Rigid Fractal Tilings

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World Numeration Seminar

1 December 2020

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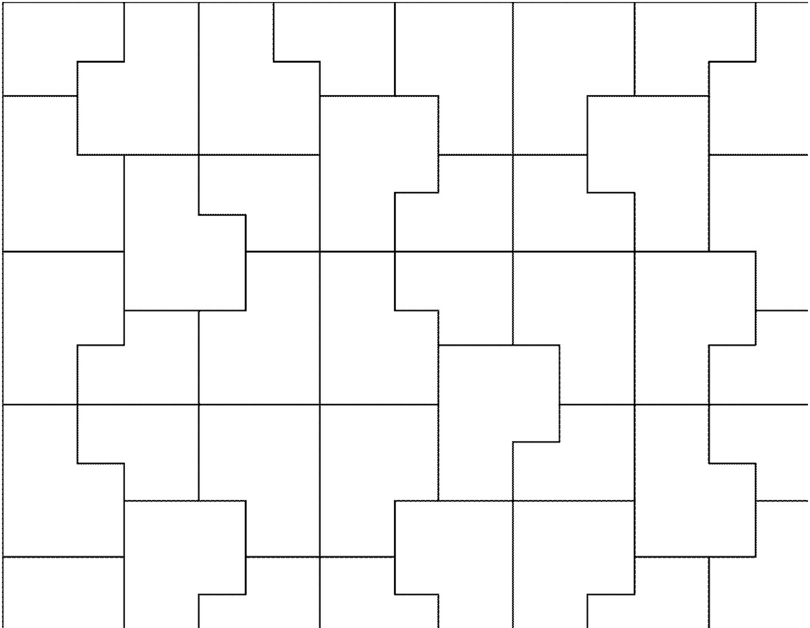
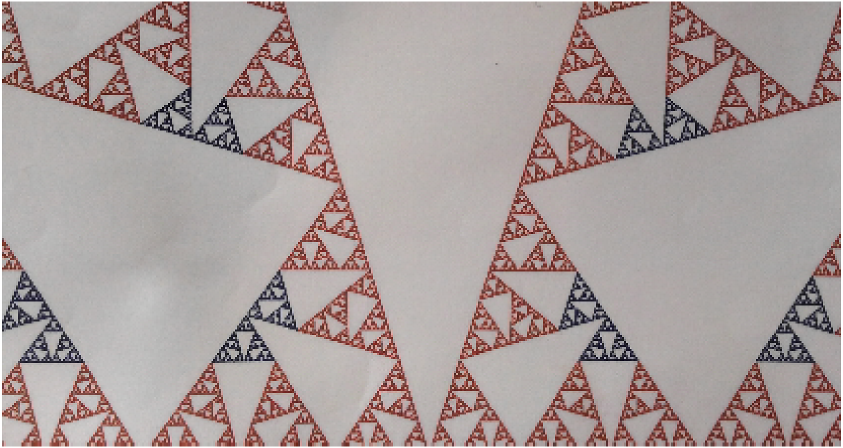
- Thanks to Christoph Bandt for advice and help.





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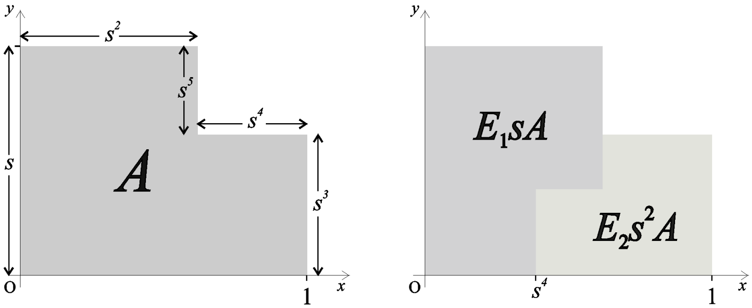
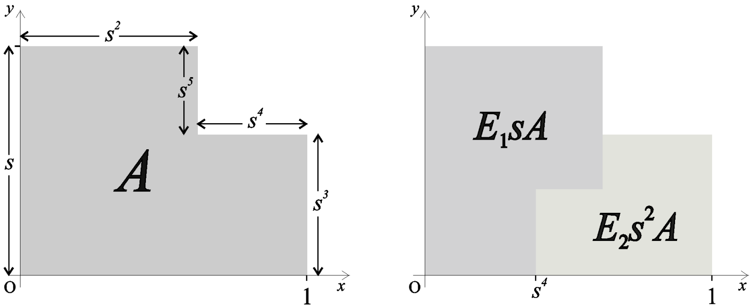
- Tiling IFS, Main Theorem



