A Rauzy fractal unbounded in all directions of the plane

Mélodie Andrieu

Advisors : Pierre Arnoux & Julien Cassaigne
Institut de Mathématiques de Marseille

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I - MOTIVATIONS : understand the dynamics of multiD continued fraction algorithms.

II - RESULTS : construction of an Arnoux Rauzy word whose Rauzy fractal is unbounded in all directions

III - MAIN IDEAS : study the set of differences of abelianized factors of all Arnoux-Rauzy words.

I - Motivations

Regular continued fraction algorithm & Sturmian words

Substractive continued fraction algorithm = iteration of the Farey map :

$$\begin{array}{rcll} (\mathbb{R}^+)^2 & \to & (\mathbb{R}^+)^2 & \\ (x,y) & \mapsto & (x-y,y) & \text{if } x \geq y, \\ & & (x,y-x) & \text{otherwise.} \end{array}$$

The symbolic trajectories under this dynamical systems give rise to the class of Sturmian words.

Sturmian words enjoy multiple [combinatorial, geometrical, dynamical] characterizations.

Balance characterization :
Sturmian words are exactly the aperiodic binary words for which any two factors of same length contain, with +/- 1, the same number of 0s.

Ex
A word starting with $w = 0010001001000100001001\ldots$ is possibly Sturmian.
A word starting with $w = 0\underline{11}011\underline{100}...$ is not.

Regular continued fraction algorithm & Sturmian words

Consequences :
1.The letters 0 and 1 are uniformly distributed with respect to a probability measure ν on $\{0, 1\}$.
2. Stronger : the difference between the observed frequency of 0s among the N first letters of w and its expected value $\nu(0)$ is bounded above by $1/N$.

Geometrically, the "broken line" made of the points $P_N := \sum_{n=0}^{N} e_{w[n]}$, where (e_0, e_1) is the usual basis of $\mathbb{R}^2$, remains at bounded distance from its average direction.

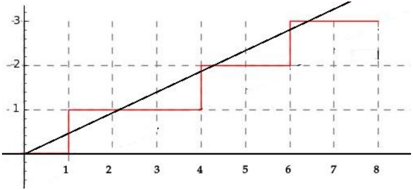

Figure – The broken line of 01000100100...

⟶ Sturmian words are used to approximate lines with irrational slopes.

MultiD continued fraction algorithms

Since Jacobi, several algorithms have been proposed to generalize continued fractions to triplets of nonnegative real numbers.
Such algorithms should make it possible to simultaneously and efficiently approach two real numbers with a sequence of pairs of rational numbers.

The Arnoux-Rauzy algorithm

$$F_{AR}: \begin{array}{ccll} (\mathbb{R}^+)^3 & \to & (\mathbb{R}^+)^3 & \\ (x,y,z) & \mapsto & (x-y-z,y,z) & \text{if } x \geq y+z, \\ & & (x,y-x-z,z) & \text{if } y \geq x+z, \\ & & (x,y,z-x-y) & \text{if } z \geq x+y. \end{array}$$

This algorithm gives rise to the class of Arnoux-Rauzy words, which are, from the combinatorial view point, the generalization of Sturmian word.
In particular, all Arnoux-Rauzy words admit a letters frequencies vector $(\nu(1), \nu(2), \nu(3))$.

$\longrightarrow$ What can we say of the 3D broken line ?

Properties of the Arnoux-Rauzy broken line ?

Old belief : "the broken line of any Arnoux-Rauzy word remains at bounded distance from its average direction ; or equivalently, all Arnoux-Rauzy *Rauzy fractals* are bounded."
$\longrightarrow$ disproved in 2000 by Cassaigne, Ferenczi and Zamboni.

Today, we barely know nothing about the geometry or the topology of these unbounded Rauzy fractals.

Modern belief : "The broken line of any Arnoux-Rauzy word remains at bounded distance from an hyperplane containing its average direction ; or equivalently, all Arnoux-Rauzy *Rauzy fractals* are trapped between two parallel lines of the plane."

$\longrightarrow$ This is suggested by the Oseledets theorem. Indeed, if the Lyapunov exponents of the product of matrices associated with w exist, one of these exponents at least is nonpositive since their sum is equal to zero.

