Automata generated topological spaces

(and self-affine tiles)

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1. Multiple addresses in numeration
2. Properties of topology-generating automata
3. Iterated function systems and automata

(Few remarks on tiles)

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1.1 Ingredients of numeration

$D = \{0, 1, \ldots, m-1\}$ alphabet, digit set

$S = D^{\mathbb{N}} = \{s = s_1 s_2 \ldots \mid s_k \in D\}$ symbolic space (or subshift)

X topological space

$\varphi : S \to X$ numeration map (onto)

assigns addresses to points $x \in X$

continuous $\to$ X compact

φ is a quotient map

1.2 Multiple addresses

topology of X is determined by the equivalence relation

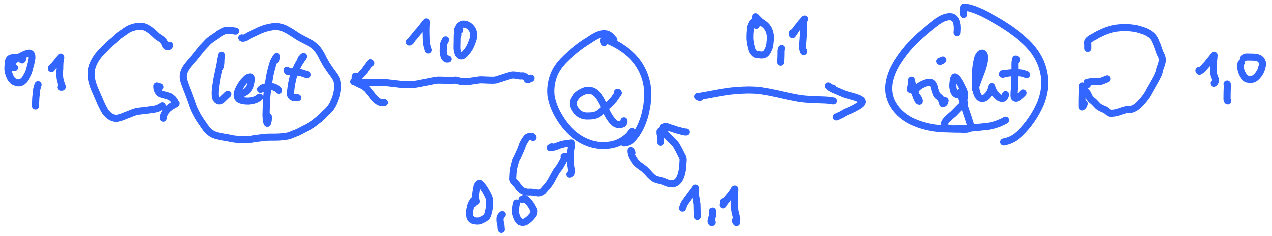
$$L = \{(s,t) \mid \varphi(s)=\varphi(t)\} \subseteq S \times S$$

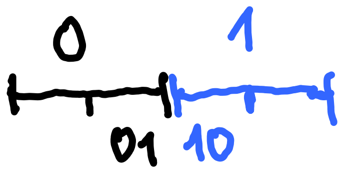
We consider the case that L is generated by an automaton

Ex. Binary numbers $\quad X=[0,1] \quad D=\{0,1\} \quad \bar{1} = 111\ldots$

$$\varphi(s_1 s_2 \ldots) = \sum_{i=1}^{\infty} s_i \cdot 2^{-i}$$

$$\varphi(0\bar{1}) = \varphi(1\bar{0})$$





1.3 Modular structure

$S = \{s_1 s_2 \ldots \mid s_i \in D\}$

word $w = w_1 \ldots w_n \in D^n$

$S_w = \{w_1 \ldots w_n s_{n+1} s_{n+2} \ldots \mid s_i \in D\}$ cylinder set

Principle of self-similarity: All S_w are treated in the same way as S

(or several types)

$$\varphi(s) = \varphi(t) \rightarrow \varphi(ws) = \varphi(wt)$$

for all words w

(simplifies our work)

1.4 Topology-generating automaton

Def. $G = (\underbrace{V}_{\text{states}}, \underbrace{E}_{\text{transitions}})$ directed graph, $\alpha \in V$ initial state

Input alphabet $D \times D$ for marking edges

Automaton accepts those sequences $(i_1, j_1), (i_2, j_2), \dots$ for which there is a path $e_1 e_2 \dots$ starting in α and edge e_k is marked (i_k, j_k).

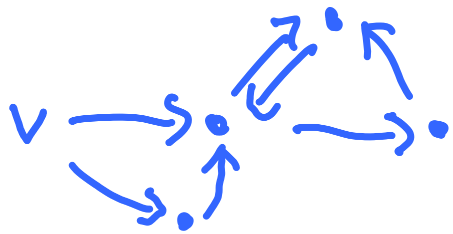
Rem. – all states accepting. Could add rejecting state ω.

– α must have loops (i,i) for all $i \in D$ (self-sim.)

– α should have no other incoming edges
(in order to get finite equiv. classes)

– from each state v there must be a path to a directed cycle
(in order to accept infinite sequences (i_k, j_k))

or



etc.

1.5 Topology generated by the automaton

$v, w \in D^n$ $\quad (v, w)$ accepted means $\varphi(S_v)$ intersects $\varphi(S_w)$

On finite level n, topology is approximated by an undirected graph $H_n = (D^n, E_n = \text{accepted } \{v, w\})$

(Hata 1986)

$s, t \in S$ $\quad (s, t)$ accepted means $\varphi(s) = \varphi(t)$

So the automaton defines the equivalence relation ~ and hence the topology of X.

$$H_n = \left(D^n,\ E_n = \text{accepted } \{v,w\}\right)$$

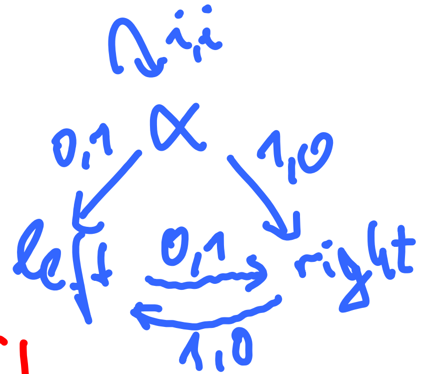
Prop. Let $X^{(n)} = D^n \cup E_n$, where points of D^n are open and points of E_n are closed. Then X is the inverse limit of the finite topological spaces $X^{(n)}$.

$$X = \varprojlim \left(X^{(n)}, \pi_n\right)$$

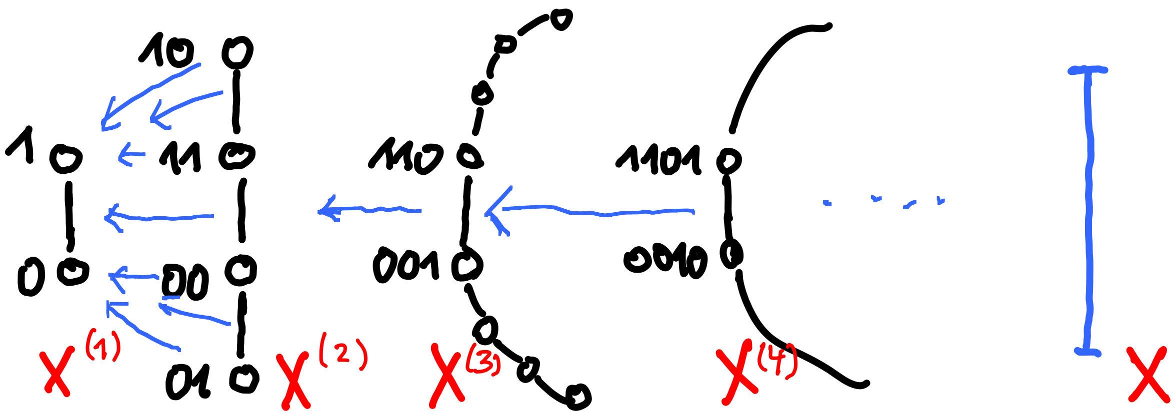
with $\pi_n : X^{(n)} \to X^{(n-1)}$, $\pi_n(w_1 \dots w_n) = w_1 \dots w_{n-1}$,

$\pi_n(v_1 \dots v_n, w_1 \dots w_n) = (v_1 \dots v_{n-1}, w_1 \dots w_{n-1})$

Ex.



$\varphi(1\overline{10}) = \varphi(0\overline{01})$



Rem.