- $A = f_1 A \cup f_2 A$ where $f_1 = E_1 s, f_2 = E_2 s^2$ and $s^2 + s^4 = 1$



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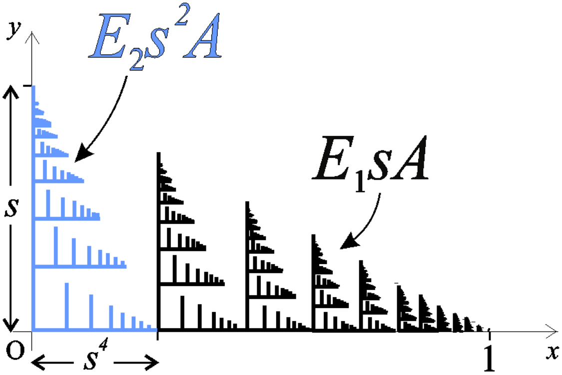

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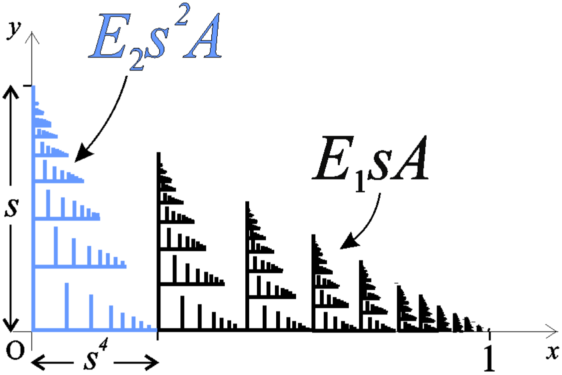

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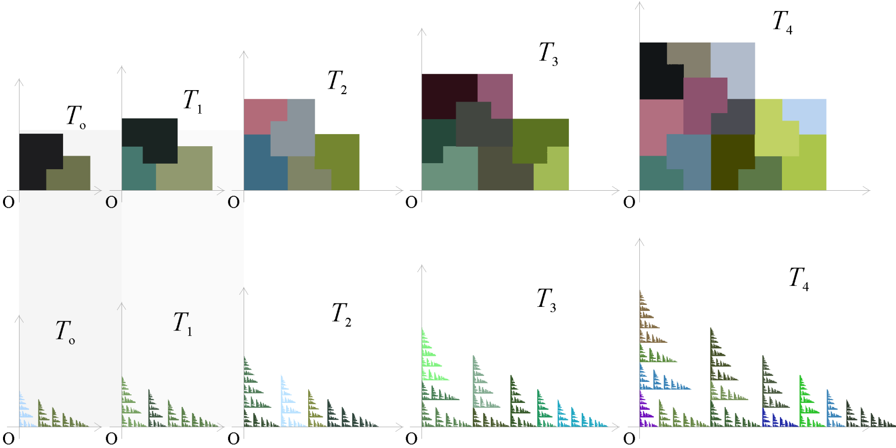
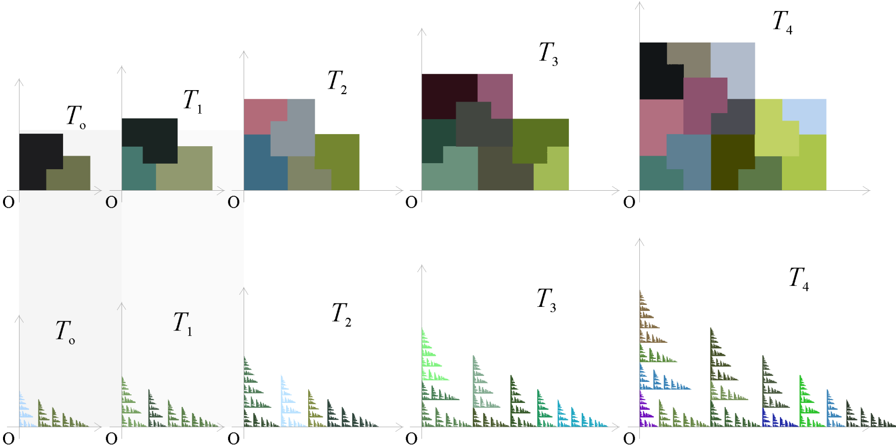


- Repeatedly: apply s^{-1} and “split” all copies of A.



