Equidistribution results for self-similar measures.

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Motivating problem: Suppose a property holds for Lebesgue almost every $x \in \mathbb{R}$ and μ is a Borel probability measure that is defined "independently" from this property. Does this property hold for μ almost every x?

A simple example where we can obtain a positive answer to this question is the following theorem.

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Theorem

Let $E \subseteq \mathbb{R}$ be a Borel set such that $\mathcal{L}(\mathbb{R} \setminus E) = 0$ and μ be a Borel probability measure. Denote by μ_t the pushforward of μ by the map $x \to x + t$. Then for Lebesgue almost every $t \in \mathbb{R}$ we have $\mu_t(E) = 1$.

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Proof.

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Proof.

Fubini's theorem. $\square$

The independence comes from our random choice of translation parameter.

In this talk we will consider this problem when the property we are interested in is defined in terms of some sequence being uniformly distributed, and the measures we are interested in are fractal measures.

Definition

A sequence $(x_n)_{n=1}^{\infty}$ of real numbers is said to be uniformly distributed modulo one if for every pair of real numbers u, v with $0 \leq u < v \leq 1$ we have

$$\lim_{N\to\infty} \frac{\#\{1 \leq n \leq N : x_n \bmod 1 \in [u, v]\}}{N} = v - u.$$

Theorem (Borel's normal number theorem)

Let $b \geq 2$ be an integer. Then for Lebesgue almost every x the sequence $(b^n x)_{n=1}^{\infty}$ is uniformly distributed modulo one.

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Several analogues of Borel's normal number theorem have been established for fractal measures.

Definition

- We call a finite set of contracting similarities $\Phi := \{\varphi_i(x) = r_i x + t_i\}_{i=1}^{k}$ an iterated function system or IFS for short ($|r_i| \in (0,1)$ and $t_i \in \mathbb{R}$).

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- For any IFS there exists a unique non-empty compact set X satisfying
$$X = \bigcup_{i=1}^{k} \varphi_i(X).$$
We call X the self-similar set of the IFS.

Example

Let $\Phi = \{\varphi_1(x) = \frac{x}{3}, \varphi_2(x) = \frac{x+2}{3}\}$. The self-similar set for this IFS is

$$C := \left\{ \sum_{i=1}^{\infty} \frac{\epsilon_i}{3^i} : \epsilon_i \in \{0, 2\} \right\}.$$

We call C the middle third Cantor set.

Definition

Let $\Phi = \{\varphi_i(x) = r_i x + t_i\}_{i=1}^k$ be an IFS and $\mathbf{p} = (p_i)_{i=1}^k$ be a probability vector. Then there exists a unique Borel probability measure $\mu_{\mathbf{p}}$ supported on the self-similar set of Φ such that

$$\mu_{\mathbf{p}} = \sum_{i=1}^{k} p_i \cdot \mu_{\mathbf{p}} \circ \varphi_i^{-1}.$$

We call $\mu_{\mathbf{p}}$ the self-similar measure corresponding to Φ and $\mathbf{p}$.

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Therefore for any self-similar measure $\mu_{\mathbf{p}}$ the sequence $(3^n x)_{n=1}^{\infty}$ is not uniformly distributed modulo one for $\mu_{\mathbf{p}}$ almost every x.

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Metaconjecture

Suppose $\mu_{\mathbf{p}}$ is a self-similar measure that is "independent" from the dynamical system $x \to bx \mod 1$. Then for $\mu_{\mathbf{p}}$ almost every x the sequence $(b^n x)_{n=1}^{\infty}$ is uniformly distributed modulo one.

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- Hochman and Shmerkin (2015): Let $\{\varphi_i(x) = r_i x + t_i\}_{i=1}^{k}$ be an IFS satisfying the open set condition. Let $b \geq 2$ be such that $\frac{\log |r_i|}{\log b} \notin \mathbb{Q}$ for some i, then for every self-similar measure $\mu_{\mathbf{p}}$, $\mu_{\mathbf{p}}$ almost every x is such that $(b^n x)_{n=1}^{\infty}$ is uniformly distributed modulo one.

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- Dajan, Ganguly, and Weiss (2020): Let $\{\varphi_i(x) = \frac{x}{b} + t_i\}_{i=1}^{k}$ be an IFS. Suppose $t_i - t_j \notin \mathbb{Q}$ for some i, j, then for every self-similar measure $\mu_{\mathbf{p}}$, $\mu_{\mathbf{p}}$ almost every x is such that $(b^n x)_{n=1}^{\infty}$ is uniformly distributed modulo one.