1) Example was number system with base -2.

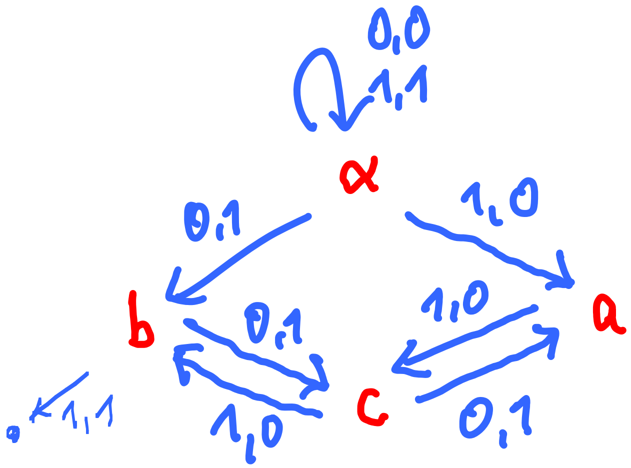
$$\varphi(s_1 s_2 \dots) = \sum_{i=1}^{\infty} s_i \cdot 2^{-i}$$

2) complete automaton – defines equivalence relation

incomplete automaton – defines relation which can be extended to equivalence relation

(easier to draw)

Ex.



$01 \sim 00$ $\quad \alpha \to \alpha \to a$

$00 \sim 11$ $\quad \alpha \to b \to c$

but $01 \not\sim 11$

automaton not complete

1.6 Motivation: Description of complex geometries by computer

3D geometry is at the centre of current science: cell biology, brain research, nano materials

great variety of complex geometric structures, even in everyday context: dust, soil, smoke, fire, ...

More network-like than manifold-like

Mathematics must find ways to model, describe, analyze such geometric phenomena by computer.

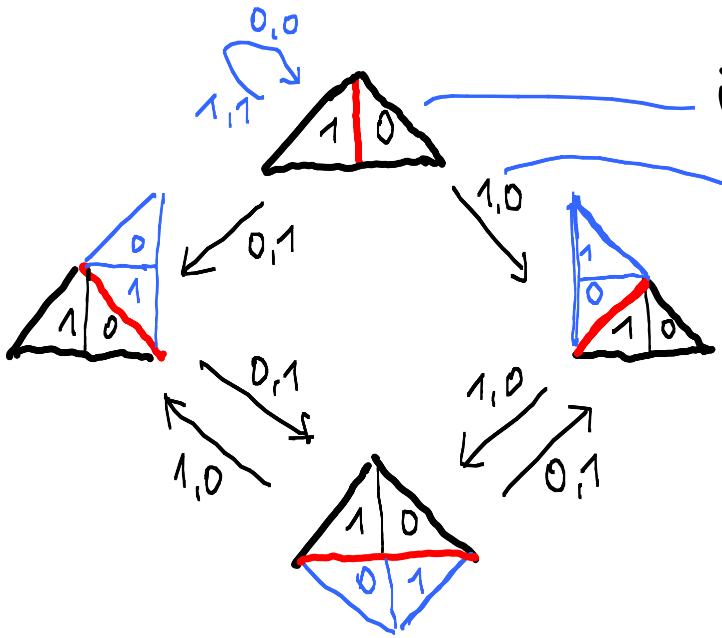
Numeration is one option.

2. Properties of topology-generating automata

2.1 Interpretation of states:

relative positions of intersecting pieces

Ex.



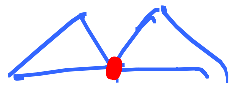
initial state α is standard position of X

first label denotes reference piece which is turned into standard position

second label for neighboring piece

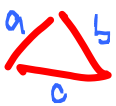
(no reflections involved

one-point intersections ignored like



incomplete automaton)

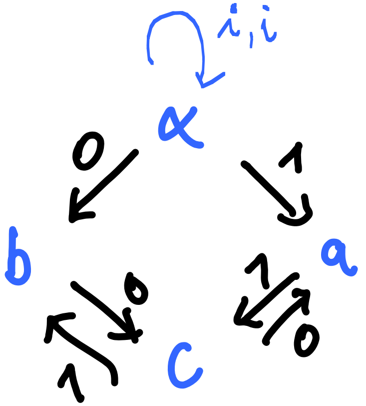
States can also be considered as



boundary edges of X

transitions go from pairs of pieces to pairs of subpieces

Rem. The automaton and its accepted language code the geometry of X.



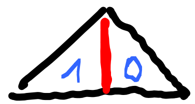
only first label, since second label is found at the "inverse" state $b = a^{-1}$, $c = c^{-1}$

(simplified)

$L_a = \{ (1(10)^* 0)^\infty \}$ addresses of boundary set

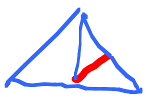
$L_b = \{ (0(01)^* 1)^\infty \}$ addresses of

$L = \{ (0 L_b , 1 L_a) \}$ multiple addresses of

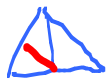


$h = 1a = 0b$

$(0,0)L$



$(1,1)L$



other multiple addresses

2.2 Cycles in the automaton

<u>Prop.</u> a state, B_a corresponding boundary set

Consider directed paths starting in a

- Path directly to terminal cycle
  $\iff$ B_a is singleton
- Path to terminal cycle through transient cycle
  $\iff$ B_a is countably infinite
- Path to at least two connected cycles
  $\iff$ B_a is uncountable

Ex:

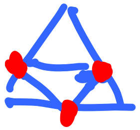
$$a \to \bullet \to \bullet \circlearrowleft$$

$$a \to \underset{\circlearrowleft}{\bullet} \to \bullet \rightleftarrows \bullet$$

$$a \to \underset{\circlearrowleft}{\bullet} \rightleftarrows \bullet$$

This also holds for initial state α and intersection sets $X_i \cap X_j$.

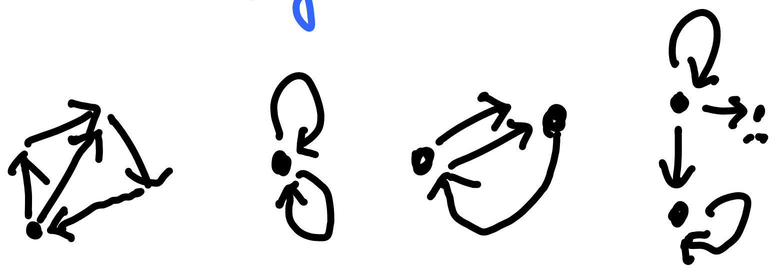
For $X = \varphi(S)$, let $X_i = \varphi(S_i)$, $i = 0, \ldots, m-1$



X is called p.c.f. (post-critically finite, used by Thurston 1989 for Julia sets)
if $X_i \cap X_j$ consist of finitely many points with eventually periodic addresses

Prop. • P.c.f. spaces are generated by automata.

• An automaton generates p.c.f. space $X \iff$ no cycles of G are connected by a directed path.