- Repeatedly: apply s^{-1} and “split” all copies of A.
- Build sequence of **canonical** tilings $\{T_n\}$

- Use $\{T_n\}$ to find sequences of isometries $\{E_{n_k}\}$ such that $E_{n_k} T_{n_k} \subset E_{n_{k+1}} T_{n_k+1}$

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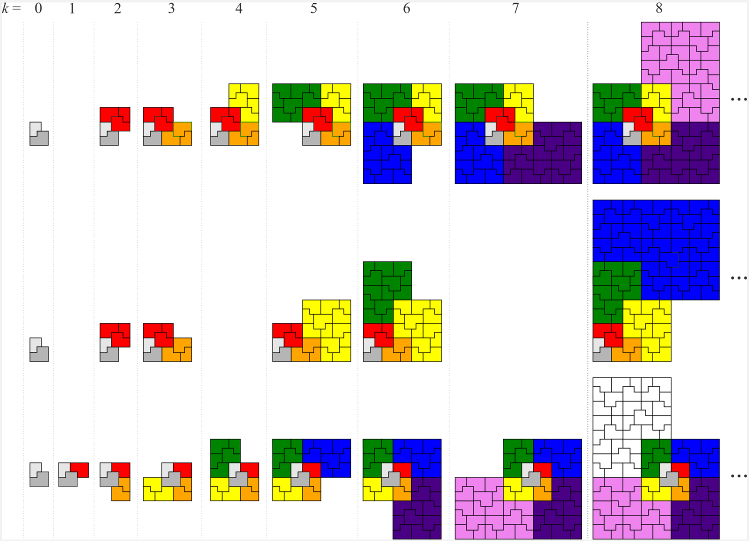
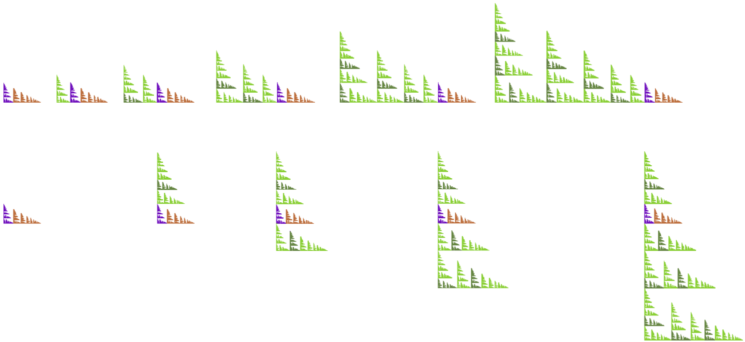
- Define unbounded tilings $T(\{E_{n_k}\}) = \bigcup E_{n_k} T_{n_k}$


$k =$ 0 1 2 3 4 5 6 7 8
...
...
...



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- DEFINITION: $\mathcal{F}$ is **rigid** w.r.t. a set of transformations $\mathcal{U}$ if Es^kT_0 meets T_0 for $E \in \mathcal{U}, k \in \mathbb{Z}$ implies $E = I$ and $k = 0$.

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- Demonstrations:

We redefine the tilings $T(\{E_{k_n}\})$ using the language of IFS. Each example is associated with an IFS $\{f_1,f_2\}$ such that there is a fixed $0 < s < 1$ so that, for all $k \in \mathbb{N} =\{1,2,...\}$, for all $x \in \mathbb{R}^2$, for all $(\theta_1, \theta_2, ...\theta_k) \in \{1,2\}^k$

$$f_{\theta_1}^{-1} \circ f_{\theta_2}^{-1} \circ ... \circ f_{\theta_k}^{-1} x = s^{-\theta_1+\theta_2+...+\theta_k} Ux + t$$

where U is unitary and t is a translation.