This belief is wrong.

II - Main results

definitions & notations

Preliminaries : finite and infinite words

An alphabet $\mathfrak{A}$ is a finite set.
A finite word of length n is an element of $\mathfrak{A}^n$.
An infinite word is an element of $\mathfrak{A}^{\mathbb{N}}$.

Following Python, $u[k]$ denotes the $(k+1)$-th letter of u.

A finite word u of length n is a factor of a word w if there exists an index i such that :

$$\text{for all } k \in \{0, ..., n-1\},\ w[i+k] = u[k].$$

$\longrightarrow$If $i = 0$, u is a prefix of w.

Notations :
- $\mathfrak{A}^*$ = the set of all finite words over $\mathfrak{A}$
- $\mathcal{F}_n(w)$ = the set of factors of w of length n of w
- $\mathcal{F}(w)$ the set of factors of all lengths.

The set $\mathfrak{A}^{\mathbb{N}}$ is endowed with the distance δ that makes it compact.

$$\delta(w, w') = \begin{cases} 2^{-n_0}, \text{ where } n_0 = \min\{n \in \mathbb{N} | w[n] \neq w'[n]\} \text{ if } w \neq w', \\ 0, \text{ otherwise.} \end{cases}$$

We are going to work with two distinct alphabets : $A = \{1, 2, 3\}$ and $AR = \{\sigma_1, \sigma_2, \sigma_3\}$.

definitions & notations

Preliminaries : Arnoux-Rauzy substitutions

A substitution is an application mapping letters to finite words : $\mathfrak{A} \mapsto \mathfrak{A}^*$, that we extend into a morphism on $\mathfrak{A}^*$ and on $\mathfrak{A}^{\mathbb{N}}$.

Three substitutions will be of high interest : σ_1, σ_2 and σ_3 defined over $A = \{1, 2, 3\}$ by :

$$\sigma_i : \quad A \to A^* \\ i \mapsto i \\ j \mapsto ij \text{ for } j \in A \backslash \{i\}.$$

They are called *Arnoux-Rauzy substitutions*; we denote $AR = \{\sigma_1, \sigma_2, \sigma_3\}$.

Ex :

$$\sigma_3(1) = 31 \qquad \sigma_1(1332) = 1131312.$$

definitions & notations

Arnoux-Rauzy words ["S-adic definition"]

Fact 1 : if $(s_n)_{n\in\mathbb{N}} \in AR^{\mathbb{N}}$ is a sequence containing infinitely many occurrences of σ_1, σ_2 and σ_3, then the sequence of finite words $(s_0 \circ \ldots \circ s_{n-1}(\alpha))$, with $\alpha \in A$, converges to an infinite word w_0 which does not depend on α.

These infinite words w_0 are called standard Arnoux-Rauzy words.

Ex : $w_{Trib} = (\sigma_1 \circ \sigma_2 \circ \sigma_3)^{\omega}(1) = 1213121121312121312112131213121312\ldots$

An infinite word w is an Arnoux-Rauzy word if it has the same set of factors than a standard Arnoux-Rauzy word w_0.

Fact 2 : the standard Arnoux-Rauzy word w_0 and the directive sequence $(s_n)_{n\in\mathbb{N}}$ associated with w are unique.

definitions & notations

Abelianization

Let $u \in A^*$, $i \in A$. Denote by $|u|_i$ the number of occurrences of i in u.

The abelianized word of u is the (column) vector $\mathrm{ab}(u) = (|u|_i)_{i \in A}$.
Observation : The sum of the entries of $\mathrm{ab}(u)$ is equal to $|u|$ the *length* of the word u.

Ex $\quad ab(1332) = (1, 1, 2)^t$.

The incidence matrix of a substitution s over A is :

$$\mathrm{ab}(s) = (|s(j)|_i)_{i,j \in A}.$$

Ex

$$\mathrm{ab}(\sigma_1) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathrm{ab}(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathrm{ab}(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \in GL_3(\mathbb{Z}).$$

Abelianized words and incidence matrices satisfy : $\mathrm{ab}(s(u)) = \mathrm{ab}(s) \cdot \mathrm{ab}(u)$.