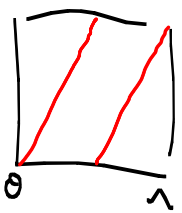
not allowed: 

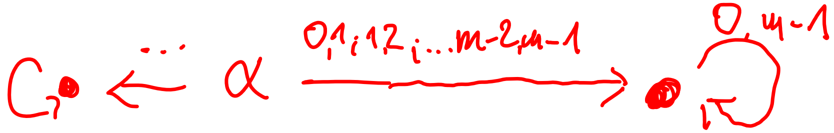
2.3 Automata with 2 states (plus α)

Prop. Each complete automaton with 2 states (plus α) and connected X describes a one-dimensional numeration system. Three cases for $D = \{0, 1, \ldots m-1\}$

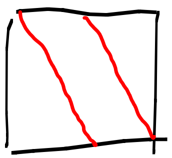
shift map

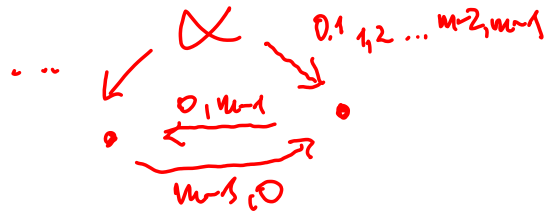
a) Numeration with base m



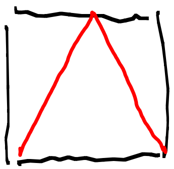


b) Numeration with base $-m$



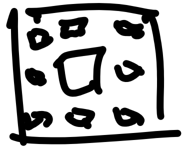


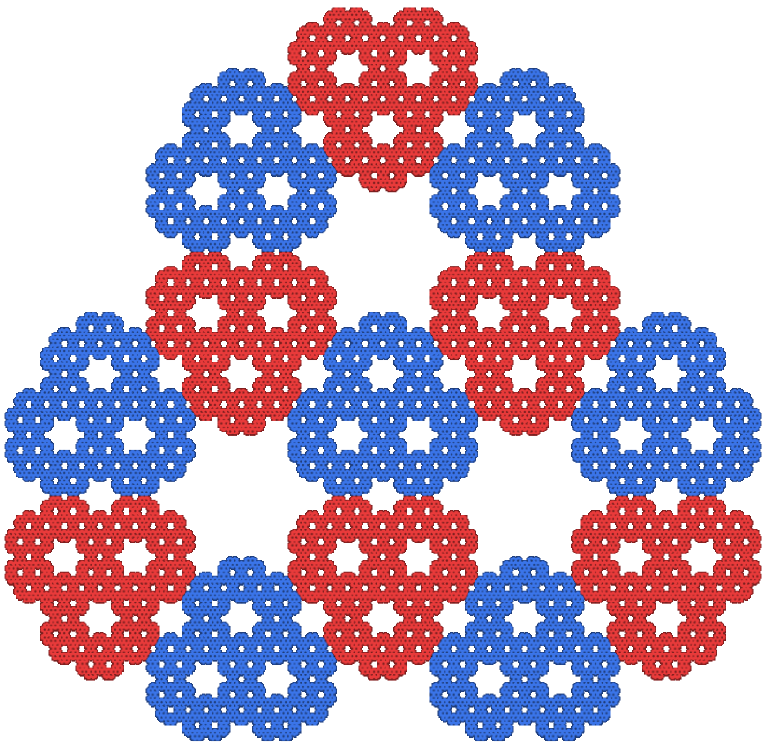
c) Numeration by paperfolding map
(tent map for $m=2$)

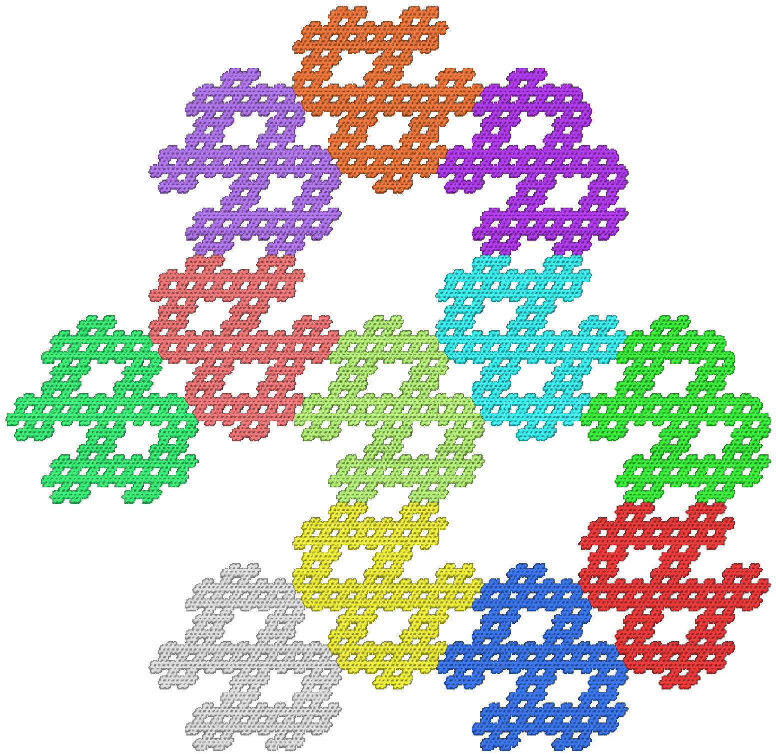


2 automata for even $|D|$
2 automata for odd $|D|$

Rem. • Already with 3 states and $m > 2$, there is a huge variety of spaces X. For 4 states, we get the "fractal squares" (generalizations of Sierpiński carpet)







2.4 Topological properties of X

Due to $X = \varprojlim X^{(n)}$ all topological properties of X can be expressed in terms of the automaton and the H_n.

- X connected $\Leftrightarrow$ H_1 connected (Hata 1986, Barnsley 1987)
  $\Downarrow$ X locally and arcwise connected (Hahn-Mazurkiewicz 1930)

- If X disconnected: connected components
  Conj: They are described by an automaton.

- Q: how to describe top. dimension? When is X homeomorphic to a ball?
  see Thuswaldner + Zhang 2020, 2022 for a special case (polyhedral structure)

2.5 Uniform structure - interior metric

Assumption: all X_w mit $|w|=n$ have "almost the same size".

Prop The following entourages form a uniform structure on X.

$U_{n,k} = \{(x,y) \mid$ projections x_n, y_n have distance $\le k$ in $H_n\}$

with $n,k \in \mathbb{N}$.

Q. Is there a natural interior metric on X? Is there a kind of "uniform dimension"? Conj. yes for p.c.f. case.

Rem. Topology-generating automaton can only determine absolute properties of the abstract space X, not the relative properties of an $X \subseteq \mathbb{R}^d$ like folding and knots.

3. Metric realization of automata-generated spaces

3.1 Iterated function systems (IFS)

$f : \mathbb{R}^d \to \mathbb{R}^d$ contractive map if $|f(x) - f(y)| < r \cdot |x-y|$ for some constant $r < 1$.

