Spectral aspects of aperiodic dynamical systems

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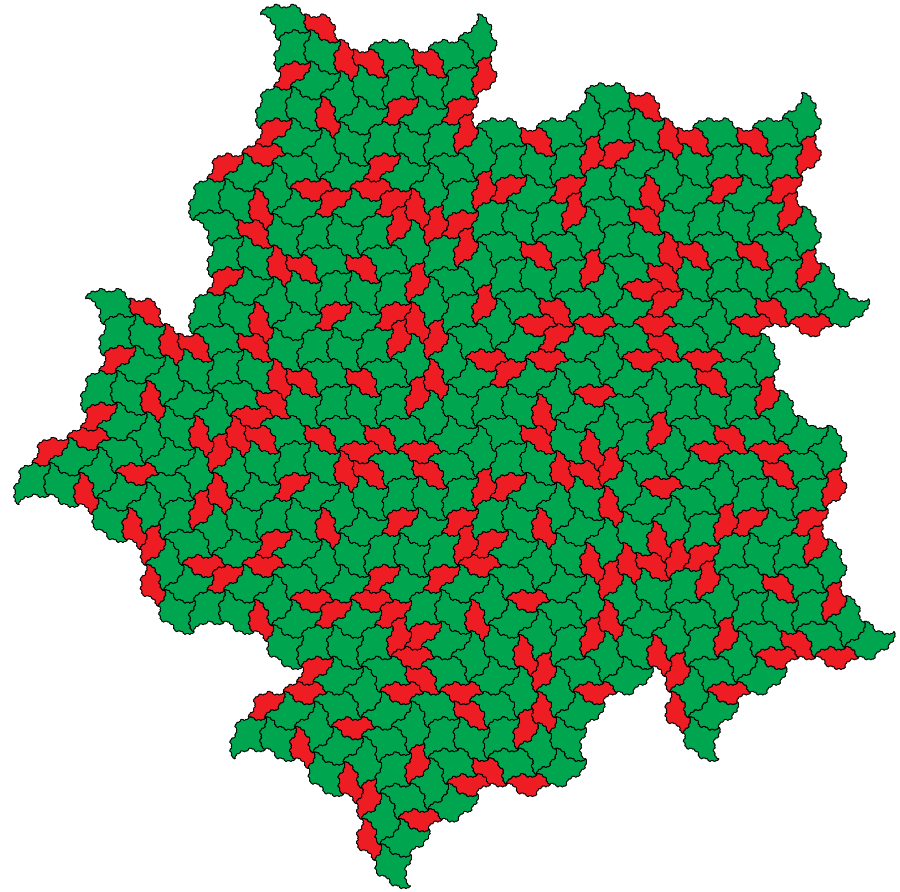
(joint work with A. Bustos, F. Gähler, U. Grimm, N. Mañibo)

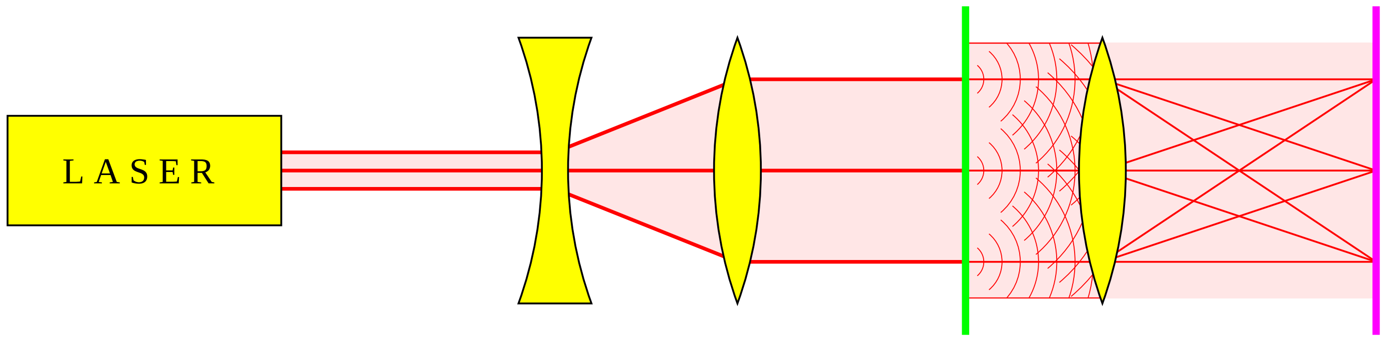
(and many others)

Menu

- Diffraction
- Pure point spectra
- Model sets, CPS
- Inflation tilings
- Renormalisation
- Visible lattice points
- Weak model sets
- Outlook



Diffraction theory



Wiener's diagram obstacle $f(x)$, with $\widetilde{f}(x) := \overline{f(-x)}$

$$\begin{array}{ccc} f & \xrightarrow{\;*\;} & f * \widetilde{f} \\ \mathcal{F}\big\downarrow & & \big\downarrow\mathcal{F} \\ \widehat{f} & \xrightarrow{\;|.|^2\;} & |\widehat{f}\,|^2 \end{array}$$

Diffraction theory

Structure translation bounded measure ω

assumed 'self-amenable' (Hof 1995)

Autocorrelation $\gamma = \gamma_\omega = \omega \circledast \widetilde{\omega} := \lim_{R\to\infty} \frac{\omega|_R * \widetilde{\omega|_R}}{\mathrm{vol}(B_R)}$

Diffraction $\boxed{\widehat{\gamma} = \big(\widehat{\gamma}\big)_{\mathsf{pp}} + \big(\widehat{\gamma}\big)_{\mathsf{sc}} + \big(\widehat{\gamma}\big)_{\mathsf{ac}}}$ (relative to λ_{L})

- pp: Bragg peaks
- ac: diffuse scattering (with RN density)
- sc: whatever remains ...

Diffraction versus dynamical spectrum

Dynamical system

$$(\mathbb{X}, \mathbb{Z}, \mu) \quad \text{with} \quad \mathbb{Z} \simeq \{T^n \mid n \in \mathbb{Z}\}$$

$\curvearrowright$ Hilbert space $\mathcal{H} = L^2(\mathbb{X}, \mu)$

$\curvearrowright$ unitary operator on $\mathcal{H}$, $\big(U_T f\big)(x) := f(Tx)$

$\curvearrowright$ spectrum of U_T (Koopman, von Neumann, Halmos)

Extension analogous definition for other groups, e.g. $\mathbb{R}^d$

Spaces shifts, tilings, Delone sets, measures, ...

(Host 1986, Queffélec 1987, Pytheas Fogg 2002)

(Radin/Wolff 1992, Robinson 1996, Solomyak 1997)

Diffraction versus dynamical spectrum

Theorem Let $(\mathbb{X}, \mathbb{R}^d, \mu)$ be an (ergodic) point set dynamical system with diffraction $\widehat{\gamma}$. Then, $\widehat{\gamma}$ is pure point iff $(\mathbb{X}, \mathbb{R}^d, \mu)$ has pure point dynamical spectrum. The latter then is the group generated by the support of $\widehat{\gamma}$, the so-called Fourier–Bohr spectrum of γ.

(Dworkin 1993, Hof 1995, Schlottmann 2000, Lee/Moody/Solomyak 2002, B/Lenz 2004, Lenz/Strungaru 2009, Lenz/Moody 2012)

Connection $\varLambda \subset \mathbb{R}^d$, $\quad \mathbb{X} = \overline{\{t + \varLambda : t \in \mathbb{R}^d\}}$, $\quad (\mathbb{X}, \mathbb{R}^d, \mu)$

FB coefficients $a_{\varLambda}(k) \; := \lim\limits_{r\to\infty} \frac{1}{\mathrm{vol}(B_r)} \sum_{x\in\varLambda_r} \mathrm{e}^{-2\pi \mathrm{i} k x}$

Eigenfunctions $\boxed{a_{t+\varLambda}(k) \; = \; \mathrm{e}^{-2\pi\mathrm{i}kt} a_{\varLambda}(k)}$ ($\neq 0$ for $k \in L^{\circledast}$)

Pure point spectra

Point measures $\delta_x\,, \quad \delta_S := \sum_{x\in S} \delta_x$

Poisson's summation formula $$\boxed{\widehat{\delta_{\varGamma}} = \mathrm{dens}(\varGamma)\, \delta_{\varGamma^*}}$$

for lattice $\varGamma$, dual lattice $\varGamma^*$

Perfect crystals $\omega = \mu * \delta_{\varGamma}$ (μ finite, $\varGamma$ maximal)

⟹ $$\gamma = \mathrm{dens}(\varGamma)\, (\mu * \widetilde{\mu}) * \delta_{\varGamma}$$

⟹ $$\boxed{\widehat{\gamma} = \bigl(\mathrm{dens}(\varGamma)\bigr)^2 \, \bigl|\widehat{\mu}\bigr|^2 \, \delta_{\varGamma^*}}$$ pure point !!

