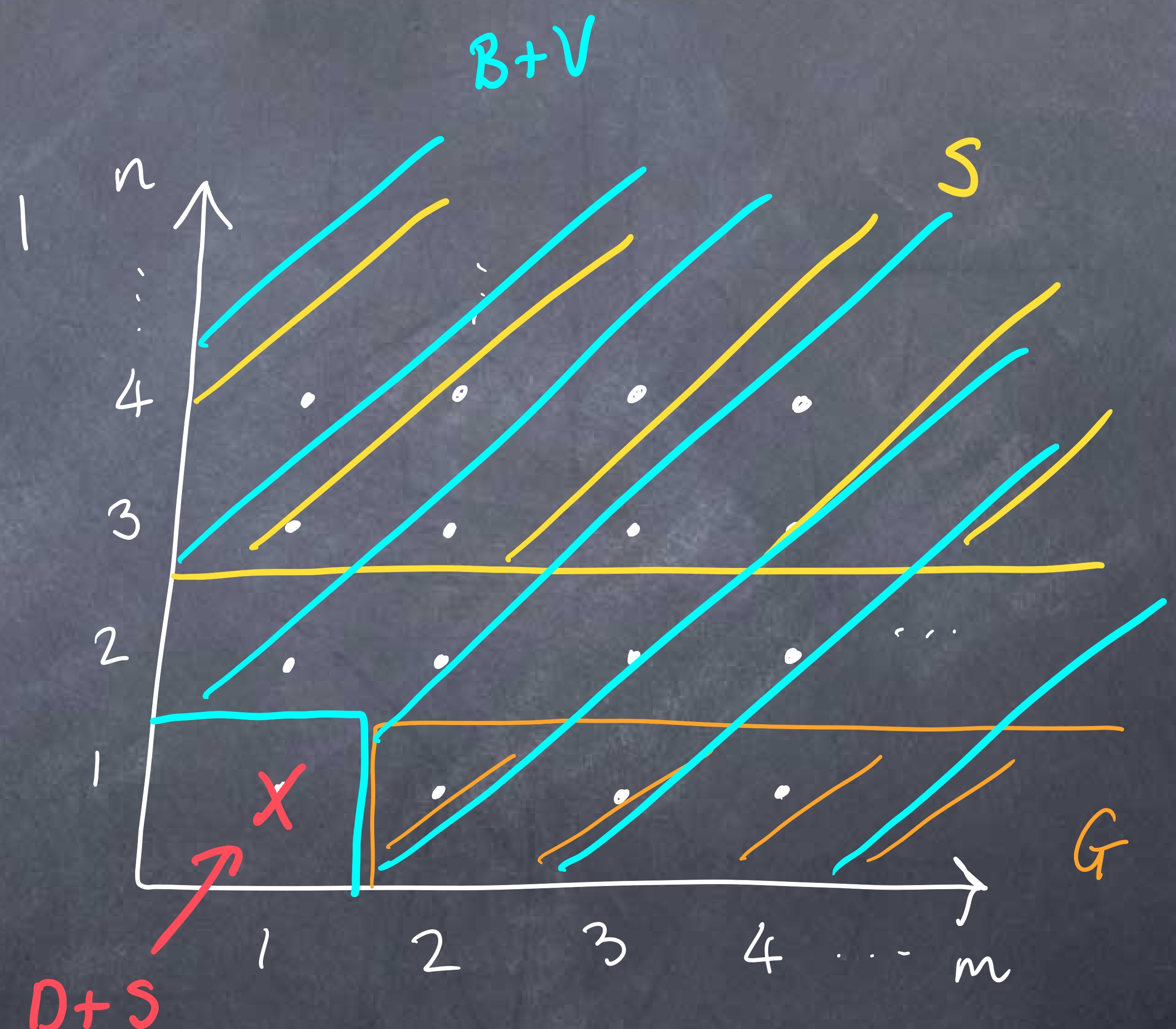
Khintchine - Groshev Theorem (1938) For any $\psi : \mathbb{N} \to [0, \infty)$,

$$\mathcal{L}(A_{n,m}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n-1} \psi(q)^m < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} \quad '' \quad = \infty \text{ \underline{and} } \psi \text{ is monotonic.} \end{cases}$$

Question Do we really need monotonicity of Ψ in the Khintchine–Groshev Theorem?