Prop. If $f_1, \dots, f_m$ are contractions on $\mathbb{R}^d$, there is a unique compact nonempty set $X \subseteq \mathbb{R}^d$ with

$$X = \bigcup_{i=1}^{m} f_i(X), \qquad (*)$$

and a numeration system $\varphi : S \twoheadrightarrow X$ with

$$\varphi(S_{w_1 \dots w_n}) = f_{w_1} \cdots f_{w_n}(X),$$

$$\varphi(s_1 s_2 \dots) = \lim_{n \to \infty} f_{s_1} \cdots f_{s_n}(0).$$

(Hutchinson 1981)

$\tau_i : S \to S$

$\tau_i(s_1 s_2 \ldots) = i s_1 s_2 \ldots$

conjugate to f_i, $i = 1..m$

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "\tau_i"'] & X \arrow[d, "f_i"] \\ S \arrow[r, "\varphi"'] & X \end{tikzcd}$$

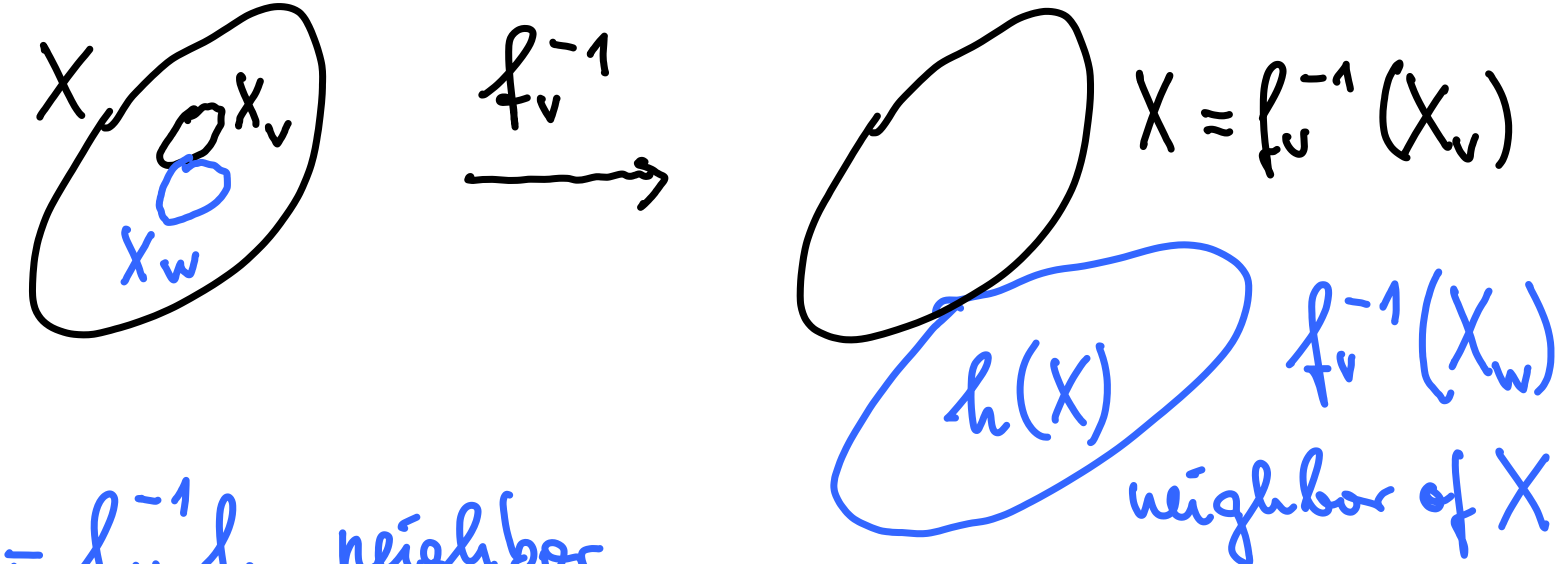
Maps f_i allow calculations in the space X.

Problem: 1) Overlap of the $f_i(X)$ can be large.

2) φ need not be automatic.

3.2 Neighbor maps

Maps that express the relative position of pieces X_w.



$h = f_v^{-1} f_w$ neighbor map

(Bandt, Graf 1991)

Rem. We restrict ourselves to proper neighbor maps for which $X \cap h(X) \neq \emptyset$.

Prop. If there are finitely many neighbor maps h, they are the states of a topology-generating automaton with transitions $h \xrightarrow{i,j} h'$ if $h' = f_i^{-1} h f_j$

"neighbor graph"

(Lau + Rao, Bandt, Thuswaldner + Scheicher, Akiyama, Feng, ...)

Rem. Need assume special form of the f_i, for example

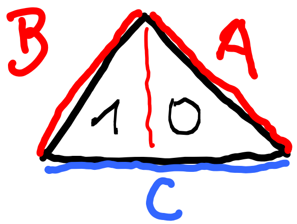
- $f_i(x) = g^{-1}(x+v_i)$, g expanding on $\mathbb{R}^d$, then $h(x) = x + b$ translations

- $f_i(z) = a_i z + b_i$ with $|a_i| = r < 1$ on $\mathbb{C}$, then $h(z) = cz + d$ are isometries $|c| = 1$

Rem. The neighbor graph can be expressed as a system of equations between the boundary sets.

(Gilbert, Duvall, Keesling, Vince, Strichartz + Wang, Ngai, Lau et al, ...)

Ex.

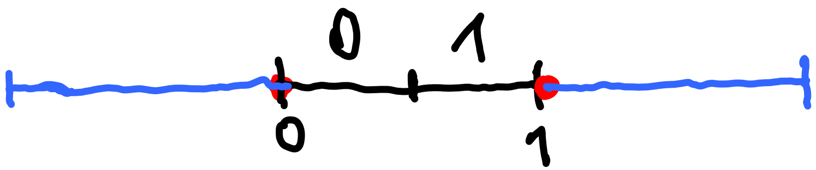


$A = f_0(C)$, $B = f_1(C)$, $C = f_0(B) \cup f_1(A)$

$f_1(B) = f_0(A)$

Rem. The neighbor maps formally describe the relation between neighboring pieces.

Ex.



$X = [0,1]$

$f_0(x) = \frac{x}{2}$ $\quad f_1(x) = \frac{x+1}{2}$

$$x-1 \xleftarrow{1,0} \boxed{\mathrm{id}} \xrightarrow{0,1} h = x+1$$

$h(x) = f_0^{-1} f_1 = 2 \cdot \frac{x+1}{2} = x+1$ translation

$h^{-1}(x) = x-1$

3.3 Number of neighbors

The number N of neighbors or boundary sets is a measure of complexity of X.

Prop For affine maps f_i on $\mathbb{R}^d$, there is a fast algorithm deciding whether $N \leq N_0$ with $N_0 \leq 2000$, say.

! ⟶ Mekhoutsev 2012-2022 ifstile.com

Rem
- The algorithm works without numerical errors if all data are from an algebraic number field.
- $N > 2000$ is practically infinite

We are interested in $N \leq 20$.

3.4 Metric realization of top. automata

Well-known: IFS $\rightsquigarrow$ automaton (neighbor graph)

Completely open:

top. automaton $\overset{??}{\rightsquigarrow}$ IFS

Q. • Which automata can be realized by mappings or affine mappings or similitudes on $\mathbb{R}^d$, $d = 1, 2, 3, \ldots$

• When is the realization unique?

Conj. If an automaton describes a space $X \subseteq \mathbb{R}^d$ with nonempty interior then there is a unique realization by affine maps f_i.

(X tile)

Rem. The automaton defines generating relations for the maps f_i which lead to equations for their coefficients.