⟹ dynamical spectrum $\varGamma^*$, also pure point

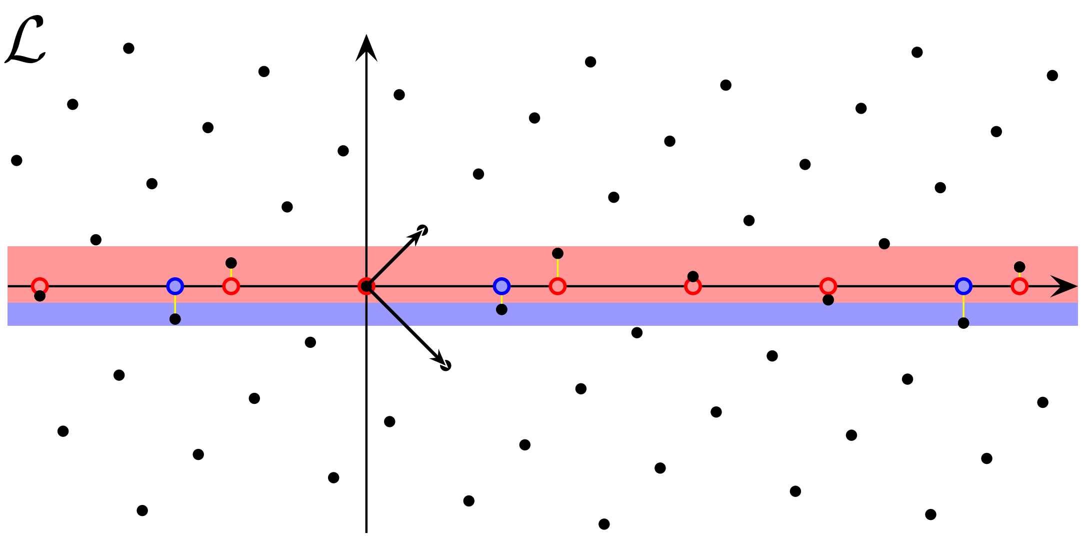
Pure point spectra

Silver mean substitution

$$a \mapsto aba \qquad b \mapsto a \qquad (\lambda_{\mathrm{PF}} = 1+\sqrt{2})$$

Inflation point set

$$\varLambda = \Big\{ x \in \mathbb{Z}[\sqrt{2}\,] : x^{\star} \in \big[-\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}\,\big] \Big\} = \varLambda_a \cup \varLambda_b$$



Window IFS

$$W_a = s\,(W_a \cup W_b) \cup s\,W_a + (1+s)$$

$$W_b = s\,W_a + s$$

$$s = \lambda_{\mathrm{PF}}^{\star} = 1 - \sqrt{2}$$

Pure point spectra

CPS

$$\begin{array}{ccccc}
\mathbb{R}^d & \xleftarrow{\;\pi\;} & \mathbb{R}^d \times \mathbb{R}^m & \xrightarrow{\;\pi_{\mathrm{int}}\;} & \mathbb{R}^m \\
\cup & & \cup & & \cup \;\text{dense} \\
\pi(\mathcal{L}) & \xleftarrow{\;1-1\;} & \mathcal{L} & \longrightarrow & \pi_{\mathrm{int}}(\mathcal{L}) \\
\| & & & & \| \\
L & & \xrightarrow{\qquad\qquad\star\qquad\qquad} & & L^\star
\end{array}$$

(Meyer 1972)

(Moody 1997)

Model set $\boxed{\varLambda = \{ x \in L : x^\star \in W \}}$ (assumed regular)

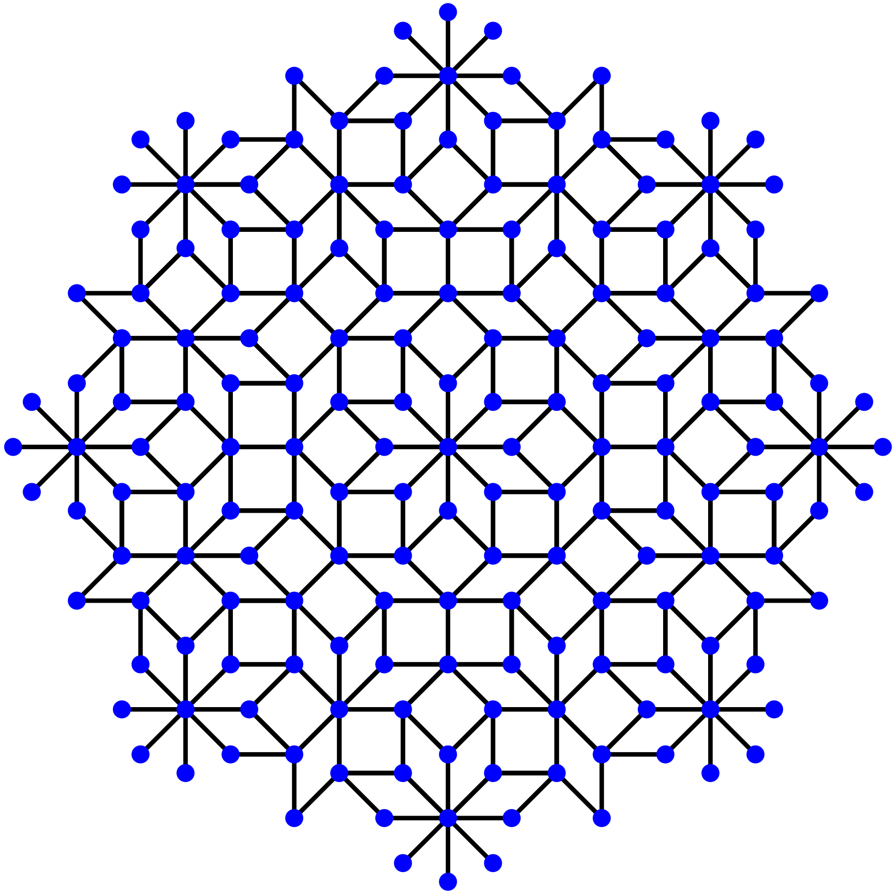
with $W \subset \mathbb{R}^m$ compact, $\lambda_{\mathrm{L}}(\partial W) = 0$

Diffraction $\boxed{\widehat{\gamma} = \sum_{k \in L^{\circledast}} \lvert A(k) \rvert^2 \, \delta_k}$ pure point !! $(\omega = \delta_\varLambda)$

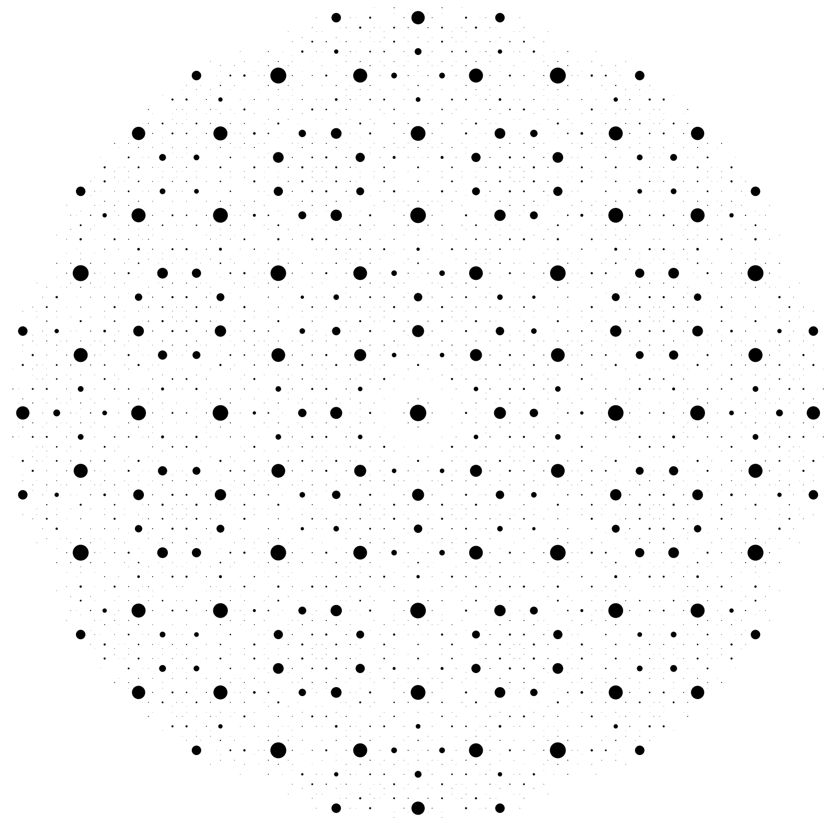
with $L^{\circledast} = \pi(\mathcal{L}^*)$ (Fourier module of $\varLambda$: spectrum)

and amplitude $A(k) = \frac{\mathrm{dens}(\varLambda)}{\mathrm{vol}(W)} \, \widehat{1_W}(-k^\star)$

