Automata generated topological spaces
(and self-affine tiles)

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1. Multiple addresses in numeration
2. Properties of topology-generating automata
3. Iterated function systems and automata

(Few remarks on tiles)

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1.1 Ingredients of numeration

\[ D = \{0, 1, \ldots, m-1\} \text{ alphabet, digit set} \]

\[ S = D^\mathbb{N} = \{s = s_1s_2\ldots | s_k \in D\} \text{ symbolic space} \]

\[ X \text{ topological space} \]

\[ \varphi : S \rightarrow X \text{ numeration map (onto)} \]

assigns addresses to points \( x \in X \)

continuous \( \rightarrow X \) compact

\( \varphi \) is a quotient map
1.2 Multiple addresses

topology of X is determined by the equivalence relation

\[ L = \{ (s, t) \mid y(s) = y(t) \} \subseteq S \times S \]

We consider the case that \( L \) is generated by an automaton.

Ex. Binary numbers \( X = \{0, 1\} \) \( D = \{0, 1\} \)

\[ y(s_1s_2 \ldots) = \sum_{i=1}^{\infty} s_i \cdot 2^{-i} \]

\[ y(0 \ 1 \overline{1}) = y(1 \overline{0}) \]

\[ 0, 1 \xrightarrow{\text{left}} \quad 1, 0 \xrightarrow{\alpha} \quad 0, 1 \xrightarrow{\text{right}} \quad 1, 0 \]

\[ \frac{0}{0, 1} \xrightarrow{1, 1} \frac{1}{1, 0} \]
1.3 Modular structure

Word \( w = w_1 \cdots w_n \in D^n \)

\[ S_w = \{ w_1 \cdots w_n s_{n+1} s_{n+2} \cdots | s_i \in D \} \] cylinder set

Principle of self-similarity: All \( S_w \) are treated in the same way as \( S \)

\[ \varphi(s) = \varphi(t) \implies \varphi(ws) = \varphi(wt) \]

(simplifies our work)
1.4 Topology-generating automaton

**Def.** \( G = (V, E) \) directed graph, \( x \in V \) initial state, \( s \) states, \( t \) transitions

Input alphabet \( D \times D \) for marking edges

Automaton accepts those sequences \((i_1, j_1), (i_2, j_2), \ldots\)
for which there is a path \( E_1E_2 \ldots \) starting in \( x \)
and edge \( E_k \) is marked \((i_k, j_k)\).
Rem. — all states accepting. Could add rejecting state w.

- A must have loops \((i,i)\) for all \(i \in D\) (self-sim.)

- \(A\) should have no other incoming edges

  (in order to get finite equiv. classes)

- From each state \(v\) there must be a path to a directed cycle

  (in order to accept infinite sequences \((i_k, y_k)\))

\[ v \rightarrow ? \quad \text{or} \quad v \rightarrow \cdots \rightarrow \cdots \quad \text{etc.} \]
1.5 Topology generated by the automaton

\( (v, w) \in \mathcal{D}^n \) accepted means \( \gamma(S_v) \) intersects \( \gamma(S_w) \)

On finite level \( n \), topology is approximated by an undirected graph

\( \mathcal{H}_n = (\mathcal{D}_n, E_n = \{ (v, w) \} ) \) (Hata 1986)

\( (s, t) \in S \) accepted means \( \gamma(s) = \gamma(t) \)

So the automaton defines the equivalence relation \( \mathcal{L} \) and hence the topology of \( X \).
Prop. Let $X^{(n)} = D^n \cup E_n$, where points of $D^n$ are open and points of $E_n$ are closed. Then $X$ is the inverse limit of the finite topological spaces $X^{(n)}$.

$$X = \lim_{\leftarrow} (X^{(n)}, \pi_n)$$

with $\pi_n : X^{(n)} \rightarrow X^{(n-1)}$, $\pi_n(w_1\ldots w_n) = w_1\ldots w_{n-1}$,

$\pi_n(v_1\ldots v_n, w_1\ldots w_n) = (v_1\ldots v_{n-1}, w_1\ldots w_{n-1})$.

Ex.

$\varphi(110) = \varphi(011)$
1) Example was number system with base $-2$.
$$f(s_1 s_2 \ldots) = \sum_{i=1}^{\infty} s_i \cdot 2^{-i}$$

2) complete automaton - defines equivalence relation
incomplete automaton - defines relation which can be extended to equivalence relation

(easier to draw)

Ex.

\[ \begin{array}{c}
\begin{array}{ccc}
0,0 & 0,1 & 1,0 \\
0,0 & \rightarrow & 0,1 \rightarrow b \\
1,0 & \rightarrow & 1,1 \\
\end{array}
\end{array} \]

\[ \begin{array}{ccc}
01 \sim 00 & \alpha \rightarrow \alpha \rightarrow a \\
00 \sim 11 & \alpha \rightarrow b \rightarrow c \\
\end{array} \]

but $01 \not\sim 11$
automaton not complete
1.6 Motivation: Description of complex geometries by computer

3D geometry is at the centre of current science: cell biology, brain research, nano materials.