Define partial tilings in terms of canonical tilings by

$$\Pi(\theta_1\theta_2...\theta_k) = f_{\theta_1}^{-1} \circ f_{\theta_2}^{-1} \circ ... \circ f_{\theta_k}^{-1} s^{\theta_1+\theta_2+...+\theta_k} T_{\theta_1+\theta_2+...+\theta_k}$$

It is a remarkable and beautiful fact that

$$\Pi(\theta_1) \subset \Pi(\theta_1\theta_2) \subset .... \subset \Pi(\theta_1\theta_2...\theta_k)$$

so that for all $\theta_1\theta_2\theta_3...$

$$\Pi(\theta_1\theta_2\theta_3...) := \bigcup \Pi(\theta_1\theta_2...\theta_k)$$

is a well-defined unbounded tiling of (possibly a subset of) $\mathbb{R}^2$.

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- Our question "When does $T(\{E_{k_n}\}) = ET(\{E'_{k'_n}\})$?" becomes: "When does $\Pi(\theta_1\theta_2\theta_3...) = E\Pi(\psi_1\psi_2\psi_3...)$?".

THEOREM:

Let $\mathcal{F} = \{f_1, f_2\}$ be as in Example 1.
(i) If $\theta = \theta_1\theta_2..., \psi = \psi_1\psi_2... \in \{1,2\}^\infty, S^p\theta = S^q\psi,$
$E = f_{-\theta|p}(f_{-\psi|q})^{-1}$*, and* $\theta_1 + \theta_2 + ... + \theta_p = \theta_1 + \theta_2 + ... + \theta_q$*, then* $\Pi(\theta) = E\Pi(\psi)$ *where E is an isometry.*
(ii) If $\Pi(\theta) = E\Pi(\psi)$ *where E is an isometry, for some pair of addresses* $\theta, \psi \in \{1,2\}^\infty$*. Then there are* $p, q \in N$ *such that* $S^p\theta = S^q\psi, E = f_{-(\theta|p)}(f_{-(\psi|q)})^{-1}$*, and* $\theta_1 + \theta_2 + ... + \theta_p = \theta_1 + \theta_2 + ... + \theta_q$.

PROOF:

- If E is known: α and α^{-1} well defined by $\alpha^{-1}ET_k = s^{-1}EsT_{k+1}$ and $\alpha ET_{k+1} = sEs^{-1}T_k$ (inflation and deflation)

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- $\bigcup \left(f_{-\theta|p}\right)^{-1} \Pi(\theta|k) = \bigcup \left(f_{-\psi|q}\right)^{-1} \Pi(\psi|k)$

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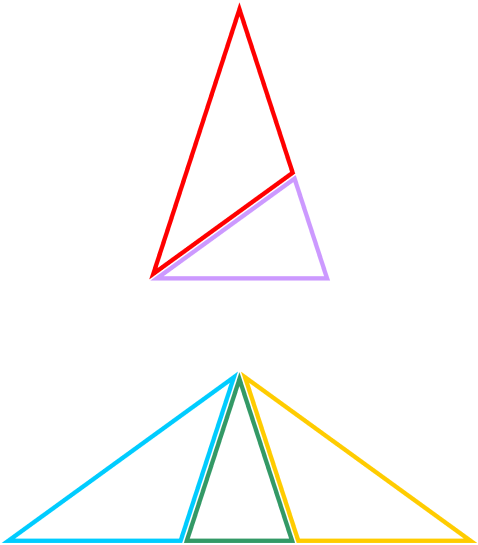
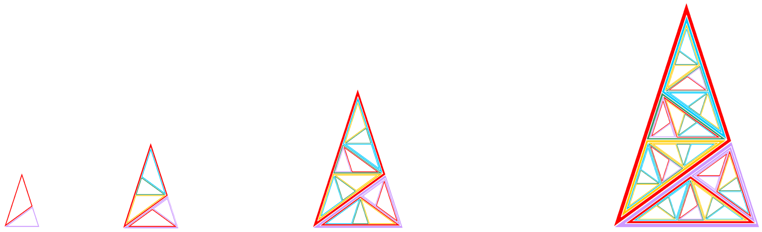
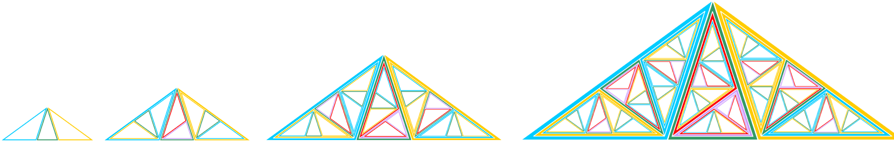
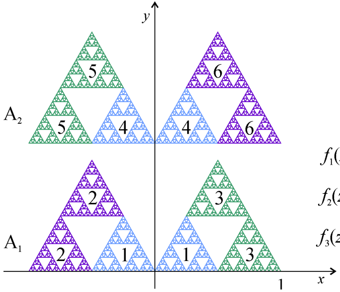
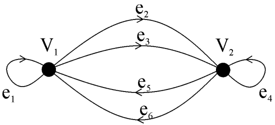
- true for all large L implies the theorem