Example: Ammann–Beenker tiling



point set



diffraction

Renormalisation of inflation tilings

Idea Inflation (group !) $\overset{?!?}{\Longrightarrow}$ Exact renormalisation

Theorem Let ϱ be a primitive inflation with inflation factor λ on the finite prototile set $\{\mathfrak{t}_1, \ldots, \mathfrak{t}_L\}$, with displacement matrix $T = (T_{ij})$. If $\nu_{ij}(z)$ is the pair correlation coefficient between tiles of type i and j at distance z, they satisfy (for all $z \in \mathbb{R}$) the identities

$$\boxed{\nu_{ij}(z) \; = \; \tfrac{1}{\lambda} \sum_{m,n=1}^{L} \sum_{r \in T_{im}} \sum_{s \in T_{jn}} \nu_{mn}\bigl(\tfrac{z+r-s}{\lambda}\bigr) .}$$

(B/Frank/Grimm/Robinson 2017, B/Gähler/Mañibo 2019, Bufetov/Solomyak 2020)

Structure Finite self-consistency part, rest is purely recursive

Renormalisation of inflation tilings

Fourier transform $\Upsilon_{ij} = \sum_z \nu_{ij}(z)\delta_z$, measure vector Υ

$f(z) = \lambda z$, Fourier matrix $B(k) = \widehat{\delta_T}(-k)$

$$\Longrightarrow \quad \boxed{\widehat{\Upsilon} = \frac{1}{\lambda^2}\big(B(.) \otimes \overline{B(.)}\,\big)\big(f^{-1}.\widehat{\Upsilon}\big)}$$

(separately for each spectral type)

RN density $\big(\widehat{\Upsilon}_{ij}\big)_{\text{ac}} = h_{ij}(.)\mu_{\text{Leb}}$, $h_{ij}(k) = \sum_\ell v_i^{(\ell)}(k)\overline{v_j^{(\ell)}(k)}$

$$\Longrightarrow \quad v(k) = \frac{1}{\sqrt{\lambda}} B(k) v(\lambda k) \quad \text{and} \quad v(\lambda k) = \sqrt{\lambda} B^{-1}(k) v(k)$$

Cocycle

$$\boxed{B^{(n)}(k) = B(k)B(\lambda k)\cdots B(\lambda^{n-1}k)}$$

Renormalisation of inflation tilings

Lyapunov exponent $\chi^B(k) = \limsup_{n\to\infty} \frac{1}{n} \log \|B^{(n)}(k)\|$

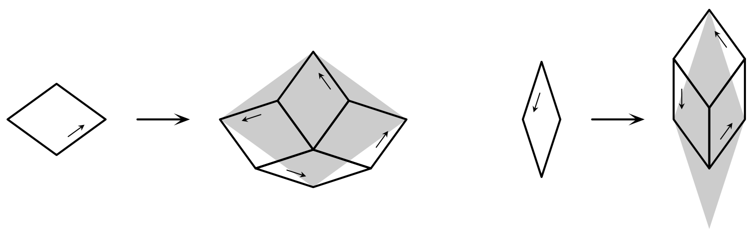
Theorem Let ϱ be a primitive inflation on a finite prototile set, with inflation multiplier λ, and let $B(k)$ be non-singular for at least one k. If, for some $\varepsilon > 0$, one has $\chi^B(k) \leqslant \frac{1}{2}\log(\lambda) - \varepsilon$ for μ_{Leb}-a.e. $k \in \mathbb{R}$, the ac part of $\widehat{\Upsilon}$ vanishes, and the diffraction is singular.

(B/Gähler/Mañibo 2019)

Extension Inflation tiling in $\mathbb{R}^d$, expansive map Q

$\curvearrowright$ criterion: $\boxed{\chi^B(k) \leqslant \frac{1}{2}\log|\det(Q)| - \varepsilon}$

Renormalisation of inflation tilings



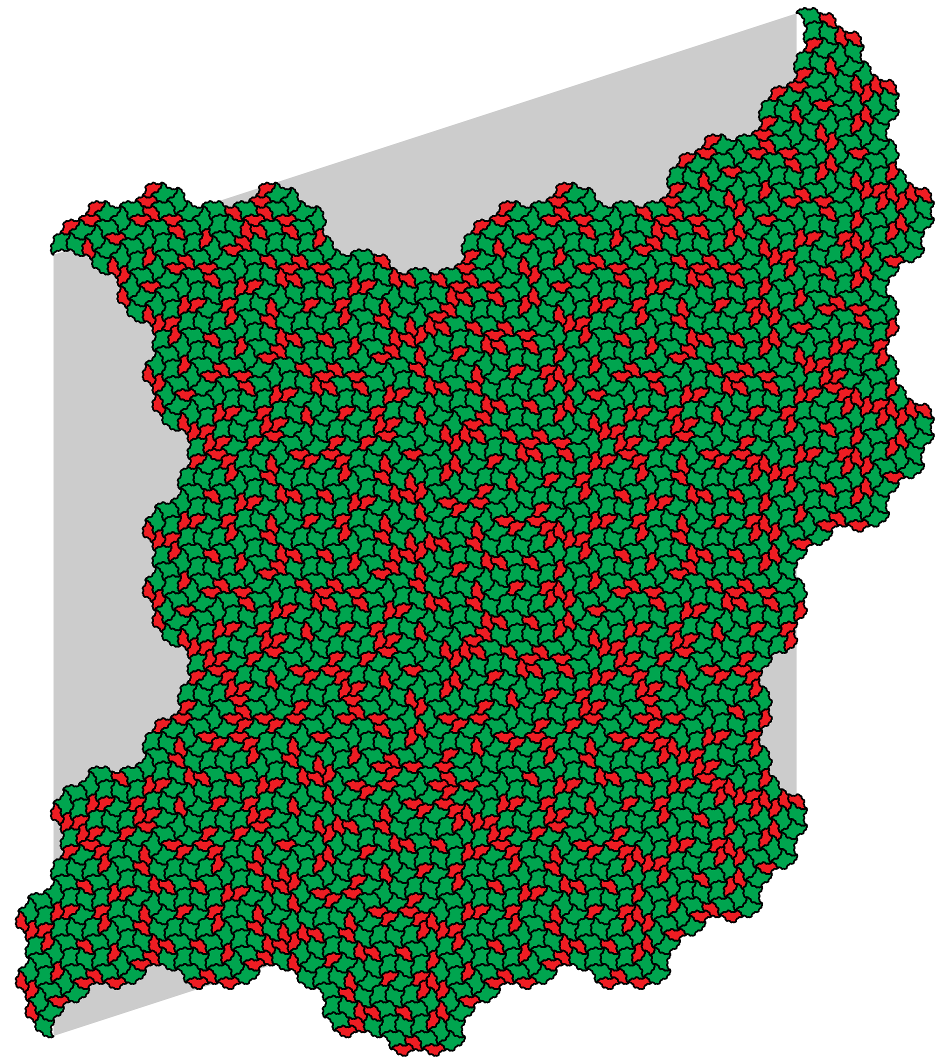
Godrèche–Lançon–Billard

(GLB) inflation rule

$\lambda = \sqrt{\frac{5+\sqrt{5}}{2}}$ is non-PV

$\Rightarrow$ trivial point spectrum

Theorem Apart from the trivial Bragg peak at 0, the spectrum of the GLB tiling is purely singular continuous.



