Metrical theory for the set of points associated with the generalized Jarník-Besicovitch set

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Diophantine approximation

Theorem (Dirichlet, 1842)

Given $x \in \mathbb{R}$ *and* $t > 1$*, there exists pair* $(p, q) \in \mathbb{Z} \times \mathbb{N}$ *such that*

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{qt} \quad \text{and} \quad 1 \leq q < t.$$

Corollary

For any $x \in \mathbb{R}$*, there exist infinitely many pairs* $(p, q) \in \mathbb{Z} \times \mathbb{N}$ *such that*

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}.$$

Uniform vs asymptotic Diophantine approximation

Theorem (Dirichlet, 1842)

Given $x \in \mathbb{R}$ and $t > 1$, there exists pair $(p, q) \in \mathbb{Z} \times \mathbb{N}$ such that

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Uniform result

Corollary

For any $x \in \mathbb{R}$, there exist infinitely many pairs $(p, q) \in \mathbb{Z} \times \mathbb{N}$ such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}.$$

Asymptotic result

Improvements in each setting?

Improvements to Dirichlet's corollary

Set of τ-well approximable numbers: For $\tau \geq 2$,

$$J(\tau) = \left\{ x \in [0,1) : \left| x - \frac{p}{q} \right| < \frac{1}{q^\tau} \text{ for i.m. } (p,q) \in \mathbb{Z} \times \mathbb{N} \right\}.$$

Theorem (Khintchine, 1924)

$$\lambda(J(\tau)) = \begin{cases} 0 & \text{if} \quad \sum_{q=1}^{\infty} q^{1-\tau} < \infty \\ 1 & \text{if} \quad \sum_{q=1}^{\infty} q^{1-\tau} = \infty. \end{cases}$$

Theorem (Jarník–Besicovitch, 1929)

For any $\tau \geq 2$,

$$\dim_{\mathrm{H}} J(\tau) = \frac{2}{\tau}.$$

$J(\tau)$ in terms of entries of continued fraction

Every irrational number $x \in [0,1)$ has a *continued fraction expansion*,

$$x = [a_1(x), a_2(x), \dots]; \quad a_i(x) \in \mathbb{Z}^+ \text{ for each i} \in \mathbb{N}.$$

In particular,

$a_1(x) = \lfloor \frac{1}{x} \rfloor$ and $a_n(x) = \lfloor \frac{1}{T^{n-1}(x)} \rfloor$ for $n \geq 2$, where the Gauss transformation $T : [0,1) \to [0,1)$ is defined as

$$T(0) = 0 \text{ and } T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \text{ for } \quad 0 < x < 1.$$

$$\frac{p_n(x)}{q_n(x)} = [a_1(x), ..., a_n(x)]$$

$(p_n, q_n$ coprime) are called the n-th *convergents* of x.

Lagrange's theorem:

$$\inf_{p \in \mathbb{Z}, q \leq q_n(x)} \left| x - \frac{p}{q} \right| \geq \left| x - \frac{p_n(x)}{q_n(x)} \right|.$$

$J(\tau)$ in terms of continued fraction entries

- *Legendre's theorem: if* $\left|x-\frac{p}{q}\right|<\frac{1}{2q^2}$ *then* $\frac{p}{q}=\frac{p_n(x)}{q_n(x)}$, for some $n\geq 1$.
- *Speed of approximation:*

$$\frac{1}{(a_{n+1}(x)+2)q_n^2(x)}\leq\left|x-\frac{p_n(x)}{q_n(x)}\right|\leq\frac{1}{a_{n+1}(x)q_n^2(x)}.$$

- $$q_n^2(x)\leq\prod_{j=0}^{n-1}|T'(T^j(x))|\leq 4q_n^2(x)$$

- For $\tau\geq 2$

$$J(\tau)=\left\{x\in[0,1):\left|x-\frac{p}{q}\right|<\frac{1}{q^\tau}\ \text{ for i.m.}(p,q)\in\mathbb{Z}\times\mathbb{N}\right\}$$

$J(\tau)$ in terms of continued fraction entries

- *Legendre's theorem: if* $\left|x - \frac{p}{q}\right| < \frac{1}{2q^2}$ *then* $\frac{p}{q} = \frac{p_n(x)}{q_n(x)}$, for some $n \geq 1$.