Ex.

$$\begin{tikzcd} h \arrow[loop right, "{i,j}"] \end{tikzcd}$$

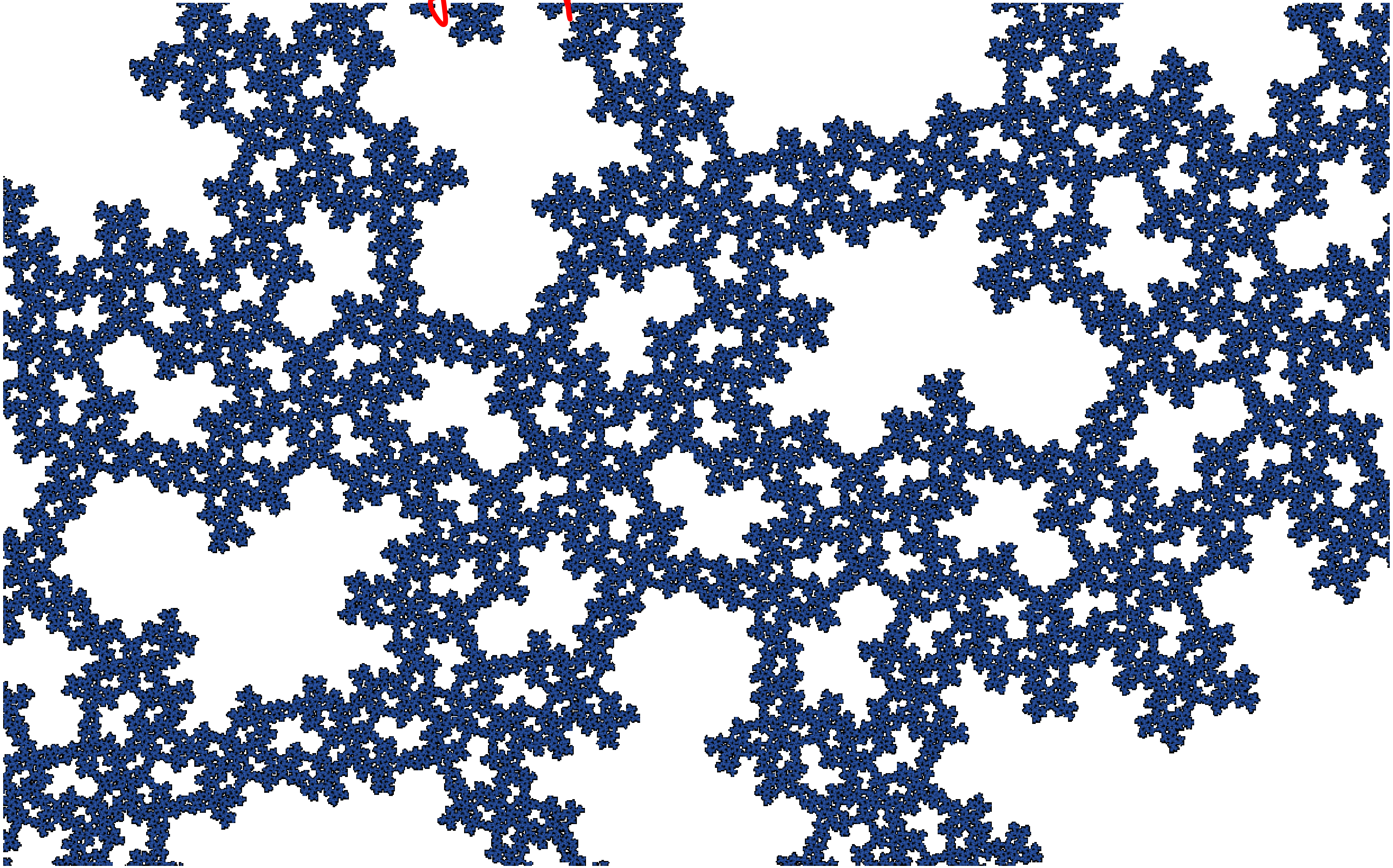
$$f_i^{-1} h f_j = h$$

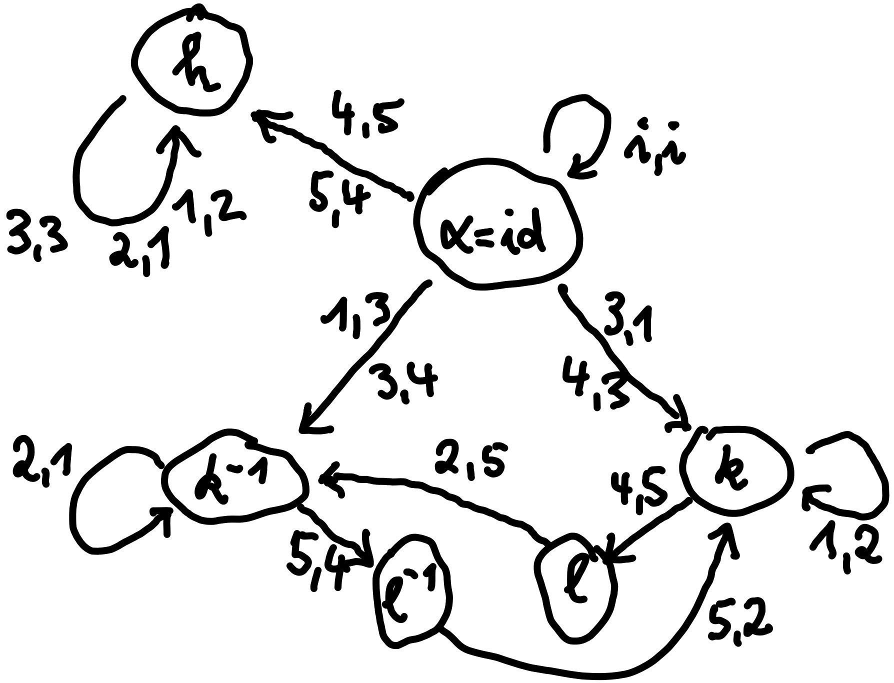
$$h f_j = f_i h$$

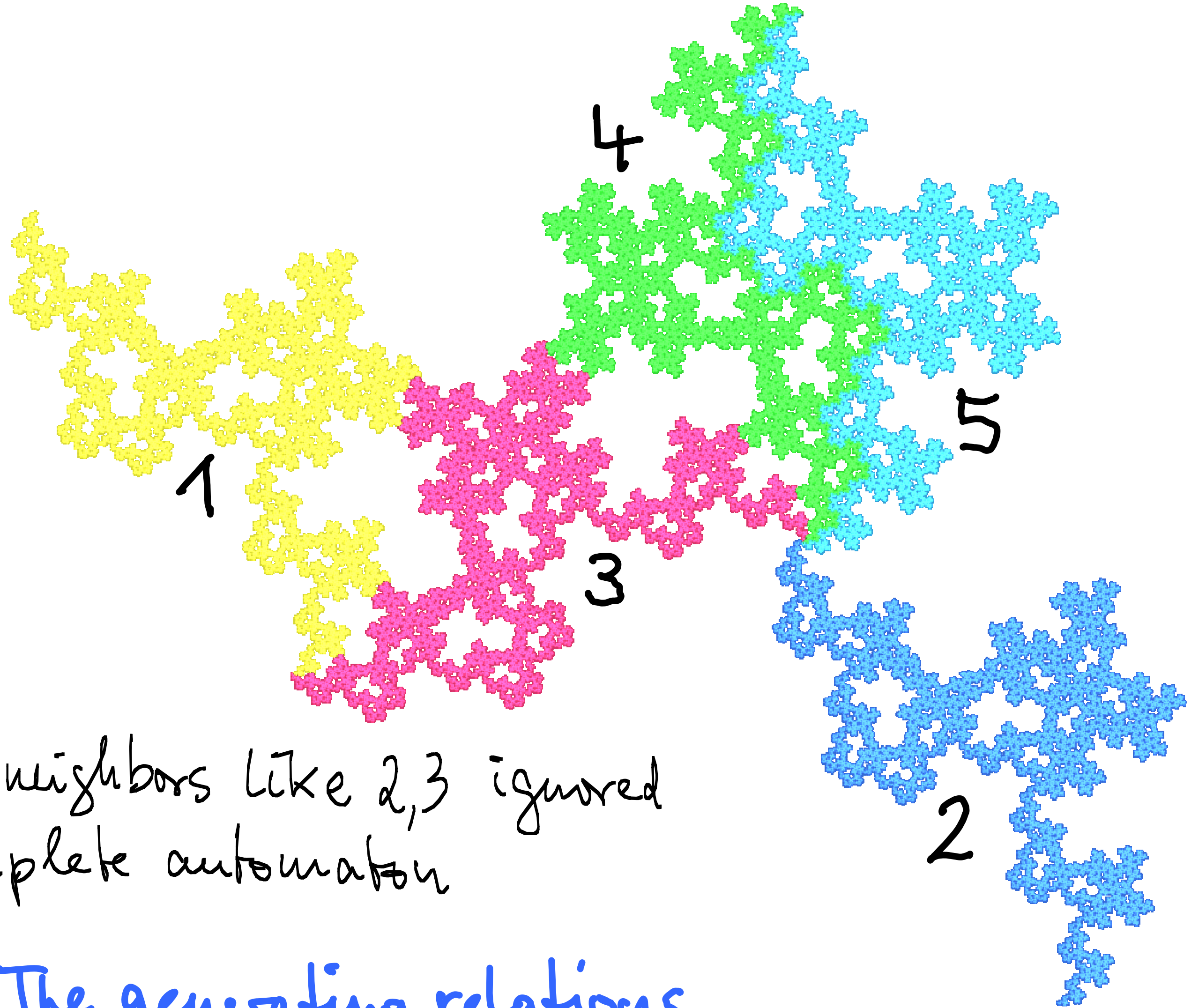
$$\begin{tikzcd} h \arrow[r, shift left, "{i,j}"] \arrow[r, shift right, "{k,l}"'] & h' \end{tikzcd}$$