Patch of the (fractal) GLB tiling

Plastic number inflation

Substitution $\varrho\colon\ a \mapsto b \mapsto c \mapsto ab$

Subst. matrix $M = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $\quad p(x) = x^3 - x - 1$

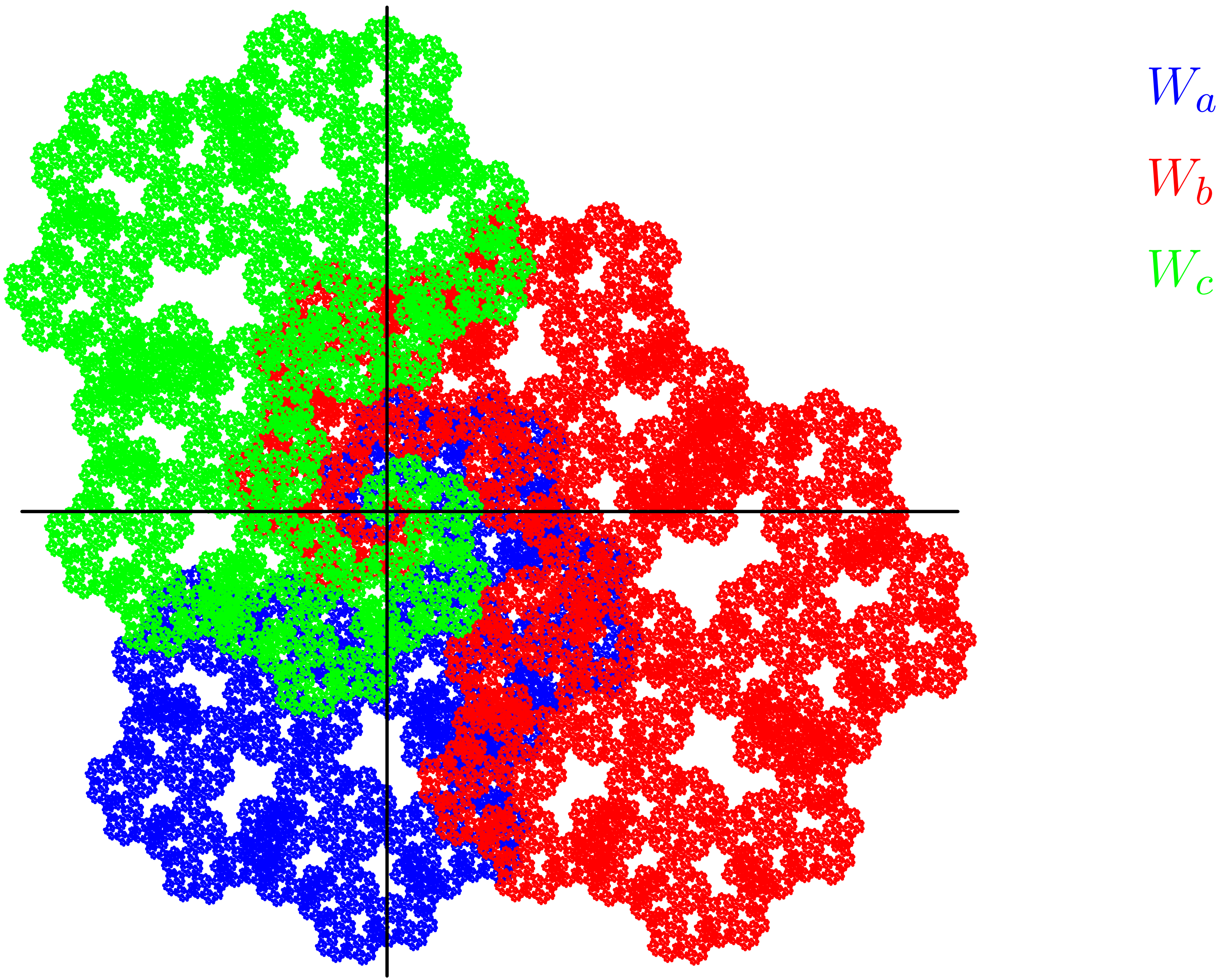
Roots $\beta = \lambda_{\mathrm{PF}} \approx 1.32472$ (min. PV number)

$\alpha, \overline{\alpha}$ (complex pair)

Inflation tiling tile lengths $1, \beta, \beta^2$ for a, b, c

- point set $\varLambda = \bigcup_i \varLambda_i \subset$ $\boxed{L = \mathbb{Z}[\beta] = \langle 1, \beta, \beta^2 \rangle_{\mathbb{Z}}}$
- model set for CPS $(\mathbb{R}, \mathbb{R}^2, \mathcal{L})$, with $\star$-map: $\beta \mapsto \alpha$
- pure point spectrum (diffraction and dynamical)

Complex windows



Spectrum and Fourier matrix

Fourier module $\boxed{L^{\circledast} = \frac{5-6\beta+4\beta^2}{23} L}$ (point spectrum)

Question How to calculate $f_i(y) := \widecheck{1_{W_i}}(y)$?

Fourier matrix $B(y) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & \mathrm{e}^{2\pi \mathrm{i} y_1} \\ 0 & 1 & 0 \end{pmatrix}$, $y = (y_1, y_2)$, $\boxed{B(0) = M}$

Internal scaling $Q = \begin{pmatrix} \mathrm{Re}(\alpha) & -\mathrm{Im}(\alpha) \\ \mathrm{Im}(\alpha) & \mathrm{Re}(\alpha) \end{pmatrix}$, $R = Q^T$, $\det(Q) = \beta^{-1}$

Lemma $\boxed{|f(y)\rangle = \beta^{-1} B(y) \, |f(Ry)\rangle}$

Fourier transform of Rauzy fractals

Cocycle $\boxed{B^{(n)}(y) = B(y)B(Ry)\cdots B(R^{n-1}y)}$ ($n \in \mathbb{N}$)

$\Longrightarrow$ $|f(y)\rangle \; = \; \beta^{-n}\, B^{(n)}(y)\, |f(R^n y)\rangle$

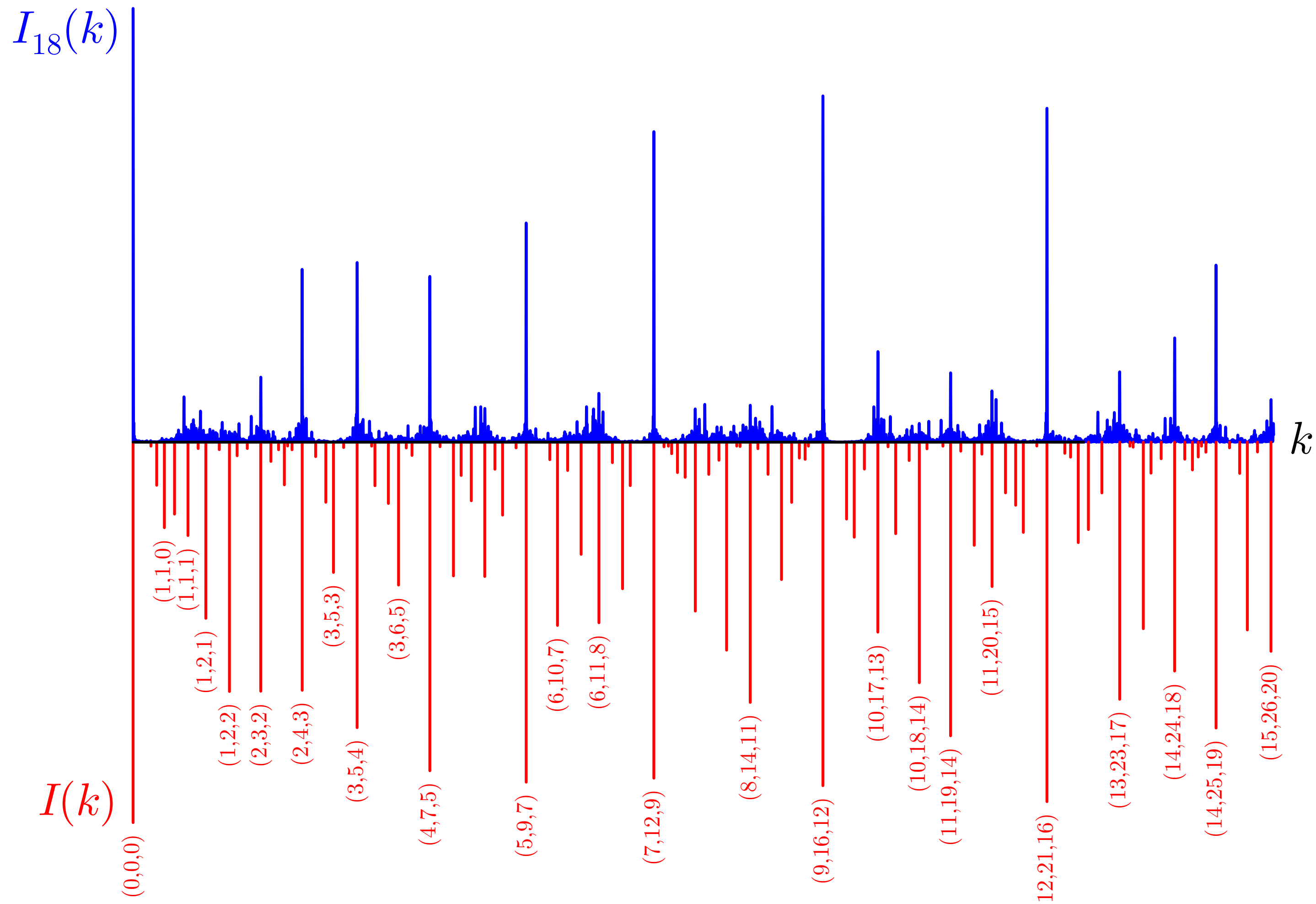
Theorem $\bigl(\beta^{-n} B^{(n)}(y)\bigr)_{n\in\mathbb{N}}$ is compactly converging on $\mathbb{R}^2$.
Thus, the matrix function $C(y) := \lim_{n\to\infty} \beta^{-n} B^{(n)}(y)$ exists and is continuous. Moreover, $C(y) = |c(y)\rangle\langle u|$ with $|c(y)\rangle = C(y)|v\rangle$, where $\langle u|v\rangle = 1$, $|v\rangle = |2-\beta^2, \beta^2-\beta, \beta-1\rangle$ and $C(0) = |v\rangle\langle u|$. This also gives $|f(y)\rangle = \overline{\ell}\, |c(y)\rangle$ with $\overline{\ell} = 4 + 2\beta - 4\beta^2$. (B/Grimm 2020)