A great variety of complex geometric structures, even in everyday context: dust, soil, smoke, fire, ...

More network-like than manifold-like

Mathematics must find ways to model, describe, analyze such geometric phenomena by computer.

Numeration is one option.
2. Properties of topology-generating automata

2.1 Interpretation of states:
relative positions of intersecting pieces

Ex.

initial state \( x \) is standard position of \( X \)

first label denotes reference piece which is turned into standard position
second label for neighboring piece

States can also be considered as boundary edges of \( X \)

(no reflections involved
one-point intersections ignored
like \( \triangle \)
incomplete automaton)
transitions go from pairs of pieces to pairs of subpieces

Rem. The automaton and its accepted language code the geometry of $X$.

only first label, since second label is found at the “inverse” state $b=a^*\cdot c=c^*$

$L_a = \{(1(10)^*0)^\infty\}$ addresses of boundary set

$L_b = \{(0(01)^*1)^\infty\}$ addresses of

$L = \{(0\cdot L_b, \cdot L_a)\}$ multiple addresses of

(0,0) L $\triangle$ (1,1) L $\triangle$ other multiple addresses

$h = 1a = 0b$
2.2 Cycles in the automaton

Prop. a state, $B_a$ corresponding boundary set

Consider directed paths starting in $a$

- Path directly to terminal cycle
  \[\Rightarrow \text{ } B_a \text{ is singleton}\]
- Path to terminal cycle through transient cycle
  \[\Rightarrow \text{ } B_a \text{ is countably infinite}\]
- Path to at least two connected cycles
  \[\Rightarrow \text{ } B_a \text{ is uncountable}\]

This also holds for initial state $\alpha$ and intersection sets $X_i \cap X_j$. 

Ex.: 

\[a \rightarrow \bullet \rightarrow \circ\]

\[a \rightarrow \circ \rightarrow \bullet\]

\[a \rightarrow \circ \leftrightarrow \bullet\]
For $X = \varphi(S)$, let $X_i = \varphi(S_i)$, $i = 0, \ldots, m - 1$. $X$ is called p.c.f. (post-critically finite, used by Thurston 1989 for Julia sets) if $X_i \cap X_j$ consist of finitely many points with eventually periodic addresses.

Prop. P.c.f. spaces are generated by automata.

- An automaton generates p.c.f. space $X$ if no cycles of $G$ are connected by a directed path.

Not allowed: 

![Diagram example]
2.3 Automata with 2 states (plus α)

Prop. Each complete automaton with 2 states (plus α) and connected X describes a one-dimensional numeration system. Three cases for $\mathbb{D} = \{0, 1, \ldots, m-1\}$

a) Numeration with base $m$

\[
G_0 \xleftarrow{\alpha} 0_{0,1,2,\ldots,m-2,m-1} \xrightarrow{\beta} G_{m-1}
\]

b) Numeration with base $-m$

\[
G_0 \xleftarrow{\alpha} 0_{0,1,2,\ldots,m-2,m-1} \xrightarrow{\beta} G_{m-1}
\]

c) Numeration by paperfolding map (tent map for $m=2$)

2 automata for even $|D|$ 2 automata for odd $|D|$
Remark. Already with 3 states and $m > 2$, there is a huge variety of spaces $X$. For 4 states, we get the “fractal squares” (generalizations of Sierpiński carpet).
2.4 Topological properties of $X$

Due to $X = \lim \ X^{(n)}$ all topological properties of $X$ can be expressed in terms of the automaton and the $H_n$.

- $X$ connected $\iff H_n$ connected (Hata 1986, Barnsley 1987)
  $\implies X$ locally and arborescent connected (Hahn-Mazurkiewicz 1930)

- If $X$ disconnected: connected components
  Conj: They are described by an automaton.

- Q: how to describe top. dimension? When is $X$ homeomorphic to a ball?
  See Thürwaldner + Zhang 2020, 2022 for a special case (polyhedral structure)
2.5 Uniform structure - interior metric

Assumption: all $x_n$ with $W_n$ have "almost the same size".

Prop The following entourages form a uniform structure on $X$:

$U_{n,k} = \{(x,y) \mid \text{projections } x_n, y_n \text{ have distance } \leq k \text{ in } H_n\}$

with $n,k \in \mathbb{N}$.