Letters frequencies

Let $w \in A^{\mathbb{N}}$ and $i \in A$.
The frequency of i in w is the limit, if it exists, of the proportion of i in the sequence of growing prefixes of w : $f_w(\alpha) = \lim_{n\to\infty} \frac{|p_n(w)|_\alpha}{n}$.

We denote by $f_w = (f_w(\alpha))_{\alpha\in\mathfrak{A}}$ the vector of letters frequencies of w, if it exists.

Fact : All Arnoux-Rauzy words admit a vector of letters frequencies.

Theorem (A. 20 ; Dynnikov, Hubert & Skripchenko 20)

The vector of letters frequencies of an Arnoux-Rauzy word has rationally independent entries.

$\longrightarrow$ This result was conjectured by Arnoux and Starosta in 2013.

definitions & notations

Discrepancy & Rauzy fractal

A natural question is to study the difference between the predicted frequencies of letters and their observed occurrences, that is called discrepancy :

$$\begin{array}{rl} \mathrm{discr}: & \mathbb{N} \to \mathbb{R} \\ & n \mapsto \max_{i\in A} ||p_n(w)|_i - nf_w(i)|. \end{array}$$

⟶ Geometrically, the discrepancy is linked to the diameter of the Rauzy fractal.

Let w an Arnoux-Rauzy word and f_w its letters frequencies vector.

Definition - The broken line of w is $\mathcal{B}_w := \{\mathrm{ab}(p_k(w)) | k \in \mathbb{N}\} \subset \mathbb{N}^3$.

Denote by π_w the (oblique) projection parallel to $\mathbb{R}f_w$, onto Δ_0 (the plane of $\mathbb{R}^3$ with equation $x + y + z = 0$).

Definition - The Rauzy fractal of w is $\mathcal{R}_w := \overline{\pi_w(\mathcal{B}_w)} \subset \Delta_0$.

Figure – Rauzy fractal of $w_{trib} = (\sigma_\mathbf{1} \circ \sigma_\mathbf{2} \circ \sigma_\mathbf{3})^\omega(1)$.

Results

Theorem (A. 20)

There exists an Arnoux-Rauzy word whose Rauzy fractal is unbounded in all directions of the plane.

A similar result holds for Cassaigne-Selmer words and for strict episturmian words over a d-letter alphabet, for $d \geq 3$.

III - Main ideas for the construction

1. Reduce the problem to a combinatorial question

Let w a finite or infinite word over A.

imbalance of w : $\operatorname{imb}(w) := \sup_{n \in \mathbb{N}} \sup_{u,v \in F_n(w)} ||\operatorname{ab}(u) - ab(v)||_\infty \quad \in \mathbb{N}$ or ∞

ex : $\operatorname{imb}(1221) = 1$

ex :
- Thue-Morse : $w_{TM} = 1221211221121221...$ $\quad \operatorname{imb}(w_{TM}) = 2$
- Fibonacci : $w_{Fib} = 1211212112112121...$ $\quad \operatorname{imb}(w_{Fib}) = 1$
- Tribonacci : $w_{Trib} = 1213121121312...$ $\quad \operatorname{imb}(w_{Trib}) = 2$

Fact : $\operatorname{discr}(w) \leq \operatorname{imb}(w) \leq 4.\operatorname{discr}(w)$

The Rauzy fractal of an infinite word is unbounded if and only if its imbalance is infinite.

$\longrightarrow$ A sufficient [combinatorial] condition that guarantees that the Rauzy fractal is unbounded in **all directions** of the plane ?

1. Reduce the problem to a combinatorial question

Theorem

Let $w \in \{1,2,3\}^{\mathbb{N}}$. If for all $\vec{d} \in \mathbb{Z}^3 \cap \Delta_0$, where Δ_0 denotes the plane of $\mathbb{R}^3$ with equation $x+y+z=0$, there exist u and $v \in \mathcal{F}(w)$ such that $\mathrm{ab}(u) - \mathrm{ab}(v) = \vec{d}$, then, for any plane Π and for any $D \in \mathbb{R}^+$, there exists $k \in \mathbb{N}$ such that the euclidean distance between the point P_k, whose coordinates are $\mathrm{ab}(p_k(w))$, and the plane Π satisfies $\mathrm{dist}_{\mathbb{R}^3}(P_k, \Pi) \geq D$.