Answer YES when $n=m=1$, NO otherwise.

- ☆ Duffin + Schaeffer (1941) : $n=m=1$
- ☆ Gallagher (1965): $n=1$, $m \geq 2$
- ☆ Sprindžuk (1979) : $n \geq 3$
- ☆ Beresnevich + Velani (2010) : $nm > 1$



Inhomogeneous Khintchine-Groshev Theorem

Let $n, m \geq 1$ be integers.

Let $\psi : \mathbb{N} \to [0,\infty)$.

Let $y \in \mathbb{R}^m$.

Define

$$A^{y}_{n,m}(\psi) = \{ \underline{x} \in [0,1]^{nm} : \| \underline{q}\underline{x} - \underline{p} - y \| < \psi(\|\underline{q}\|) \text{ for i.m. } (\underline{p},\underline{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \}.$$

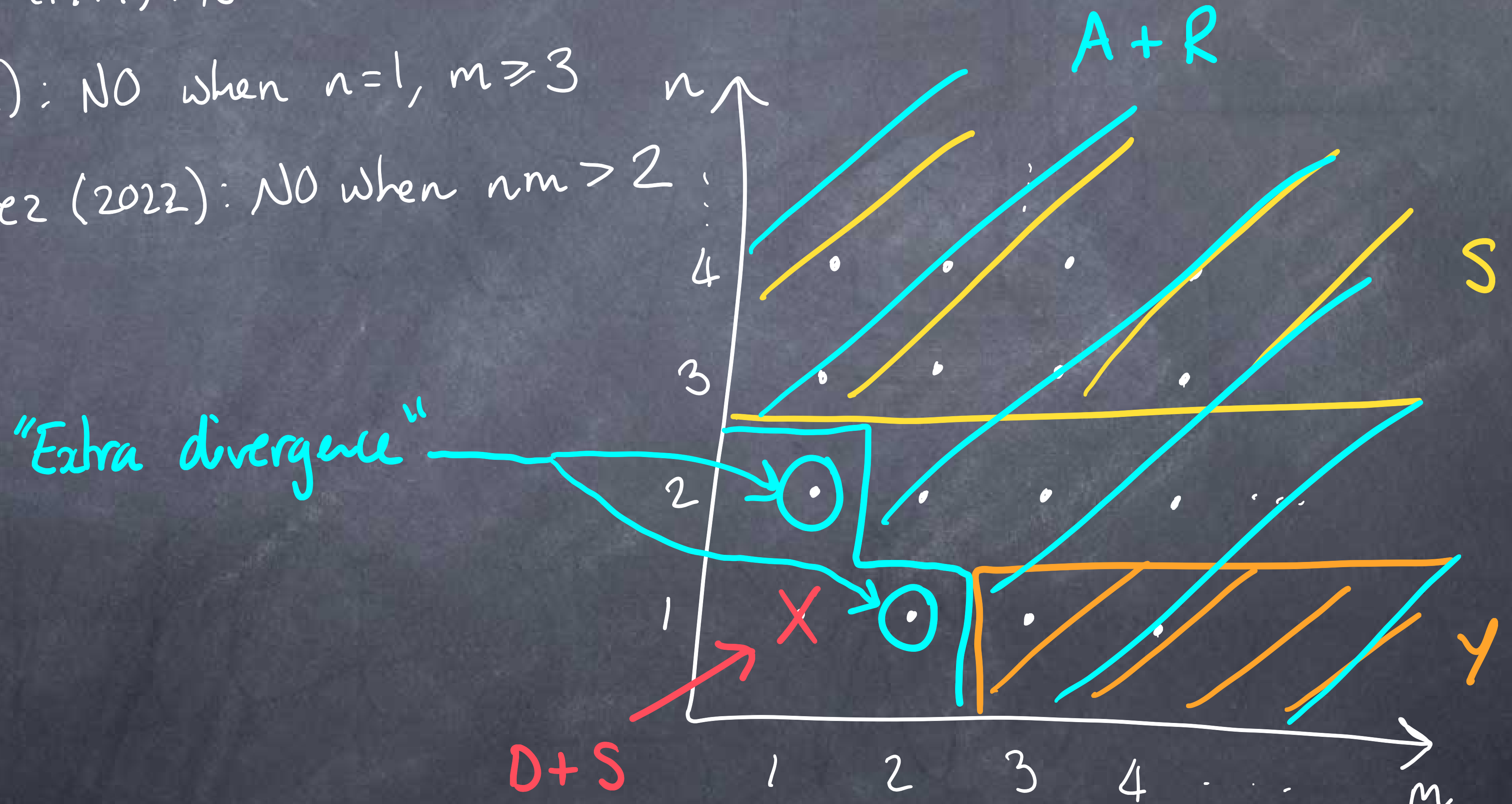
Inhomogeneous KGT (Szüzs (1958), Schmidt (1964), Sprindžuk (1979))