$$f_i^{-1} h f_j = f_k^{-1} h f_l$$

3.5 The dog carpet with irrational rotation







$f_4^{-1} f_5 = h$

$f_3^{-1} h f_3 = h$

$f_2^{-1} h f_1 = h$

$f_3^{-1} f_1 = f_4^{-1} f_3 = k$

$f_1^{-1} k f_2 = k$

$f_2^{-1} f_4^{-1} k f_5 f_5 = k^{-1}$

one-point neighbors like 2,3 ignored

incomplete automaton

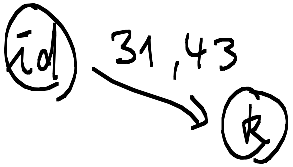
Prop. The generating relations determine the IFS.

(linear mappings in $\mathbb{C}$)

Proof. Let $g(z) = \lambda z$, $f_k = g^{-1} h_k$ with $h_k(z) = a_k z + b_k$ $k = 1, \dots, 5.$ $|a_k| = 1$

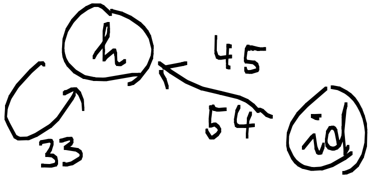
Choice of coordinate system: $h_3 = z$ (origin)

$h_1 = az + 1$ (unit point)



$k = f_3^{-1} f_1 = f_4^{-1} f_3 \Rightarrow h_1 = h_4^{-1}$, $h_4 = h_1^{-1} = \bar{a}(z-1)$

$k = h_1$



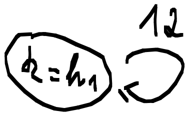
h must be self inverse, $h = -z + v$

$f_3^{-1} h f_3 = h$, $h f_3 = f_3 h$: h must have the same fixed point as $f_3 = g^{-1}$, that is 0.

$h = -z$

$h_4^{-1} h_5 = h \Rightarrow h_5 = h_4 h = -\bar{a}(z+1)$

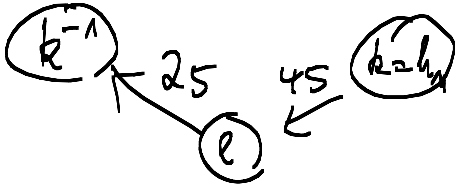
$f_1^{-1} h f_2 = h$ $\quad f_2 = h f_1 h \Rightarrow h_2 = az - 1$



$$h_1 f_2 = f_1 h_1 = \frac{1}{\lambda}(a^2 z + a + 1) \quad \Rightarrow \lambda = 2a + 1$$

$$\| $$

$$\frac{a}{\lambda}(az - 1) + 1$$



$$f_2^{-1} f_4^{-1} h_1 f_5^2 = h_1^{-1} = h_4 = \bar{a}(z-1)$$

$$h_1 f_5^2 = f_4 f_2\left(\bar{a}(z-1)\right) = \frac{\bar{a}}{\lambda^2}(z-2) - \frac{\bar{a}}{\lambda}$$

$$= \frac{\bar{a}}{\lambda^2}(z+1) - \frac{1}{\lambda} + 1$$

$$\lambda^2 + \lambda(\bar{a}-1) + 3\bar{a} = 0 \qquad \cdot a$$

$$a\lambda^2 + \lambda(1-a) + 3 = 0, \text{ note } a = \frac{\lambda-1}{2}$$

$$\lambda^3 - 2\lambda^2 + 3\lambda + 6 = 0, \quad /(\lambda+1)$$

$$\lambda^2 - 3\lambda + 6 = 0$$

$$\lambda = \frac{3+i\sqrt{15}}{2}, \quad a = \frac{1+i\sqrt{15}}{4}$$

Note that a denotes the rotation between pieces X_3 and X_1.

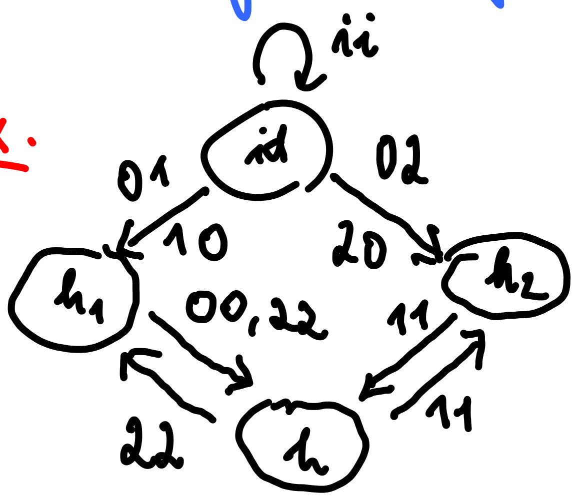
The angle $\arg a$ is irrational!

No tiling with this property known.

3.6 Crystallographic tilings

Prop. Suppose that an automaton can be realized by affine mappings f_x, and that each state has incoming edges with first label i for each digit i. Then the neighbor maps generate a crystallographic group. (and conversely)

Ex.



Dog carpet: only the state $h(x) = -x$ fulfills the condition (permanent neighbor).