- *Speed of approximation:*

$$\frac{1}{(a_{n+1}(x)+2)q_n^2(x)} \leq \left|x - \frac{p_n(x)}{q_n(x)}\right| \leq \frac{1}{a_{n+1}(x)q_n^2(x)}.$$

- *For* $\tau \geq 2$

$$J(\tau) = \left\{x \in [0,1) : \left|x - \frac{p}{q}\right| < \frac{1}{q^\tau} \text{ for i.m.} (p,q) \in \mathbb{Z} \times \mathbb{N}\right\}$$

$$J(\tau) = \left\{x \in [0,1) : a_n(x) \geq e^{((\tau-2)/2)(\log|T'(x)|+\cdots+\log|T'(T^{n-1}(x))|)} \text{ for i.m. } n \in \mathbb{N}\right\}.$$

Improvements to Dirichlet's theorem

Let $\psi : [t_0, \infty) \to \mathbb{R}_+$ be a non-increasing function with $t_0 \geq 1$ fixed. Define the set of ψ-Dirichlet improvable numbers by

$$D(\psi) := \left\{ x \in \mathbb{R} : \begin{array}{c} \exists\, N : \text{ the system } |qx - p| < \psi(t),\ |q| < t \\ \text{has a non trivial integer solution for all } t > N \end{array} \right\}.$$

Criteria for Dirichlet non-improvable numbers

Lemma (Kleinbock–Wadleigh, 2018)

Let $x \in [0,1) \setminus \mathbb{Q}$, and let $\psi : [t_0, \infty) \to \mathbb{R}_+$ be non-increasing function with $t\psi(t) < 1$ for all $t \geq t_0$ and $\Psi(t) = \frac{1}{1 - t\psi(t)} - 1$. Then

(i) *$x \in D(\psi)$ if $a_{n+1}(x)a_n(x) \leq \Psi(q_n)/4$ for all sufficiently large n.*

(ii) *$x \in D(\psi)^c$ if $a_{n+1}(x)a_n(x) > \Psi(q_n)$ for infinitely many n.*

$$G(\Psi) = \left\{ x \in [0,1) : a_n(x)a_{n+1}(x) \geq \Psi(q_n) \text{ for i.m. } n \in \mathbb{N} \right\} \subset D(\psi)^c.$$

Improvements to Dirichlet's theorem

Theorem (Kleinbock–Wadleigh, 2018)

Let ψ be a non-increasing positive function with $t\psi(t) < 1$ for all large t and $\Psi(t) = \frac{1}{1 - t\psi(t)} - 1$.

$$\lambda(G(\Psi)) = \begin{cases} 0 & \text{if} \quad \sum_{t=1}^{\infty} \frac{\log \Psi(t)}{t\Psi(t)} < \infty \\ 1 & \text{if} \quad \sum_{t=1}^{\infty} \frac{\log \Psi(t)}{t\Psi(t)} = \infty. \end{cases}$$

Theorem (Hussain–Kleinbock–Wadleigh–Wang, 2018)

$$\dim_{\mathrm{H}} G(\Psi) = \frac{2}{2+\tau}, \text{ where } \tau = \liminf_{t\to\infty} \frac{\log \Psi(t)}{\log t}.$$

$$G(\Psi) = \left\{ x \in [0,1) : a_n(x)a_{n+1}(x) \geq \Psi(q_n) \text{ for i.m. } n \in \mathbb{N} \right\} \subset D(\psi)^c \subset G(\Psi/4).$$

$$\mathcal{K}(3\Psi) := \left\{ x \in [0,1) : |qx - p| < \frac{1}{3q\Psi(q)} \text{ for i.m. } (p,q) \in \mathbb{Z}\times\mathbb{N} \right\} \subset G(\Psi).$$

Question

How big is the set $G(\Psi) \setminus \mathcal{K}(3\Psi)$?

How big is the set $G(\Psi) \setminus \mathcal{K}(C\Psi)$?

Let $E := G(\Psi) \setminus \mathcal{K}(\Psi)$. Then

$$E = \left\{ x \in [0,1) : \begin{array}{r} a_{n+1}(x)a_n(x) \geq \Psi(q_n) \text{ for infinitely many } n \in \mathbb{N} \text{ and} \\ a_{n+1}(x) < \Psi(q_n) \text{ for all sufficiently large } n \in \mathbb{N} \end{array} \right\}.$$

Theorem (B.–Bos–Hussain, 2020)

Let $\Psi : [1, \infty) \to \mathbb{R}_+$ *be a non-decreasing function and* $C > 0$*. Then*

$$\dim_{\mathrm{H}} \Big(G(\Psi) \setminus \mathcal{K}(C\Psi) \Big) = \frac{2}{\tau + 2}, \text{ where } \tau = \liminf_{q \to \infty} \frac{\log \Psi(q)}{\log q}.$$

Generalized Jarník-Besicovitch set

For any $r \in \mathbb{N}$ define the set

$$\mathcal{R}_r(\tau) := \left\{x \in [0,1) : a_{n+1}(x)a_{n+2}(x)\cdots a_{n+r}(x) \geq e^{\tau(x)(h(x)+\cdots+h(T^{n-1}x))} \text{ for i.m. } n \in \mathbb{N}\right\},$$

where $\tau(x)$ and $h(x)$ are positive continuous functions defined on $[0,1]$.