Q. Is there a natural interior metric on $X$? Is there a kind of "uniform dimension"? **Conj.** yes for p.c.f. case.

Rem. Topology-generating automaton can only determine absolute properties of the abstract space $X$, not the relative properties of an $X \subseteq \mathbb{R}^d$ like folding and knots.
3. Metric realization of automata-generated spaces

3.1 Iterated function systems (IFS)

\( f : \mathbb{R}^d \to \mathbb{R}^d \) contractive map if \( |f(x) - f(y)| < r \cdot |x - y| \) for some constant \( r < 1 \).

Prop. If \( f_1, \ldots, f_m \) are contractions on \( \mathbb{R}^d \), there is a unique compact non-empty set \( X = \mathbb{R}^d \) with

\[
X = \bigcup_{i=1}^{m} f_i(X),
\]

and a enumeration system \( \varphi : S \to X \) with

\[
\varphi(s_1, s_2, \ldots) = f_{s_1} \circ \cdots \circ f_{s_m}(X),
\]

\[
\varphi(s_1, s_2, \ldots) = \lim_{n \to \infty} f_{s_1} \circ \cdots \circ f_{s_n}(0). \quad (Hutchinson 1981)
\]
\[ \tau_i : S \rightarrow S \]
\[ \tau_i (s_1 s_2 \ldots) = i s_1 s_2 \ldots \]
conjugate to \( f_i \), \( i = 1 \ldots m \)

Maps \( f_i \) allow calculations in the space \( X \).

**Problem:**
1) Overlap of the \( f_i(X) \) can be large.
2) \( \varphi \) need not be automatic.
3.2 Neighbor maps

Maps that express the relative position of pieces $X_w$.

\[ X \circlearrowleft_{X_w} \xrightarrow{f^{-1}_v} f^{-1}_v(X_v) \xrightarrow{} X = f^{-1}_v(X_v) \]

\[ h = f^{-1}_v f_w \text{ neighbor map} \]

Rem. We restrict ourselves to proper neighbor maps for which $X \cap h(X) \neq \emptyset$.

(Bandt, Graf 1991)
Prop. If there are finitely many neighbor maps \( h_i \), they are the states of a topology-generating automaton with transitions \( \xrightarrow{i,j} h \) if \( h' = f_i \cdot h \cdot f_j \).

("neighbor graph")

( Lau + Rao, Bandt, Thuswaldner + Scheicher, Akiyama, Feng, ...)

Rem. Need assume special form of the \( f_i \), for example

- \( f_i(x) = g^{-1}(x + v_i) \), \( g \) expanding, then \( h_i(x) = x + b \)
  on \( \mathbb{R}^d \), translations

- \( f_i(z) = a_i z + b_i \) with \( |a_i| = r < 1 \) on \( \mathbb{C}_1 \)
  then \( h_i(z) = cz + d \) are isometries \( |c| = 1 \)
Rem. The neighbor graph can be expressed as a system of equations between the boundary sets.

\[ A = f_0(C), \quad B = f_1(C), \quad C = f_0(B) \cup f_1(A) \]

\[ f_1(B) = f_0(A) \]

Ex.

Rem. The neighbor maps formally describe the relation between neighboring pieces.

Ex.

\[ X = [0,1] \]

\[ f_0(x) = \frac{x}{2}, \quad f_1(x) = \frac{x+1}{2} \]

\[ h(x) = \begin{cases} 
-1 & \text{if } x < 0 \\
0 & \text{if } 0 \leq x \leq 1 \\
\end{cases} \]

\[ h^{-1}(x) = x - 1 \]

\[ h(x) = f_0 \circ f_1 = 2 \cdot \frac{x+1}{2} = x+1 \text{ translation} \]
3.3 Number of neighbors

The number $N$ of neighbors or boundary sets is a measure of complexity of $X$.

**Prop** For affine maps $f_i$ on $\mathbb{R}^d$, there is a fast algorithm deciding whether $N \leq N_0$ with $N_0 \leq 2000$, say.

REM The algorithm works without numerical errors if all data are from an algebraic number field.

- $N > 2000$ is practically infinite
- We are interested in $N \leq 20$. 
3.4 Metric realization of top. automata

Well-known: $\text{IFS} \rightarrow \text{automaton} \rightarrow \text{neighbor graph}$

Completely open: $\text{top. automaton} \xrightarrow{??} \text{IFS}$

Q. Which automata can be realized by mappings or affine mappings or similitudes on $\mathbb{R}^d$, $d=1,2,3$?

Q. When is the realization unique?
Conj. If an automaton describes a space $X \subseteq \mathbb{R}^d$ with nonempty interior then there is a unique realization by affine maps $f_i$.