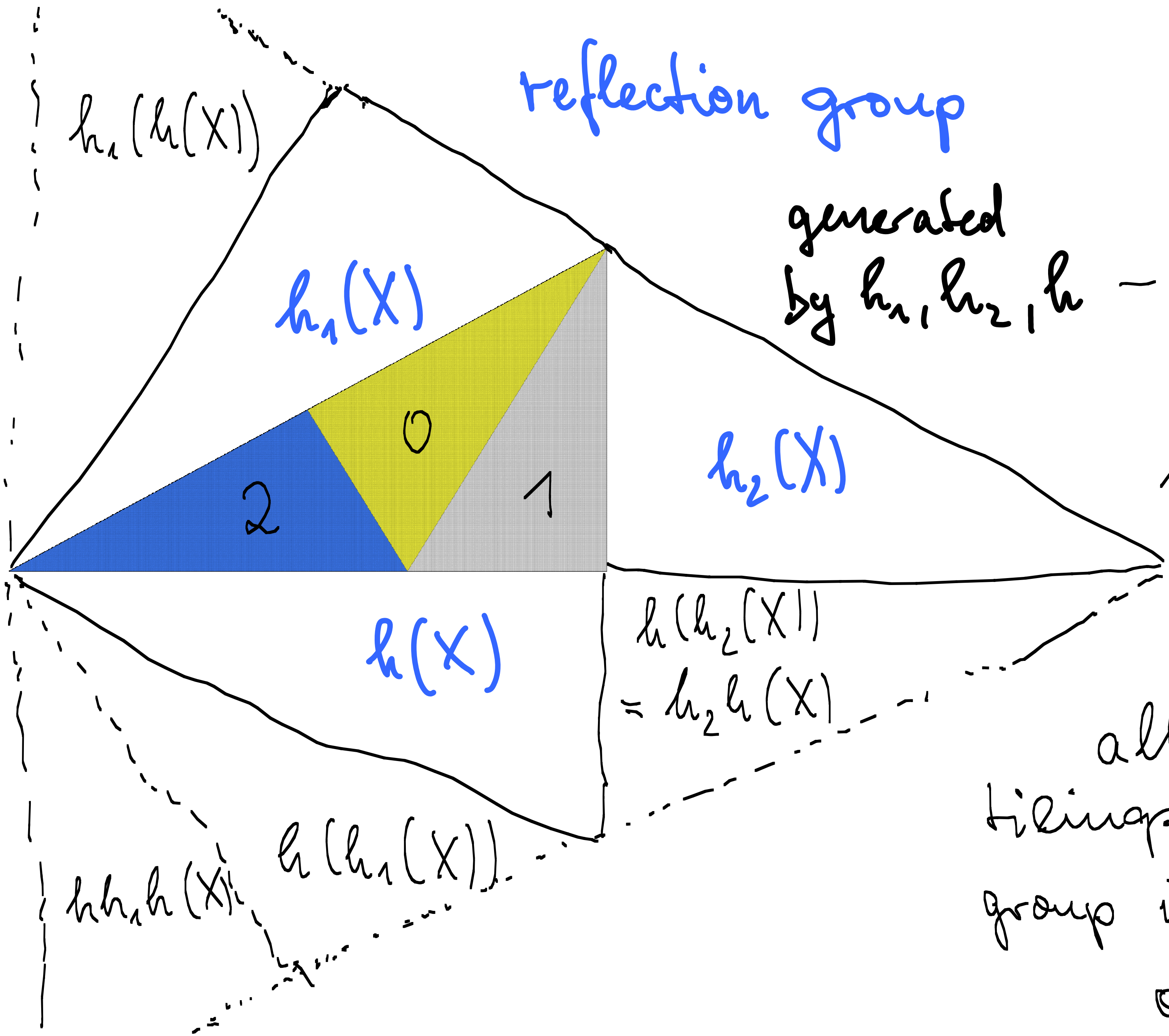
(assume $h_0 = id$)

reflection group

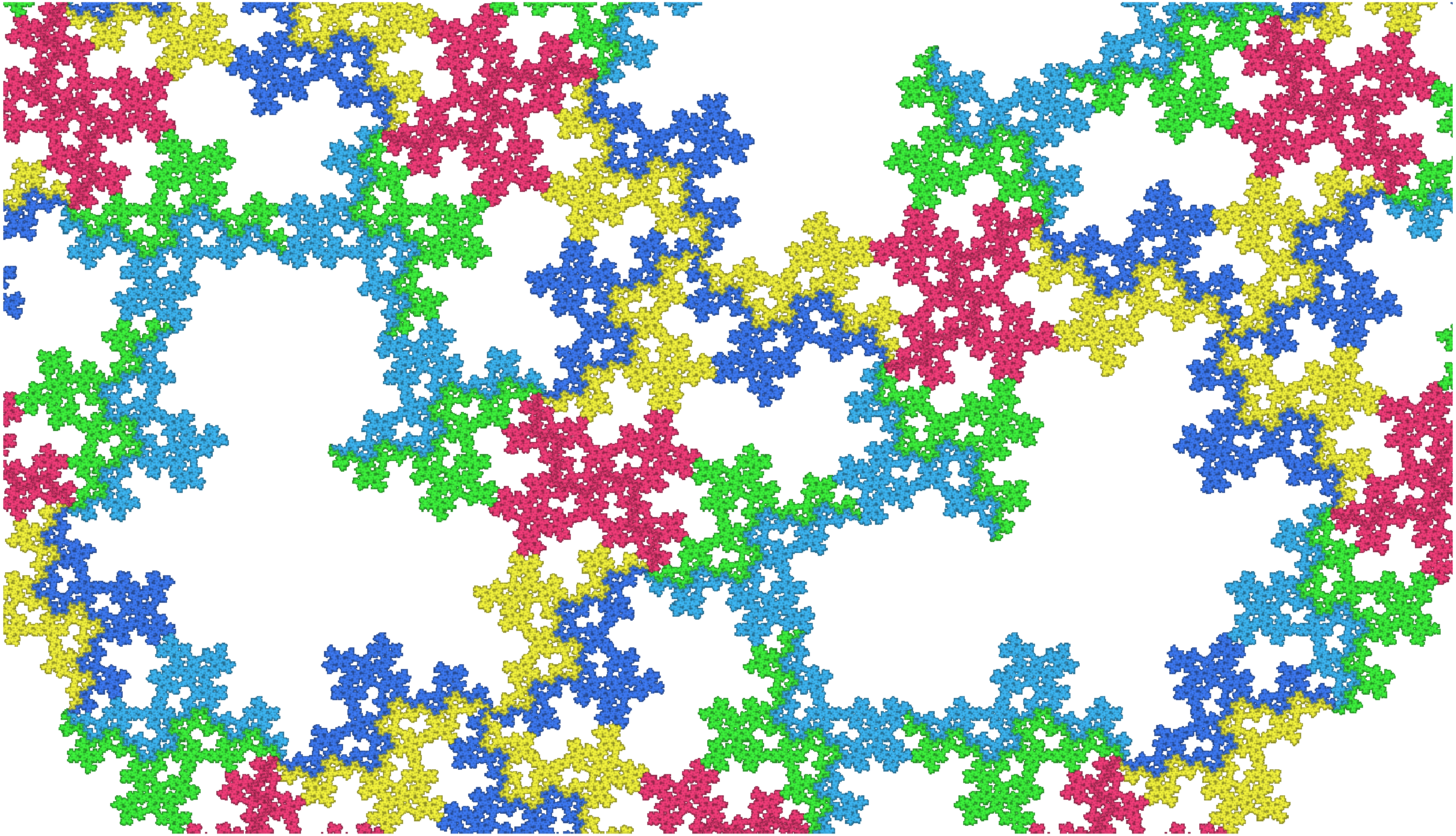
generated by h_1, h_2, h — all reflections

$h_1 \circ h$ 60° rotation

$h h_2 = h_2 h$ 180° rotation



all crystallographic tilings are periodic = group includes a lattice of translations