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- Other important works due to Kaufman, Queffélec and Ramaré, and Jordan and Sahlsten.

The motivation behind what I am discussing comes from these results and the following theorem.

Theorem (Koksma (1935))

For Lebesgue almost every $x > 1$ the sequence $(x^n)_{n=1}^{\infty}$ is uniformly distributed modulo one.

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Does an analogue of Koksma's theorem hold for self-similar measures?

The following conjecture seems plausible.

Conjecture

Let $\mu_{\mathbf{p}}$ be a non-atomic self-similar measure with support contained in $[1,\infty)$. Then for $\mu_{\mathbf{p}}$ almost every x the sequence $(x^n)_{n=1}^{\infty}$ is uniformly distributed modulo one.

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This seems reasonable because an IFS consists of affine maps and the maps $x \to x^n$ are not affine (for $n \geq 2$).

In other words, the IFS and its self-similar measures are defined "independently" from the maps $x \to x^n$.

Definition

- We say that an IFS $\Phi = \{\varphi_i(x) = r_i x + t_i\}_{i=1}^k$ is equicontractive if there exists $r \in (-1, 0) \cup (0, 1)$ such that $r_i = r$ for all i.

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- We say that an IFS $\Phi = \{\varphi_i\}_{i=1}^k$ satisfies the convex strong separation condition if

$$\varphi_i(\mathrm{conv}(X)) \cap \varphi_j(\mathrm{conv}(X)) = \emptyset \, \forall i \neq j.$$

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The middle third Cantor set is constructed using an equicontractive IFS satisfying the convex strong separation condition.

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- $f_n \in C^3(\mathrm{conv}(X), \mathbb{R})$ for each n.
- There exists $C_1, C_2 > 0$ such that for any m, n with $m < n$ we have
$$|f_n'(x) - f_m'(x)| \leq C_1 n^{C_2} x^{n-1}$$
for all $x \in \mathrm{conv}(X)$.

- There exists $C_3 > 0$ such that for all n sufficiently large, for any $m < n$ we have:

$$|f_n''(x) - f_m''(x)| \geq C_3 x^{n-2}$$

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for all $x \in \text{conv}(X)$.

- For any m, n with $m < n$ we have either

$$f_n'''(x) - f_m'''(x) \geq 0$$

for all $x \in \text{conv}(X)$, or

$$f_n'''(x) - f_m'''(x) \leq 0$$

for all $x \in \text{conv}(X)$.

It is straightforward to construct sequences of functions satisfying these four properties. For example:

- $f_n(x) = x^n$
- $f_n(x) = x^n + x^{n-1} + \cdots + x + 1$ for all n.
- We fix a polynomial g with strictly positive coefficients and let $f_n(x) = g(x)x^n$ for all n.
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Theorem (B, 2020)

Let $\{\varphi_i(x) = rx + t_i\}_{i=1}^k$ be an equicontractive iterated function system satisfying the convex strong separation condition with self-similar set X contained in $[1, \infty)$. Let $(f_n)_{n=1}^\infty$ be a sequence of functions satisfying the aforementioned properties. Moreover let $\mathbf{p} = (p_i)_{i=1}^k$ be a probability vector satisfying

$$\frac{1}{2} < \frac{-\sum_{i=1}^k p_i \log p_i}{-\log |r|}.$$

Then for $\mu_{\mathbf{p}}$ almost every x the sequence $(f_n(x))_{n=1}^\infty$ is uniformly distributed modulo one.

The following corollaries follow from this theorem:

Corollary (B)

Let $\{\varphi_i(x) = rx + t_i\}_{i=1}^k$ *be an equicontractive iterated function system satisfying the convex strong separation condition with self-similar set* X *contained in* $[1,\infty)$. *Moreover let* $\mathbf{p} = (p_i)_{i=1}^k$ *be a probability vector satisfying*

$$\frac{1}{2} < \frac{-\sum_{i=1}^k p_i \log p_i}{-\log |r|}.$$

Then for $\mu_{\mathbf{p}}$ *almost every* x *the sequence* $(x^n)_{n=1}^\infty$ *is uniformly distributed modulo one.*

Corollary (B, 2020)

Let C be the middle third Cantor set. Then for any $t \geq 1$, with respect to the Cantor-Lebesgue measure on $C + t$, for almost every x the sequence $(x^n)_{n=1}^{\infty}$ is uniformly distributed modulo one.

Here we use that the Cantor-Lebesgue measure coincides with $\mu_{(1/2,1/2)}$.

The proof differs significantly from earlier works on the sequence $(b^n x)$.