Rem. The automaton defines generating relations for the maps $f_i$ which lead to equations for their coefficients.

Ex. \[ h \circ i^+ j \quad f_i^{-1} h f_j = h \quad h f_d = f_i h \] \[ h \xrightarrow{i_+ f} h' \quad f_i^{-1} h f_j = f_k h f_l \]
3.5 The dog carpet with irrational rotation
one-point neighbors like 2,3 ignored incomplete automaton

Prop. The generating relations determine the IFS. (linear mappings in C)
**Proof.** Let \( g(z) = \lambda z \), \( f_k = g^{-1} h_k \) with 
\[ h_k(z) = a_k z + b_k \quad k = 1, \ldots, 5. \]

**Choice of coordinate system:**
- \( h_3 = z \) (origin)
- \( h_4 = az + 1 \) (unit point)
- \( |a_k| = 1 \)

\[ h = f_3^{-1} f_1 = f_4^{-1} f_3 \quad \Rightarrow \quad h_4 = h_4^{-1} h_1, \quad h_4 = h_1 = \lambda (z-1) \]

\( h \) must be self inverse, \( h = -z + \nu \)

\[ f_3^{-1} h f_3 = h, \quad h f_3 = f_3 h \quad h \text{ must have the same fixed point as } f_3 = g^{-1}, \text{ that is } 0. \]

\[ h_5 = h_5 h_4^{-1} = h \quad \Rightarrow \quad h_5 = h_4 h = -\lambda (z+1) \]

\[ f_1^{-1} h f_2 = h, \quad f_2 = h f_1 h \quad \Rightarrow \quad h_2 = a_2 z - 1 \]

\[ h_1 f_2 = f_1 h_1 = \frac{a}{\lambda} (a^2 z + a + 1) \quad \Rightarrow \quad \lambda = 2a + 1 \]
\[ f_2^{-1} f_4^{-1} h_4 f_5^2 = h_4^{-1} = h_4 = \bar{a}(z-1) \]
\[ h_4 f_5^2 = f_4 f_2 (\bar{a}(z-1)) = \frac{a}{\bar{a}} \left( \frac{z-2}{z} - \frac{\bar{a}}{a} \right) \]
\[ = \frac{a}{\bar{a}} (\bar{a}^{-1} - \frac{1}{\bar{a}}) + 1 \]
\[ \lambda^2 + \lambda (\bar{a}^{-1} - 1) + 3 \bar{a} = 0 \quad -a \]
\[ a \lambda^2 + \lambda (1-a) + 3 = 0 \quad \text{note } a = \frac{\lambda^{-1}}{\bar{a}} \]
\[ \lambda^3 - 2\lambda^2 + 3 \lambda + 6 = 0 \quad \lambda \]
\[ \lambda^2 - 3 \lambda + 6 = 0 \]

Note that \( a \) denotes the rotation between pieces \( X_3 \) and \( X_1 \).

The angle \( \text{arg} \, a \) is irrational!

Not tiling with this property known.
Prop. Suppose that an automaton can be realized by affine mappings \( f_i \), and that each state has incoming edges with first label \( i \) for each digit \( i \). Then the neighbor maps generate a crystallographic group. (and conversely)

Ex. Dog carpet: only the state \( h(x) = -x \) fulfills the condition (permanent neighbor).

(assume \( h_0 = id \))
reflection group

generated by $h_1, h_2, h$ — all reflections

$h_1 h$ 60° rotation

$h_2 h = h_2 h$ 180° rotation

all crystallographic tilings are periodic = group includes a lattice of translations
permanent neighbor $h(2) = -2$ pairs 1 with 2, 4 with 5, 3 with 3

$\begin{array}{c}
\frac{12}{2} \oplus \begin{array}{c} \frac{3}{3} \\ \approx \frac{45}{54} \oplus \frac{\alpha}{\alpha} \end{array}
\end{array}$
Proof. First show that in fractal cases, with arbitrary large holes, not all neighbors can be permanent.

If $X \subseteq \mathbb{R}^d$ has non-empty interior, there is a tiling corresponding to the automaton. All tiles must have the same proper neighbor maps $h_1, \ldots, h_n$. They must include inverses $h_i^{-1}$. All improper neighbors are obtained by repeatedly applying the maps $h_i$. So the proper and improper neighbor maps form the group generated by the $h_i$. 

3.7 An aperiodic tile

**Prop.** Already the relations at vertex $k$ identify the lps (overdetermined system of equations)

permanent nbs: $k, k^{-1}$