For any $\psi : \mathbb{N} \to [0,\infty)$ and any $y \in \mathbb{R}^m$,

$$\mathcal{L}(A^{y}_{n,m}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} \text{ " } = \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$

Question: Do we really need monotonicity of Ψ in the inhomogeneous Khintchine–Groshev Theorem?

- ☆ Duffin + Schaeffer (1941): YES when $n=m=1$
  (Ramirez (2017))
- ☆ Sprindžuk (1979): NO when $n \geq 3$
- ☆ Yu (2021): NO when $n=1$, $m \geq 3$
- ☆ A.–Ramirez (2022): NO when $nm > 2$



Comments on our proof (divergence part)

- For $\underline{q} \in \mathbb{Z}^n$, let $A^{\underline{y}}_{n,m}(\underline{q},\psi) = \{ \underline{x} \in [0,1]^{nm} : \|\underline{q}\underline{x} - \underline{p} - \underline{y}\| < \psi(\|\underline{q}\|)$ for some $\underline{p} \in \mathbb{Z}^m \}$.
- Then $\mathcal{A}^{\underline{y}}_{n,m}(\psi) = \limsup_{\|\underline{q}\| \to \infty} A^{\underline{y}}_{n,m}(\underline{q},\psi)$.

Proof strategy

- Aim 1 – QIA If we could show that the sets $A^{\underline{y}}_{n,m}(\underline{q},\psi)$ were quasi-independent on average (QIA), it would follow from the Second Borel-Cantelli Lemma that $\mathcal{L}(\mathcal{A}^{\underline{y}}_{n,m}(\psi)) > 0$.

- Aim 2 – Full measure Lebesgue density type argument.

- Key new idea: Independence Inheritance.

Define a notion of ω-weak QIA (which quantifies how much independence sets have):

$$0\text{-}\mathrm{QIA} = \mathrm{QIA} \Rightarrow 1\text{-}\mathrm{QIA} \Rightarrow 2\text{-}\mathrm{QIA} \Rightarrow \dots$$

Informal Theorem (A.-Ramírez) If the sets underlying the (n,m)-dimensional inhomogeneous Khintchine-Groshev Theorem are ω-QIA, then for $k \geq 0$, the sets underlying the $(n+k,m)$-dimensional inhomogeneous Khintchine-Groshev Theorem are $\max(0,\omega-k)$-QIA.

- $(1,m)$, $m \geq 3$, QIA $\Rightarrow$ QIA for (n,m), $n \geq 1$, $m \geq 3$
- $(1,2)$, 1-QIA $\Rightarrow$ QIA for $(n,2)$, $n \geq 2$
- $(1,1)$, 2-QIA $\Rightarrow$ QIA for $(n,1)$, $n \geq 3$

Thank

You