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We do not use the Fourier transform of the self-similar measure. (Cassels)

We do not rely on techniques from Ergodic Theory. (Hochman and Shmerkin, Dajan, Ganguly, and Weiss).

Our proof exploits the fact that our sequence of functions $(f_n)_{n=1}^{\infty}$ are not affine maps.

Combining Weyl's criterion for uniform distribution with a result of Davenport, Erdős, and LeVeque we have the following useful proposition.

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Proposition

Let μ be a Borel probability measure on $\mathbb{R}$ and $(f_n)_{n=1}^{\infty}$ be a sequence of continuous real valued functions. If for any $l \in \mathbb{Z} \setminus \{0\}$ the series

$$\sum_{N=1}^{\infty} \frac{1}{N} \int \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i l f_n(x)} \right|^2 d\mu$$

converges, then for μ almost every x the sequence $(f_n(x))_{n=1}^{\infty}$ is uniformly distributed modulo one.

As such our proof reduces to now showing that for any $l \in \mathbb{Z} \setminus \{0\}$ we have

$$\sum_{N=1}^{\infty} \frac{1}{N} \int \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i l f_n(x)} \right|^2 d\mu < \infty.$$

- The first part of the proof of the theorem involves bounding certain integrals that are defined with respect to the self-similar measure from above in terms of the L^2 norm of functions that are integrated with respect to the Lebesgue measure. This argument is an adaptation of one given by Jordan and Sahlsten.

- The first part of the proof of the theorem involves bounding certain integrals that are defined with respect to the self-similar measure from above in terms of the L^2 norm of functions that are integrated with respect to the Lebesgue measure. This argument is an adaptation of one given by Jordan and Sahlsten.
- The second part is where we use our assumptions on (f_n). It involves some careful analysis and an application of the van der Corput lemma. This is where the non-affineness of the sequence (f_n) is used.

In what follows we restrict our attention to the IFS $\{\varphi_0(x)=\frac{x+2}{3},\varphi_1(x)=\frac{x+4}{3}\}$ and the uniform $(1/2,1/2)$ self-similar measure. For this IFS the attractor is $C+1$.

In what follows we restrict our attention to the IFS $\{\varphi_0(x) = \frac{x+2}{3}, \varphi_1(x) = \frac{x+4}{3}\}$ and the uniform $(1/2, 1/2)$ self-similar measure. For this IFS the attractor is $C + 1$.

For any $\mathbf{a} \in \{0, 1\}^M$ we let

$$\varphi_{\mathbf{a}} := \varphi_{a_1} \circ \cdots \circ \varphi_{a_M}.$$

We also let $C_{\mathbf{a}} = \varphi_{\mathbf{a}}(C + 1)$.

To successfully apply the criteria of Davenport, Erdős, and LeVeque we are naturally led to consider:

$$\int \exp(l(f_n(x) - f_m(x)))\, d\mu$$

for $n > m$ and l is fixed. Here $\exp(x) = e^{2\pi i x}$.

Using the self-similarity of μ we have

$$\int \exp(l(f_n(x) - f_m(x)))\, d\mu = \int W_M(x)\, d\mu$$

where

$$W_M(x) = \sum_{\mathbf{a}\in\{0,1\}^M} \frac{\exp(l(f_n(\varphi_{\mathbf{a}}(x)) - f_m(\varphi_{\mathbf{a}}(x))))}{2^M}.$$

M is a parameter that is chosen carefully during our proof.

Fix $\delta > 0$ to be some small real number. Let

$$R_M := \left\{ \mathbf{a} \in \{0,1\}^M : \sup_{x \in C_{\mathbf{a}}} |W_M(x)| \geq 2 \cdot 3^{-\delta n} \right\}.$$

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We have the inequality

$$\left| \sum_{\mathbf{a} \in R_M^c} \int_{C_{\mathbf{a}}} W_M(x)\, d\mu \right| \leq 2 \cdot 3^{-\delta n}.$$

Fact: For the right choice of M, if $\mathbf{a} \in R_M$ then $|W_M(x)| \geq 3^{-\delta n}$ for all $x \in \varphi_{\mathbf{a}}([1,2])$.