Diffraction $\widehat{\gamma} \; = \sum_{k\in L^{\circledast}} \bigl|\sum_i h_i\, A_i(k)\bigr|^2\, \delta_k$

Amplitudes $A_i(k) = \mathrm{dens}(\varLambda) c_i(k^{\star})$ ($k \in L^{\circledast}$, $i \in \{a,b,c\}$)

Diffraction intensities

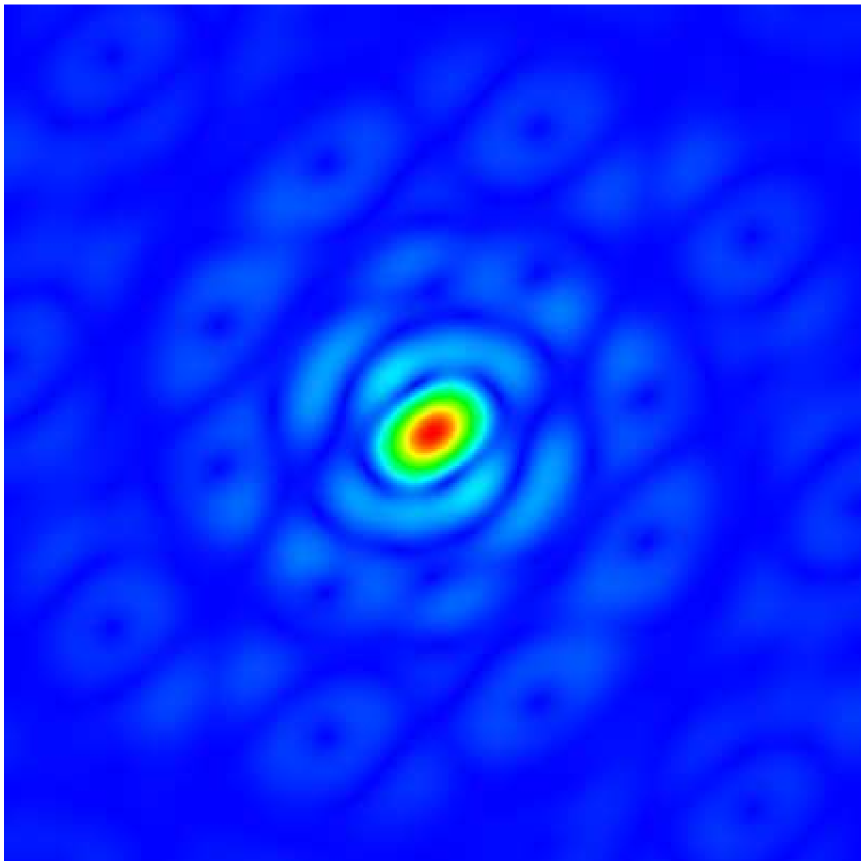
From finite system $\varrho^{18}(a)$



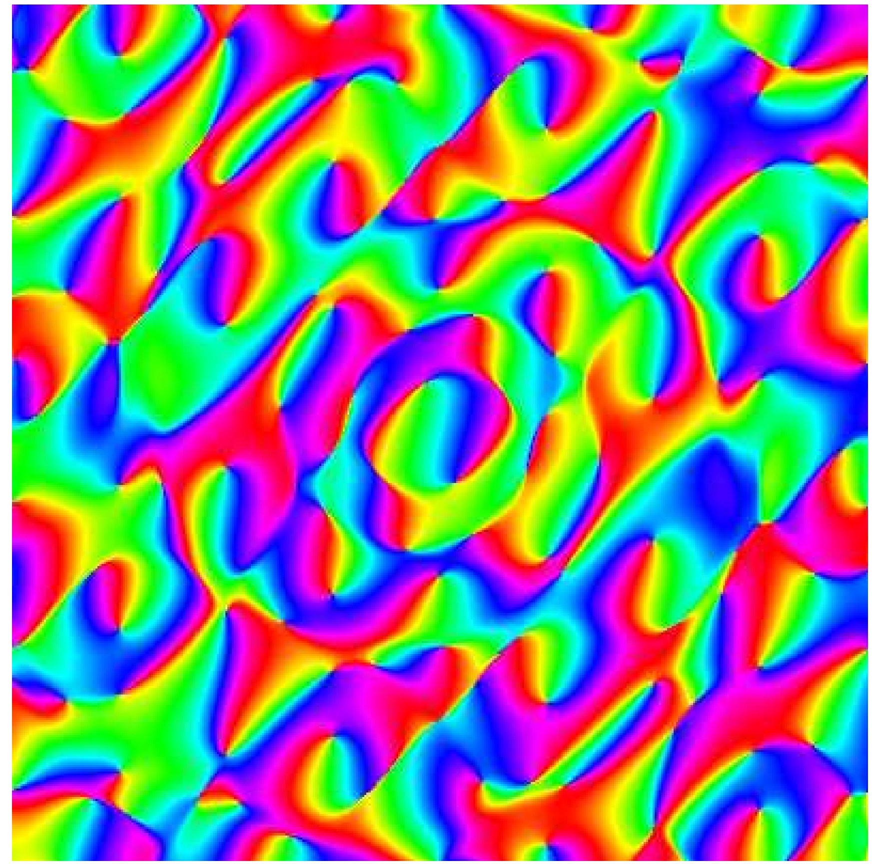
From cocycle approach

Fourier transform

Inverse Fourier transform of the window W_b