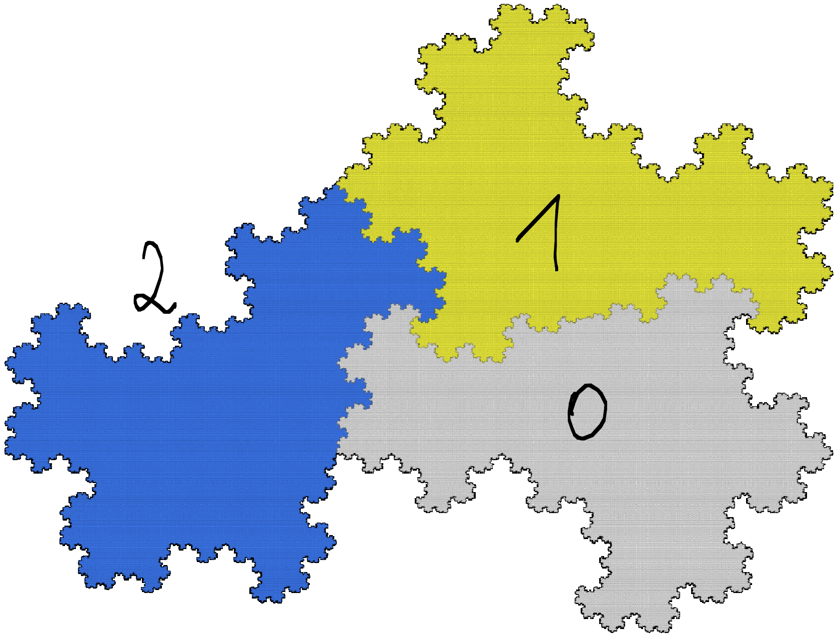
permanent neighbor $h(z) = -z$ pairs 1 with 2, 4 with 5, 3 with 3

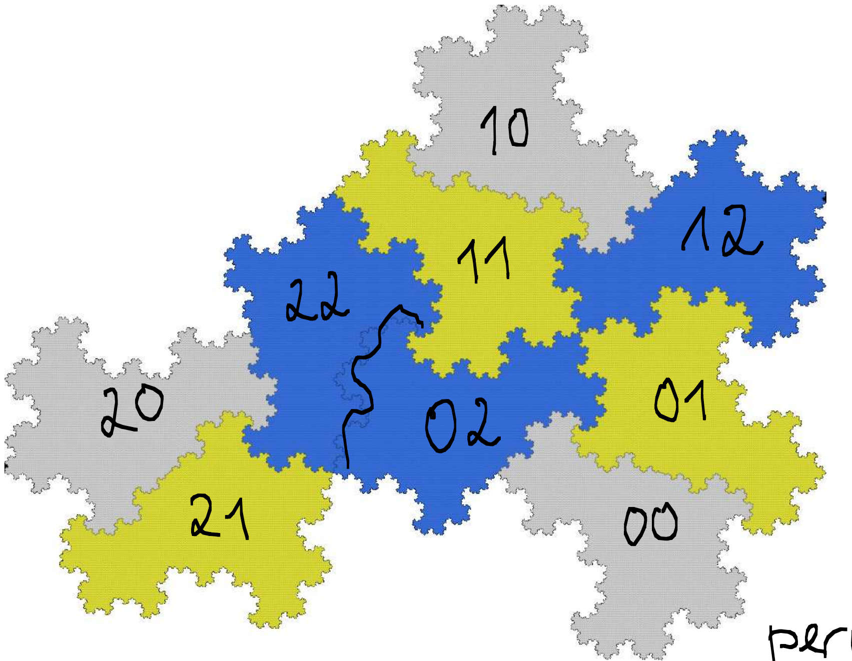
$$\begin{tikzcd} h \arrow[loop left, "\substack{12\\21\\33}"] & \alpha \arrow[l, "45"', "54"] \end{tikzcd}$$

Proof. First show that in fractal cases, with arbitrary large holes, not all neighbors can be permanent.

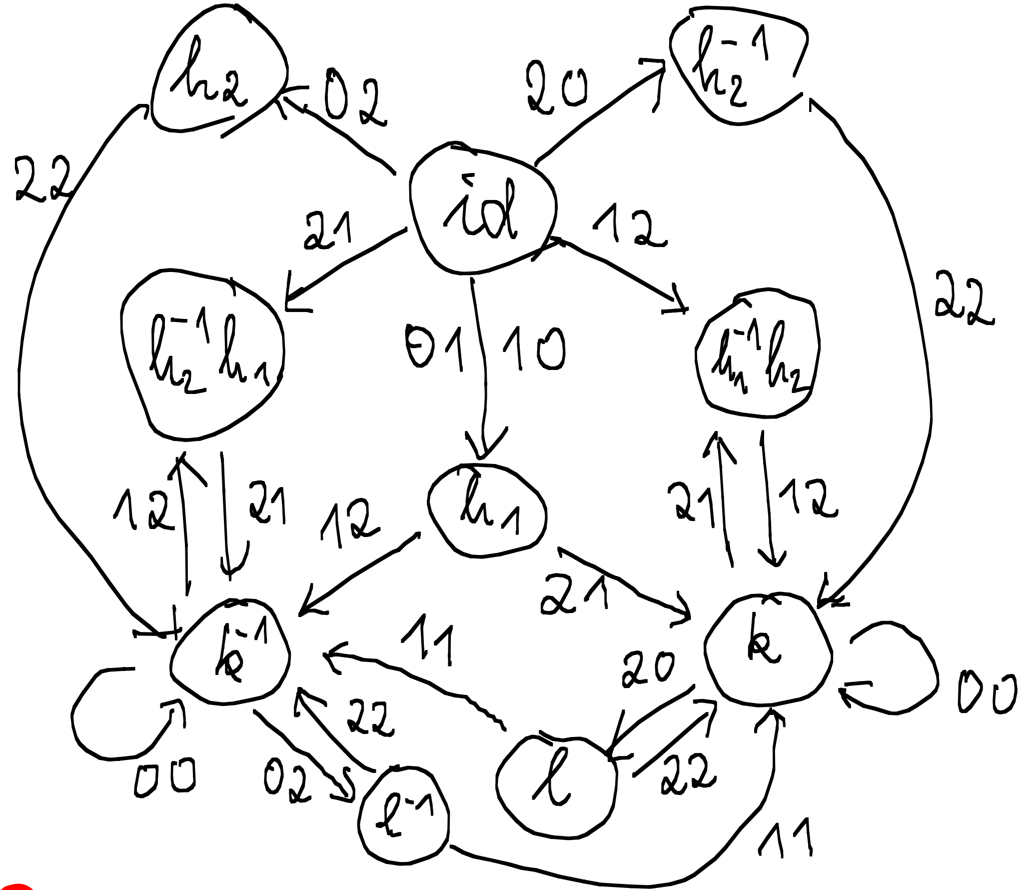
If $X \subseteq \mathbb{R}^d$ has non-empty interior, there is a tiling corresponding to the automaton. All tiles must have the same proper neighbor maps $h_1, \ldots, h_n$. They must include inverses h_i^{-1}. All improper neighbors are obtained by repeatedly applying the maps h_i. So the proper + improper neighbor maps form the group generated by the h_i.

3.7 An aperiodic tile





permanent nbs: k, k^{-1}



Prop. Already the relations at vertex k identify the IFS (overdetermined system of equations)