Using this fact we have the following

$$\begin{aligned}
\left|\sum_{\mathbf{a}\in R_M}\int_{C_{\mathbf{a}}} W_M(x)\,d\mu\right| &\leq \sum_{\mathbf{a}\in R_M}\frac{1}{2^M}\\
&= \frac{3^M}{2^M\cdot 3^{-2\delta n}}\sum_{\mathbf{a}\in R_M}\frac{3^{-2\delta n}}{3^M}\\
&\leq \frac{3^M}{2^M\cdot 3^{-2\delta n}}\sum_{\mathbf{a}\in R_M}\int_{\varphi_{\mathbf{a}}([1,2])}|W_M(x)|^2\,dx\\
&\leq \frac{3^M}{2^M\cdot 3^{-2\delta n}}\int_1^2|W_M(x)|^2\,dx.
\end{aligned}$$

Combining the above we've shown that

$$\left|\int \exp(I(f_n(x)-f_m(x)))\,d\mu\right| \leq \frac{3^M}{2^M\cdot 3^{-2\delta n}}\int_1^2 |W_M(x)|^2\,dx+2\cdot 3^{-\delta n}.$$

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To overcome it we need the lower bound

$$\frac{1}{2} < \frac{-\sum_{i=1}^{k} p_i \log p_i}{-\log |r|}$$

appearing in the assumptions of the theorem.

To complete our proof it remains to obtain good upper bounds for the integral

$$\int_1^2 |W_M(x)|^2\, dx.$$

Recall that

$$W_M(x) = \sum_{\mathbf{a}\in\{0,1\}^M} \frac{\exp(l(f_n(\varphi_{\mathbf{a}}(x)) - f_m(\varphi_{\mathbf{a}}(x))))}{2^M}.$$

Recall that

$$W_M(x) = \sum_{\mathbf{a}\in\{0,1\}^M} \frac{\exp(I(f_n(\varphi_{\mathbf{a}}(x)) - f_m(\varphi_{\mathbf{a}}(x))))}{2^M}.$$

It should not be surprising that to obtain good bounds for $\int_1^2 |W_M(x)|^2\,dx$ it suffices to bound integrals of the form

$$\int_1^2 \exp(I(f_n(\varphi_{\mathbf{a}}(x)) - f_m(\varphi_{\mathbf{a}}(x)) - f_n(\varphi_{\mathbf{b}}(x)) + f_m(\varphi_{\mathbf{b}}(x)))\,dx$$

for distinct $\mathbf{a}, \mathbf{b} \in \{0,1\}^M$.

To bound these integrals we use the following lemma due to van der Corput.

Lemma (van der Corput lemma)

Let $\phi : [a, b] \to \mathbb{R}$ be differentiable. Assume that $|\phi'(x)| \geq \gamma$ for all $x \in [a, b]$, and ϕ' is monotonic on $[a, b]$. Then

$$\left| \int_a^b e^{2\pi i \phi(x)}\, dx \right| \leq \gamma^{-1}.$$

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$$\left| \int_a^b e^{2\pi i \phi(x)}\, dx \right| \leq \gamma^{-1}.$$

This is the part of our proof which uses the fact our functions (f_n) are not affine and that our IFS is equicontractive.

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Observe that if $f_n(x) = 3^n x$ and

$$\phi(x) = l(f_n(\varphi_{\mathbf{a}}(x)) - f_m(\varphi_{\mathbf{a}}(x)) - f_n(\varphi_{\mathbf{b}}(x)) + f_m(\varphi_{\mathbf{b}}(x))$$

then

$$\phi'(x) = l(3^{n-M} - 3^{m-M} - 3^{n-M} + 3^{m-M}) = 0.$$

So we cannot use van der Corput's lemma

To finish I'd like to emphasise one feature of this argument that is specific to self-similar measures.

Suppose we want to bound the integral

$$\left|\int \exp(l(f_n(x) - f_m(x)))\, d\mu\right|.$$

Taking the absolute value inside the integral provides a trivial upper bound which is not very useful.

For self-similar measures we may use their defining property then take the absolute value inside:

$$\left|\int \exp(l(f_n(x)-f_m(x)))\,d\mu\right|$$
$$=\left|\int \sum_{\mathbf{a}\in\{0,1\}^M} \frac{\exp(l(f_n(\varphi_{\mathbf{a}}(x))-f_m(\varphi_{\mathbf{a}}(x))))}{2^M}\,d\mu\right|$$
$$\leq \int \left|\sum_{\mathbf{a}\in\{0,1\}^M} \frac{\exp(l(f_n(\varphi_{\mathbf{a}}(x))-f_m(\varphi_{\mathbf{a}}(x))))}{2^M}\right|\,d\mu.$$

It is reasonable to expect that

$$\left|\sum_{\mathbf{a}\in\{0,1\}^M}\frac{\exp(I(f_n(\varphi_{\mathbf{a}}(x))-f_m(\varphi_{\mathbf{a}}(x))))}{2^M}\right|$$

will be small.

Thank you for listening.