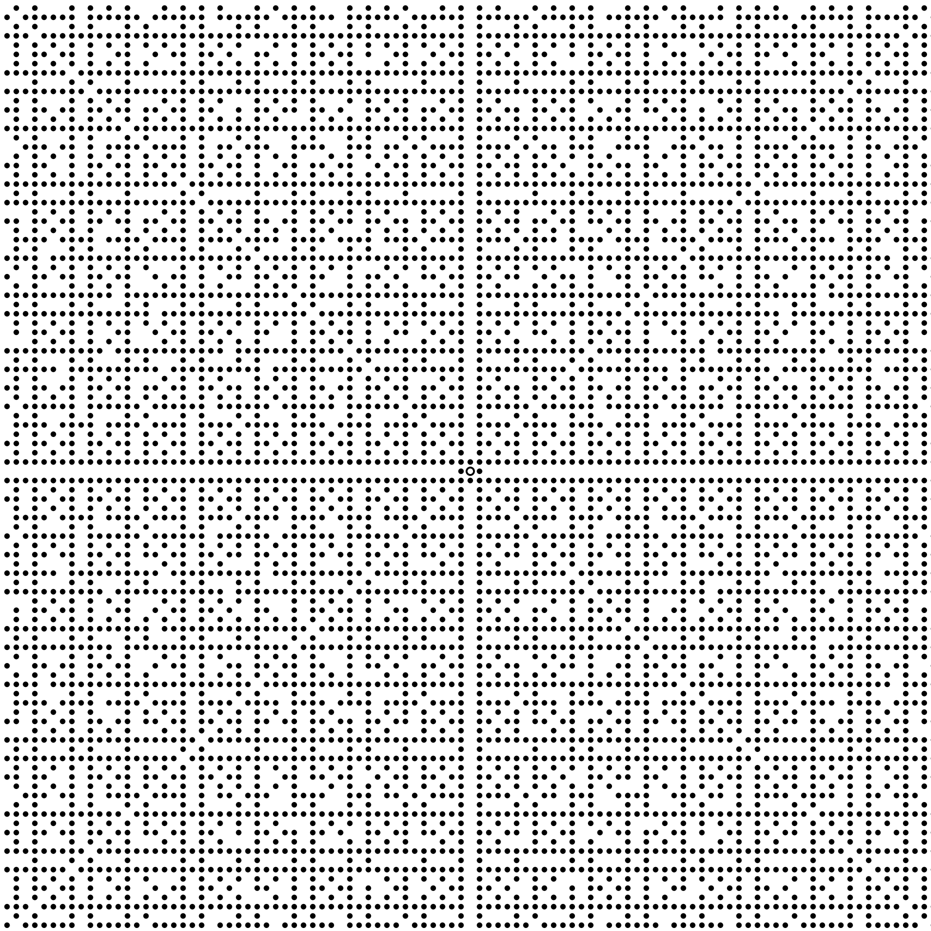


$$\left|\check{1}_{W_b}(y_1, y_2)\right|$$



$$\arg\left(\check{1}_{W_b}(y_1, y_2)\right)$$

Visible lattice points



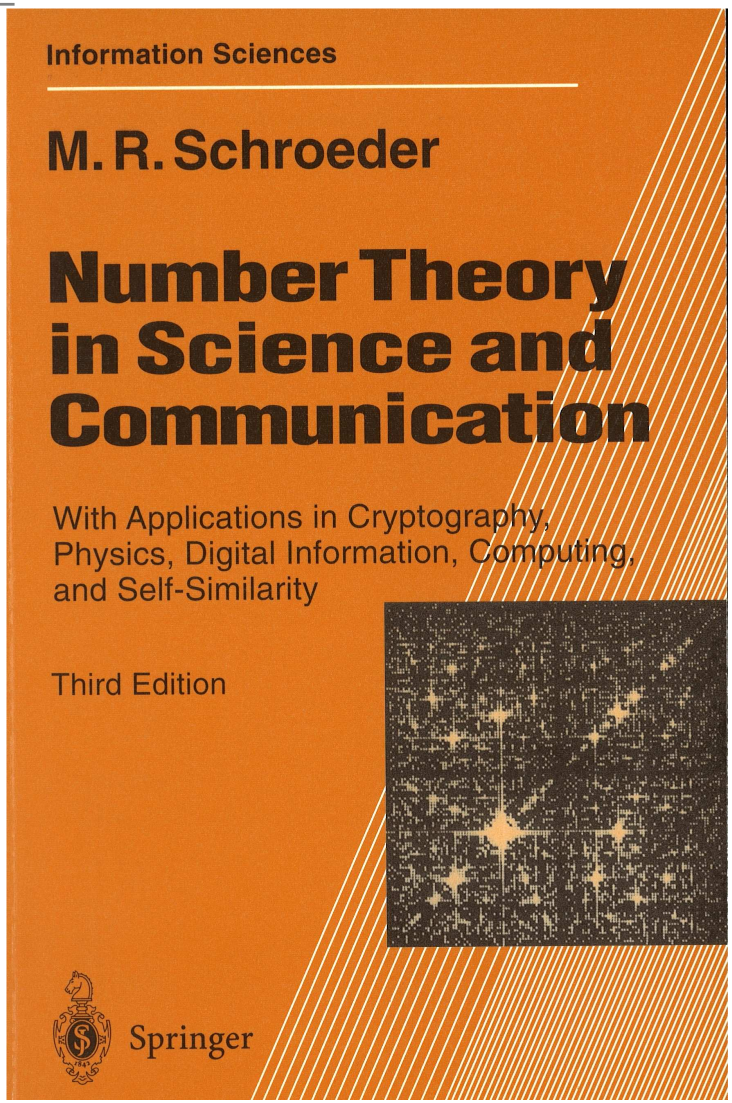
$$V = \{x \in \mathbb{Z}^2 \mid \gcd(x) = 1\}$$

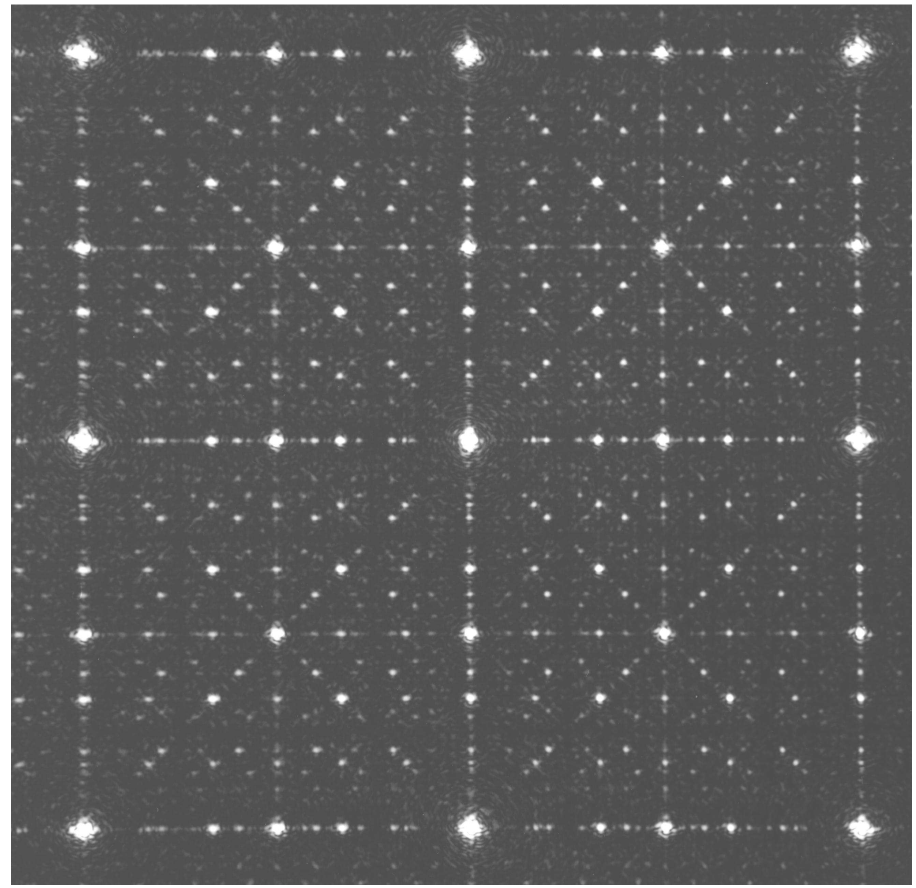
Properties

- $\mathrm{dens}(V) = 6/\pi^2$
- V not Delone
- $V - V = \mathbb{Z}^2$
- pure point diffraction
- weak model set
- $h_{\mathrm{top}}(V) > h_{\mathrm{m}}(V) = 0$