We would like to construct an Arnoux-Rauzy word w_∞ such that :

$$\Delta_0 \cap \mathbb{Z}^3 \subset \{ab(u) - ab(v) | u, v \in \mathcal{F}(w_\infty)\}...$$

1. Reduce the problem to a combinatorial question

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We would like to construct an Arnoux-Rauzy word w_∞ such that :

$$\Delta_0 \cap \mathbb{Z}^3 \subset \{ab(u)-ab(v) | u,v \in \mathcal{F}(w_\infty)\}\ldots$$

$\longrightarrow$ We are going to construct an Arnoux-Rauzy word w_∞ such that :

$$\{ab(u)-ab(v) | u,v \in \mathcal{F}(w_\infty)\} = \mathbb{Z}^3.$$

The Rauzy fractal of w_∞ will be unbounded in all directions of the plane !

Construction of w_∞ : outline

Technical part

Lemma (1)

For any $(a,b,c) \in \mathbb{Z}^3$*, there exists* $s \in AR^*$ *and there exist* $u, v \in \mathcal{F}(s(1))$ *that satisfy* $\mathrm{ab}(u) - \mathrm{ab}(v) = (a,b,c)$*.*

$\longrightarrow$ In particular, all Arnoux-Rauzy words whose directive sequence starts with p contain these two factors u and v.

Straightforward part :

Lemma (2)

For any $p \in AR^*$ *and any* $(a,b,c) \in \mathbb{Z}^3$*, there exists* $s \in AR^*$ *and there exist* $u, v \in \mathcal{F}(p \cdot s(1))$ *that satisfy* $\mathrm{ab}(u) - \mathrm{ab}(v) = (a,b,c)$*.*

Theorem (3)

There exists an Arnoux-Rauzy word w_∞ *such that for all* $(a,b,c) \in \mathbb{Z}^3$*, there exist* u *and* $v \in \mathcal{F}(w_\infty)$ *satisfying* $\mathrm{ab}(u) - \mathrm{ab}(v) = (a,b,c)$*.*

Construction of w_∞ : straightforward part

Lemma (1)

For any $(a, b, c) \in \mathbb{Z}^3$*, there exists* $s \in AR^*$ *and there exist* $u, v \in \mathcal{F}(s(1))$ *that satisfy* $\mathrm{ab}(u) - \mathrm{ab}(v) = (a, b, c)$*.*

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Proof.

- Observe that p is a finite composition of Arnoux-Rauzy substitutions, so $\mathrm{ab}(p) \in GL_3(\mathbb{Z})$.
- Take the s, u and v given by Lemma (1) for the vector $\mathrm{ab}(p)^{-1} \cdot d$.
- We have $u, v \in \mathcal{F}(p.s(1))$ and $\mathrm{ab}(u) - \mathrm{ab}(v) = \mathrm{ab}(p) \cdot \mathrm{ab}(p)^{-1} \cdot d = d$. QED

Remark. Here its crucial to have all $\mathbb{Z}^3$, and not just $\Delta_0 \cap \mathbb{Z}^3$, as set of possible differences of abelianized factors.

Construction of w_∞ : straightforward part

Lemma (2)

For any $p \in AR^*$ and any $(a,b,c) \in \mathbb{Z}^3$, there exists $s \in AR^*$ and there exist $u, v \in \mathcal{F}(p \cdot s(1))$ that satisfy $\mathrm{ab}(u) - \mathrm{ab}(v) = (a,b,c)$.

Theorem (3)

There exists an Arnoux-Rauzy word w_∞ such that for all $(a,b,c) \in \mathbb{Z}^3$, there exist u and $v \in \mathcal{F}(w_\infty)$ satisfying $\mathrm{ab}(u) - \mathrm{ab}(v) = (a,b,c)$.