$f_1(z) = \frac{1}{2} z$

$f_2(z) = \frac{1}{2} e^{\frac{-2i\pi}{3}} (z - i) - \frac{1}{4}(3 - \sqrt{3} i)$

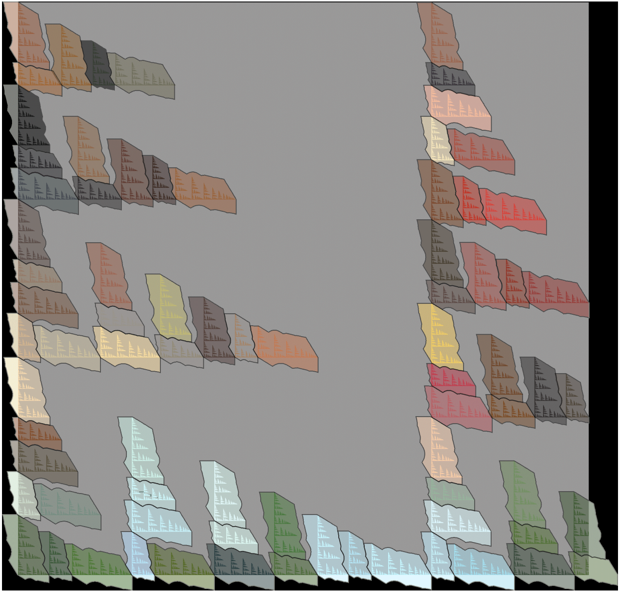
$f_3(z) = \frac{1}{2} e^{\frac{2i\pi}{3}} (z - i) + \frac{1}{4}(3 + \sqrt{3} i)$

$f_4(z) = \frac{1}{2} z + \frac{i}{2}$

$f_5(z) = \frac{1}{2} e^{\frac{-2i\pi}{3}} (z) - \frac{1}{4}(3 - i(4 + \sqrt{3}))$

$f_6(z) = \frac{1}{2} e^{\frac{2i\pi}{3}} (z) + \frac{1}{4}(3 + i(4 + \sqrt{3}))$

Rigid Tiling Theorem *Let* $(\mathcal{F}, \mathcal{G})$ *be a tiling IFS.*
(i) If $\theta, \psi \in \Sigma_{\infty}^{\dagger}$, $S^p\theta = S^q\psi$, $E = f_{-\theta|p}(f_{-\psi|q})^{-1}$, $(\theta|p)^{+} = (\psi|q)^{+}$, *and* $\xi(\theta|p) = \xi(\psi|q)$, *then* $\Pi(\theta) = E\Pi(\psi)$ *where* E *is an isometry.*
(ii) Let $(\mathcal{F}, \mathcal{G})$ *be rigid, and let* $\Pi(\theta) = E\Pi(\psi)$ *where* $E \in \mathcal{U}$ *is an isometry, for some pair of addresses* $\theta, \psi \in \Sigma_{\infty}^{\dagger}$. *Then there are* $p, q \in N$ *such that* $S^p\theta = S^q\psi$,
$E = f_{-(\theta|p)}(f_{-(\psi|q)})^{-1}$, $(\theta|p)^{+} = (\psi|q)^{+}$, *and* $\xi(\theta|p) = \xi(\psi|q)$.



Summary

Thank you for your attention.