Theorem (B., 2020)

Let $h : [0,1] \to \mathbb{R}_+$ and $\tau : [0,1] \to \mathbb{R}_+$ be positive continuous functions. Then for any $r \in \mathbb{N}$,

$$\dim_{\mathrm{H}} \mathcal{R}_r(\tau) = \inf\{s \geq 0 : \lim_{n\to\infty} \frac{1}{n} \log \sum_{a_1,\ldots,a_n \in \mathbb{N}} \frac{1}{e^{g_r(s)\tau_{min} S_n h(x)} q_n^{2s}} \leq 0\},$$

where

$$\tau_{min} = min\{\tau(x) : x \in [0,1]\}, g_1(s) = s \text{ and } g_r(s) = \frac{s g_{r-1}(s)}{1 - s + g_{r-1}(s)} \ \forall r \geq 2.$$

Outline of Proof

The sketch of proof is divided into three parts.

1. First, calculate the upper bound for the Hausdorff dimension
   - for the posited dimensions s, it is enough to evaluate $\sum_{i=1}^{\infty} |U_i|^s$ for specific coverings $\{U_i\}_{i\geq 1}$ of a set say X.
2. Second, calculate the lower bound for the dimension using
   - A lower bound requires showing that $\sum_{i=1}^{\infty} |U_i|^s$ is greater than some positive constant for all δ-coverings of X.

   1. a Cantor-type construction,
   2. the mass distribution principle.
3. Use a limiting process to show these are the same.

Concluding remarks and open problem

- $r = 1$

Theorem (Wang–Wu–Xu, 2016)

$$\dim_{\mathrm{H}} \left\{ x \in [0,1) : a_{n+1}(x) \geq e^{\tau(x)(h(x)+\cdots+h(T^{n-1}(x)))} \text{ for i.m. } n \in \mathbb{N} \right\} = s_{\mathbb{N}}^{(1)}.$$

- $r = 1, \tau(x) = 1$ and $h(x) = \log B$

Theorem (Wang–Wu, 2008)

For any $B > 1$,

$$\dim_{\mathrm{H}} \{ x \in [0,1) : a_{n+1}(x) \geq B^n \text{ for i.m. } n \in \mathbb{N} \} = s_B^{(1)}$$

where

$$s_B^{(1)} = \inf\{ s \geq 0 : \mathrm{P}(T, -s \log B - s \log |T'|) \leq 0 \}.$$

- $\tau(x) = 1$ and $h(x) = \log B$

Theorem (Huang–Wu–Xu, 2020)

For any $B > 1$,

$$\dim_{\mathrm{H}} \{ x \in [0,1) : a_{n+1}(x) a_{n+2}(x) \cdots a_{n+r}(x) \geq B^n \text{ for i.m. } n \in \mathbb{N} \} = s_B^{(r)},$$

where

$$s_B^{(r)} = \inf\{ s \geq 0 : \mathrm{P}(T, -g_r(s) \log B - s \log |T'|) \leq 0 \}.$$

Open problems

- Metrical theory associated with the weighted version of set $\mathcal{R}_r(\tau)$?
- Can we obtain something similar for Lüroth expansion?

For $x \in [0,1)$, let $[d_1(x), d_2(x), \dots]$ be its Lüroth expansion and $\left\{\frac{p_n}{q_n}\right\}$, $n \geq 1$ be its sequence of convergents.
Define $r \in \mathbb{N}$ the set

$$\mathcal{F}_r(\phi) := \{x \in [0,1) : d_n(x)d_{n+1}(x)\cdots d_{n+r-1}(x) \geq \phi(n) \text{ for i.m. } n \in \mathbb{N}\},$$

where $\phi : \mathbb{N} \to [2, \infty)$ be a positive function.

Question

How big is set $\mathcal{F}_r(\phi)$?

- In fact for r=2,

Theorem (Tan–Zhou, 2021)

$$\lambda(\mathcal{F}_2(\phi)) = \begin{cases} 0 & \text{if} \quad \sum_{n=1}^{\infty} \frac{\log \phi(n)}{\phi(n)} < \infty \\ 1 & \text{if} \quad \sum_{n=1}^{\infty} \frac{\log \phi(n)}{\phi(n)} = \infty. \end{cases}$$

THANK YOU!