Proof.
Let $\varphi : \mathbb{N} \to \mathbb{Z}^3$ a bijection.
We construct by recurrence the directive sequence of w_∞ :
1. Set $p_0 = \epsilon$ (empty word)
2. For $k \in \mathbb{N}$, we set $p_{k+1} = p_k.\sigma_1.\sigma_2.\sigma_3.s$, where $s \in AR^*$ is given by Lemma (2) to the word $p_k.\sigma_1.\sigma_2.\sigma_3 \in AR^*$ and the vector $\varphi(k+1) \in \mathbb{Z}^3$.

Thus, the sequence of finite words $(p_k)_k$ converges to an infinite sequence $d \in AR^{\mathbb{N}}$, which defines (as directive sequence) a unique Arnoux-Rauzy word : w_∞.
By construction, $\{ab(u) - ab(v) | u, v \in \mathcal{F}(w_\infty)\} = \mathbb{Z}^3$. QED

Construction of w_∞ : technical part (glimpse)

Lemma (1)

For any $(a, b, c) \in \mathbb{Z}^3$*, there exists* $s \in AR^*$ *and there exist* $u, v \in \mathcal{F}(s(1))$ *that satisfy* $\mathrm{ab}(u) - \mathrm{ab}(v) = (a, b, c)$.

Let $\mathcal{G}$ the infinite oriented graph $\mathcal{G}$ whose set of vertices is $\mathbb{Z}^3$ and whose edges maps triplets to their images by one the 15 following applications.
For $\delta \in \{-2, -1, 0, 1, 2\}$ and $i \in \{1, 2, 3\}$:

$$\begin{array}{rccc} \tau_{i,\delta} : & \mathbb{Z}^3 & \to & \mathbb{Z}^3 \\ & (x_j)_{j\in\{1,2,3\}} & \mapsto & (y_j)_{j\in\{1,2,3\}} \end{array} \quad \text{where } \begin{cases} y_i = x_1 + x_2 + x_3 + \delta \\ y_j = x_j \text{ for } j \neq i. \end{cases}$$

Ex for $i = 1$ and $\delta = -2$:

$$\begin{array}{rccc} \tau_{1,-2} : & \mathbb{Z}^3 & \to & \mathbb{Z}^3 \\ & (a, b, c) & \mapsto & (a + b + c - 2, b, c) \end{array}$$

Construction of w_∞ : technical part (glimpse)

Why this graph ?

Fact : If $\tau_{i,\delta}(a,b,c) = (d,e,f)$ and if (a,b,c) is the difference of two abelianized factors of an Arnoux-Rauzy word w,
then (d,e,f) is the difference of two abelianized factors of [the Arnoux-Rauzy word] $\sigma_i(w)$.

A generic example.

For $i = 1$ and $\delta = -2$.
If $u, v \in F(w)$ are nonempty and satisfy $\mathrm{ab}(u) - \mathrm{ab}(v) = (a,b,c)$, then the word $\sigma_1(u)$ starts with the letter 1, and each occurrence of $\sigma_1(v)$ in $\sigma_1(w)$ is immediately followed by the letter 1.
Therefore, the words $\tilde{u} = 1^{-}\sigma_1(u)$ (the word u without its initial 1) and $\tilde{v} = \sigma(v)1$, are factors of $\sigma_1(w)$ and satisfy
$\mathrm{ab}(\tilde{u}) - \mathrm{ab}(\tilde{v}) = (a+b+c-2, b, c) = \tau_{1,-2}(\mathrm{ab}(u) - \mathrm{ab}(v))$.

$\longrightarrow$ We are going to study the paths in this graph...

Construction of w_∞ : technical part (glimpse)

A careful study of $\mathcal{G}$ then shows :

Lemma

All triplets in $\mathbb{Z}^3$ can be reached from the vertex $(0,0,0) \in \mathbb{Z}^3$, moving through a finite number of edges.

We conclude the proof by observing that $(0,0,0)$ is the difference between $\mathrm{ab}(u)$ and itself - where u is an Arnoux-Rauzy factor that can be chosen as long as we need.

Remark. $\mathcal{G}$ is not strongly connected.

Thank you !