Theorem PP dynamical spectrum, trivial top. point spectrum

Visible lattice points

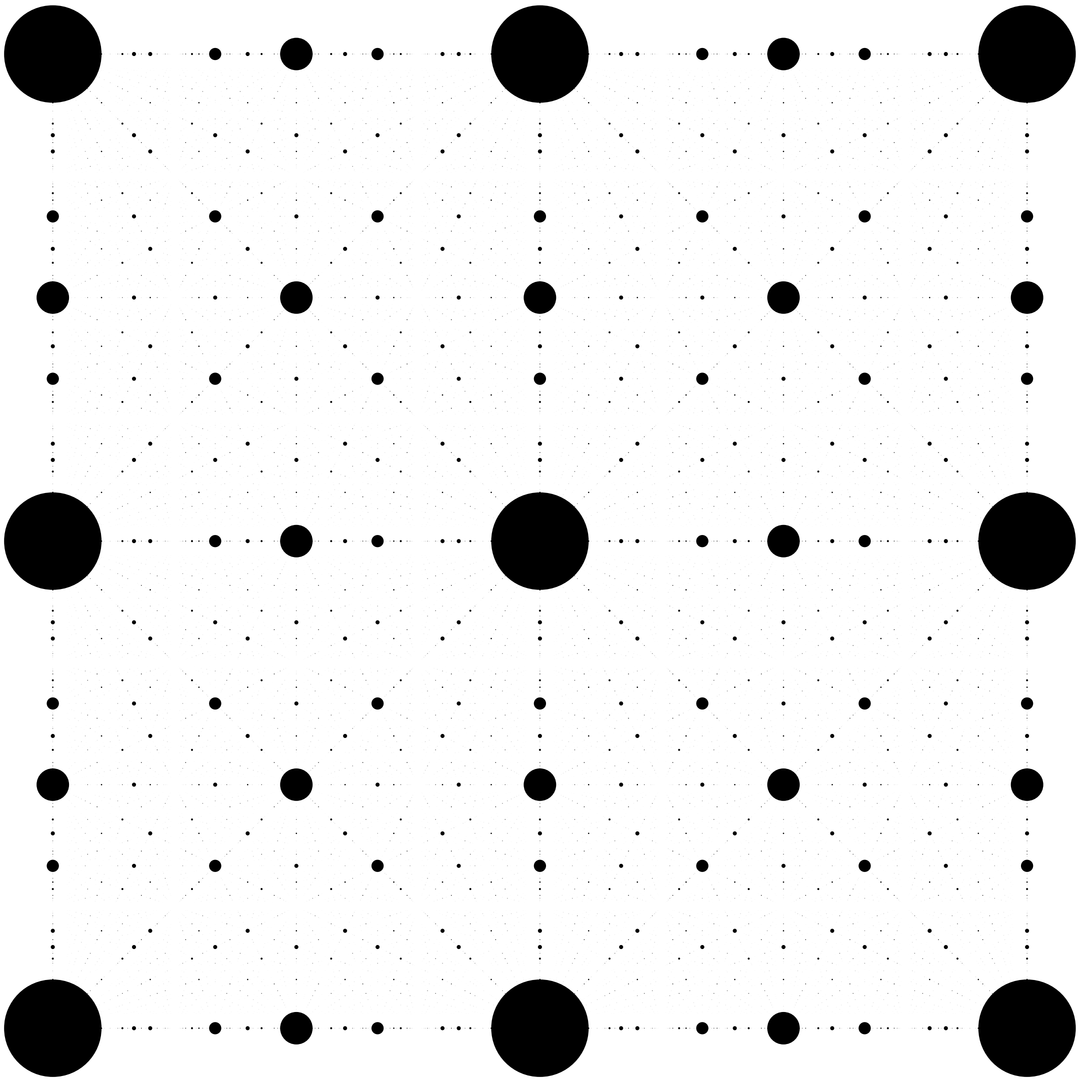




B/Grimm/Warrington 1994

Schroeder 1982, Mosseri 1992, B/Moody/Pleasants 2000, B/Huck 2013

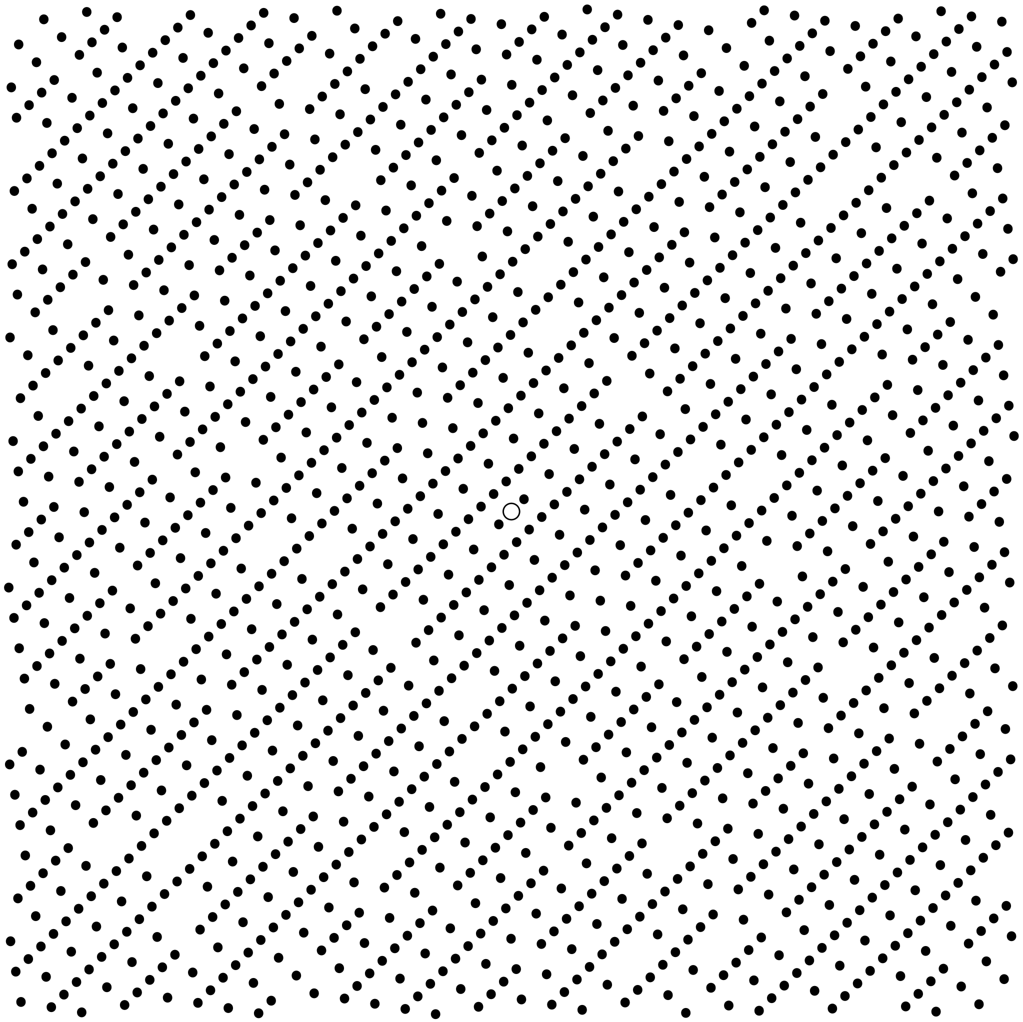
Visible lattice points



Properties

- $\mathbb{Z}^2$-periodic
- D_4-symmetric
- $\mathrm{GL}(2,\mathbb{Z})$-invariant
- support of $\widehat{\gamma}$: $S = \{k \in \mathbb{Q}^2$ with $\mathrm{den}(k)$ square-free$\}$: FB spectrum
- intensity for $k \in S$ with $\mathrm{den}(k) = q$ $\left(\frac{6}{\pi^2}\right)^2 \prod_{p|q} \frac{1}{(p^2-1)^2}$

Squarefree integers in $\mathbb{Z}[\sqrt{2}\,]$



$V = \{(x, x') \mid x \text{ sq.-free}\}$

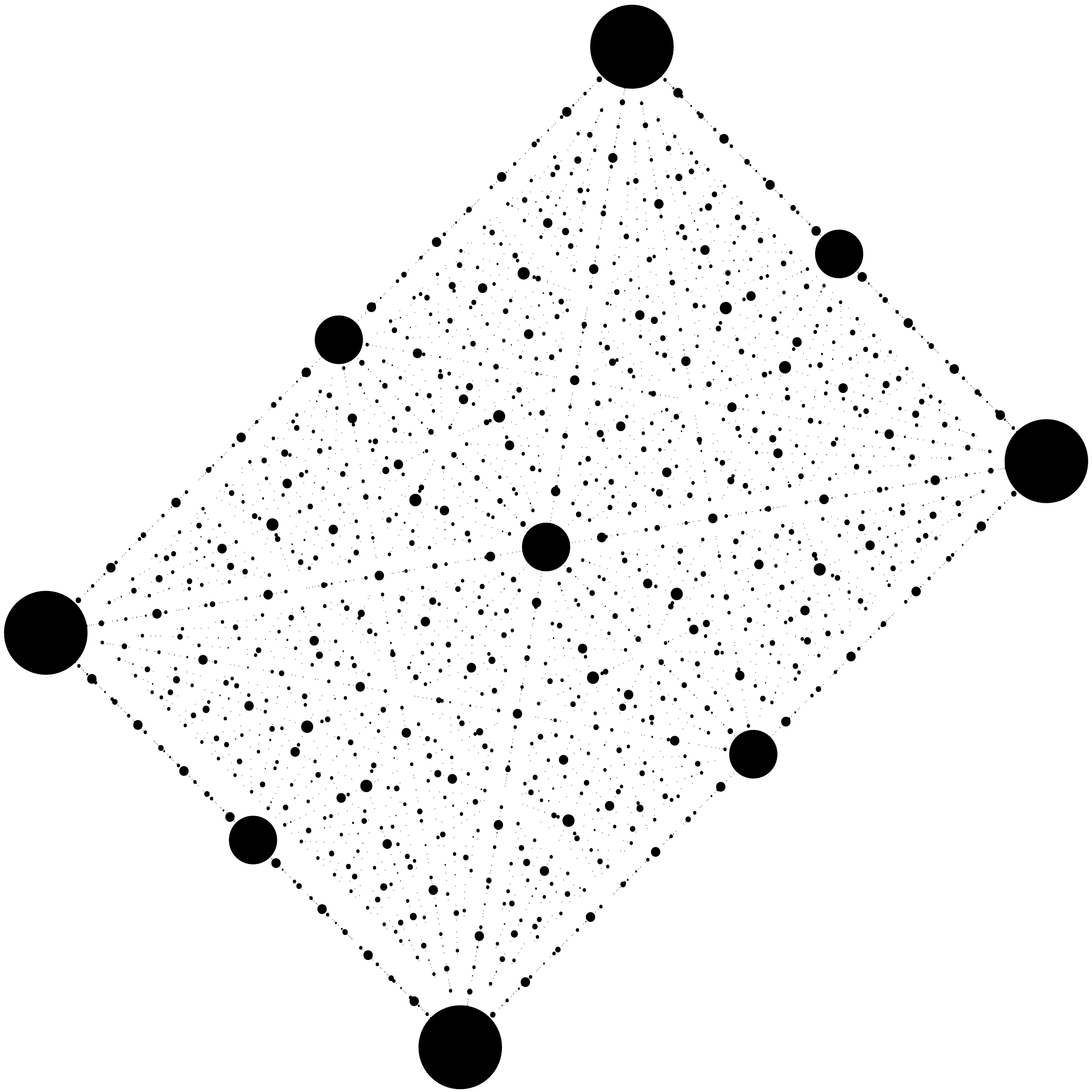
Properties

- $\mathrm{dens}(V) = \frac{24}{\pi^4} = \frac{\mathrm{dens}(\mathcal{L})}{\zeta_K(2)}$
- V not Delone
- $V - V = \langle V \rangle = \mathcal{L}$
- pure point diffraction
- weak model set
- $h_{\mathrm{top}}(V) > h_{\mathrm{m}}(V) = 0$

Theorem PP dynamical spectrum, trivial top. point spectrum

(Cellarosi/Vinogradov 2013, B/Huck 2013)

Squarefree integers in $\mathbb{Z}[\sqrt{2}\,]$



Properties

- $\mathcal{L}^*$-periodic
- $C_2 \times C_2$-symmetric
- $\mathrm{GL}(2, \mathbb{Z})$-invariant
- support of $\widehat{\gamma}$:
  $S = \{k \in \mathbb{Q}\mathcal{L}^*$ with $\mathrm{den}(k)$ cube-free$\}$:
  FB spectrum
- intensity for $k \in S$
  with $\mathrm{den}(k) = q$:
  $\left(\frac{24}{\pi^4}\right)^2 \prod_{p|q} \frac{1}{(N(p)^2 - 1)^2}$

(B/Huck 2013)

Weak model sets

CPS

$$\begin{array}{ccccc} G & \xleftarrow{\pi} & G \times H & \xrightarrow{\pi_{\text{int}}} & H \\ \cup & & \cup & & \cup \ \text{dense} \\ \pi(\mathcal{L}) & \xleftarrow{1-1} & \mathcal{L} & \longrightarrow & \pi_{\text{int}}(\mathcal{L}) \\ \| & & & & \| \\ L & & \xrightarrow{\hspace{3em}\star\hspace{3em}} & & L^{\star} \end{array}$$

G: σ-compact

H: comp. gen.

$\mathcal{A}$: van Hove in G

WMS

$$\boxed{\varLambda = \{ x \in L \mid x^{\star} \in W \}}$$

with $W \subset H$ compact, $\theta_H(W) > 0$

max. density: $\mathrm{dens}(\varLambda) = \mathrm{dens}(\mathcal{L})\, \theta_H(W)$

$$\gamma_{\varLambda} := \lim_{n\to\infty} \frac{\delta_{\varLambda \cap A_n} * \delta_{-(\varLambda \cap A_n)}}{\theta_G(A_n)}$$ (exists !!)

(B/Huck/Strungaru 2015, Lenz/Spindeler/Strungaru 2020)

Weak model sets

Diffraction $\boxed{\widehat{\gamma} = \sum_{k\in L^0} |A(k)|^2\, \delta_k}$ pure point !! $(\omega = \delta_{\varLambda})$

with $L^0 = \pi(\mathcal{L}^0)$ (annihilator of $\mathcal{L}$ in dual CPS)

amplitude $A(k) = \frac{\mathrm{dens}(\varLambda)}{\theta_H(W)}\, \widehat{1_W}(-k^\star)$

Hull $\boxed{\mathbb{X}_{\varLambda} = \overline{G + \varLambda}}$

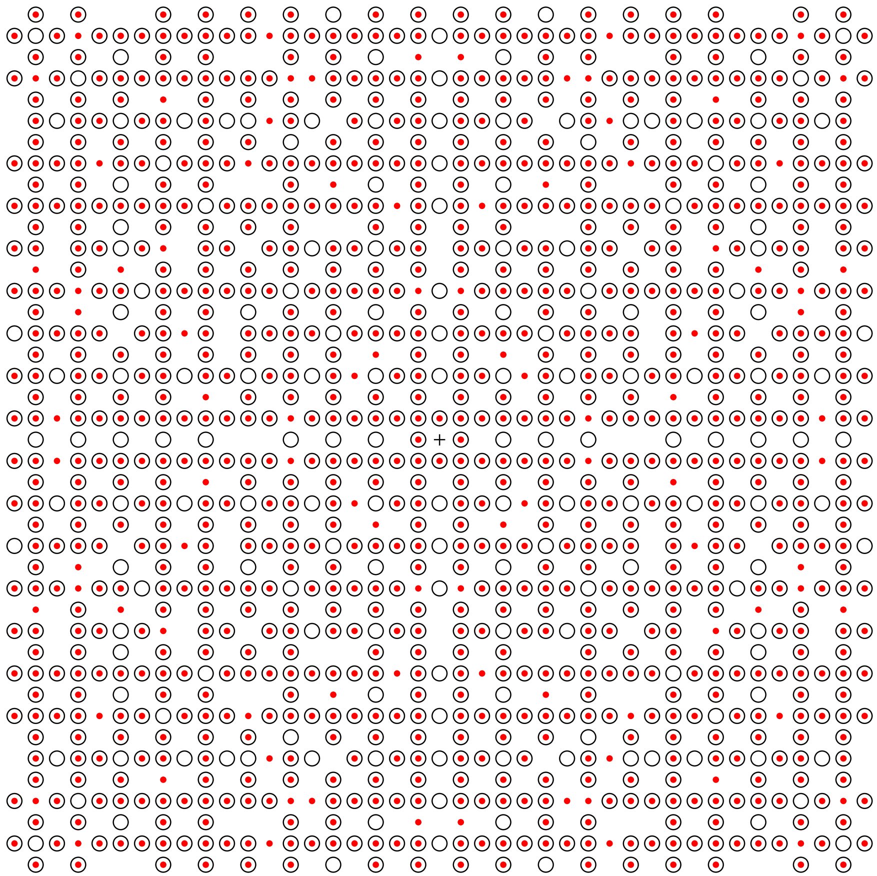
with patch frequency measure μ

μ is ergodic, $\varLambda$ is generic for μ

Theorem $(\mathbb{X}_{\varLambda}, G, \mu)$ has pure point dynamical spectrum: L^0

(Keller/Richard 2015, B/Huck/Strungaru 2015)

Topological conjugacy



Visible lattice points (red dots) versus

square-free Gaussian integers (circles)

Topological conjugacy

Systems $(\mathbb{X}_{\mathrm{G}}, \mathbb{Z}^2)$ versus $(\mathbb{X}_{\mathrm{V}}, \mathbb{Z}^2)$

not measure-theor. isomorphic, by Halmos-von Neumann theorem

Top. invariants $\mathcal{S} = \mathrm{cent}_{\mathrm{Aut}(\mathbb{X})}(\mathbb{Z}^2)$ $\quad \mathrm{Aut}(\mathbb{X}) = \mathrm{Homeo}(\mathbb{X})$

$$\mathcal{R} = \mathrm{norm}_{\mathrm{Aut}(\mathbb{X})}(\mathbb{Z}^2)$$

Theorem $\mathcal{S}_{\mathrm{V}} = \mathcal{S}_{\mathrm{G}} = \mathbb{Z}^2$ together with

$$\boxed{\mathcal{R}_{\mathrm{G}} = \mathbb{Z}^2 \rtimes D_4} \quad \text{and} \quad \boxed{\mathcal{R}_{\mathrm{V}} = \mathbb{Z}^2 \rtimes \mathrm{GL}(2, \mathbb{Z})}$$

Question General structure ?

Topological conjugacy

System $(\mathbb{X}, \mathbb{Z}^2)$ with $\mathbb{X} = \overline{\mathbb{Z}^2 + V_2}$ and

$$V_2 = \{m + n\sqrt{2} \text{ is } k\text{-free} \mid m, n \in \mathbb{Z}\} \qquad \text{any } k \leqslant 2$$

Theorem $\boxed{\mathcal{S} = \mathbb{Z}^2}$ and $\boxed{\mathcal{R} = \mathcal{S} \rtimes \mathcal{H}}$ with

$$\mathcal{H} \simeq \mathcal{O}^{\times} \rtimes \mathrm{Gal}\big(\mathbb{Q}(\sqrt{2}\,)/\mathbb{Q}\big) \simeq C_2 \times D_\infty$$

(B/Bustos/Huck/Lemańczyk/Nickel 2020)

Extensions Same type of result for all quadratic fields

(B/Bustos/Nickel 2021)

Outlook

- Higher-dimensional inflation tilings
- Eigenfunctions under Rauzy fractals
- Connections with almost periodicity
- Systems with mixed spectrum
- Non-minimal systems from number theory
- Efficient topological invariants
- Quadratic and cyclotomic fields
- General $\mathcal{B}$-free systems for $d > 1$

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