Infinite ergodic theory and a tree of rational pairs

Claudio Bonanno

University of Pisa

One World Numeration Seminar

January 25, 2022

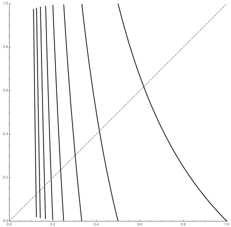
Infinite ergodic theory and the Farey tree

The *regular continued fraction* expansion of $x \in \mathbb{R}$ is

$$x = [a_0; a_1, a_2, a_3, \dots] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

with $a_0(x) = \lfloor x \rfloor \in \mathbb{Z}$, and $a_k(x) \in \mathbb{N}$ for $k \geq 1$.

Let $G : [0,1] \to [0,1]$ be the *Gauss map*, $G(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$, and $G(0) = 0$



Infinite ergodic theory and the Farey tree

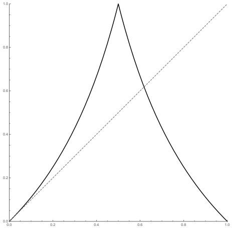
Properties of the Gauss map

- if $x = [a_1, a_2, a_3, a_4, \dots]$ then $G(x) = [a_2, a_3, a_4, \dots]$;
- $x \in \mathbb{Q} \cap [0, 1]$ implies $G^k(x) = 0$ for some $k \geq 0$, and viceversa;
- x is periodic or pre-periodic if and only if x is a quadratic irrational (Lagrange);
- $d\mu(x) = \frac{1}{(1+x)\log 2} dx$ is an ergodic G-invariant *probability* measure;
- $\#\{1 \leq k \leq n \, : \, a_k(x) = M\}/n \to \log(1 + 1/M(M+2))/\log 2$ for all $M \geq 1$ and a.e. x (Lévy);
- $(a_1(x) a_2(x) \dots a_n(x))^{1/n} \to K$ for a.e. x (Khintchine);
- $(a_1(x) + a_2(x) + \cdots + a_n(x))/n \to +\infty$ for a.e. x.

Infinite ergodic theory and the Farey tree

Let $F : [0,1] \to [0,1]$ be the *Farey map*

$$F(x) := \begin{cases} \frac{x}{1-x}\,, & x \in \left[0, \frac{1}{2}\right] \\ \frac{1}{x} - 1\,, & x \in \left[\frac{1}{2}, 1\right] \end{cases}$$



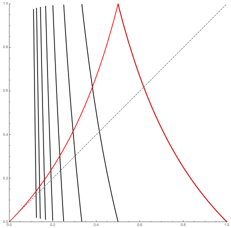
$$x = [a_1, a_2, a_3, a_4, \dots] \longmapsto F(x) = \begin{cases} [a_1 - 1, a_2, a_3, a_4, \dots], & x \in \left[0, \frac{1}{2}\right] \\ [a_2, a_3, a_4, \dots], & x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Infinite ergodic theory and the Farey tree

G is the *jump transformation* of F on $C = (\frac{1}{2}, 1]$, that is

$$G(x) = F^{\tau(x)}(x)$$

where $\tau(x) := 1 + \min\{k \geq 0 \,:\, F^k(x) \in C\} = a_1(x)$.



Infinite ergodic theory and the Farey tree

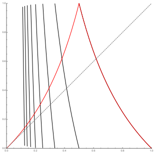
G is the *jump transformation* of F on $C = (\frac{1}{2}, 1]$, that is

$$G(x) = F^{\tau(x)}(x)$$

where $\tau(x) := 1 + \min\{k \geq 0 \, : \, F^k(x) \in C\} = a_1(x)$. Then

$$\frac{a_1(x) + a_2(x) + \cdots + a_n(x)}{n} = \frac{a_1(x) + a_1(G(x)) + \cdots + a_1(G^{n-1}(x))}{n} =$$

$$\frac{\tau(x) + \tau(G(x)) + \cdots + \tau(G^{n-1}(x))}{n} = \frac{N}{\sum_{j=0}^{N-1} \chi_C(F^j(x))}$$



Infinite ergodic theory and the Farey tree

Properties of the Farey map

- $d\nu(x) = 1/x\, dx$ is the unique F-invariant *absolutely continuous* measure, is ergodic and $\nu([0,1]) =$ *infinite*;
- $\frac{1}{N}\sum_{j=0}^{N-1} h(F^j(x)) \to 0$ for a.e. x and all $h \in L^1(\nu)$;
- using $h = \chi_C$ implies $(a_1(x) + a_2(x) + \cdots + a_n(x))/n \to +\infty$ for a.e. x;
- $\mathbb{P}(|\frac{\log N}{N}\sum_{j=0}^{N-1} h(F^j(x)) - \int h\, d\nu)| > \epsilon) \to 0$ for all $\varepsilon > 0$ and all $h \in L^1(\nu)$;
- using $h = \chi_C$ implies $(a_1(x) + a_2(x) + \cdots + a_n(x))/(n \log_2 n) \to 1$ in probability (Khinchin weak law).

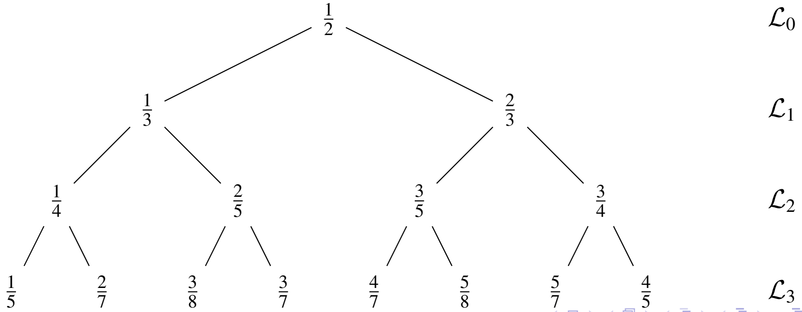
Infinite ergodic theory and the Farey tree

The *Farey tree* is a complete binary tree of fractions in $(0,1)$ generated by

- the Farey sum $\frac{a}{c} \oplus \frac{b}{d} := \frac{a+b}{c+d}$;
- the Farey map F;
- matrices L and R.

$$\mathcal{F}_{-1} = \left\{ \frac{0}{1}, \frac{1}{1} \right\}, \mathcal{F}_0 = \left\{ \frac{0}{1}, \frac{0}{1} \oplus \frac{1}{1}, \frac{1}{1} \right\} = \left\{ \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \right\}, \mathcal{F}_1 = \left\{ \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right\}$$

$$\mathcal{L}_n := \mathcal{F}_n \setminus \mathcal{F}_{n-1}, \forall n \geq 0$$

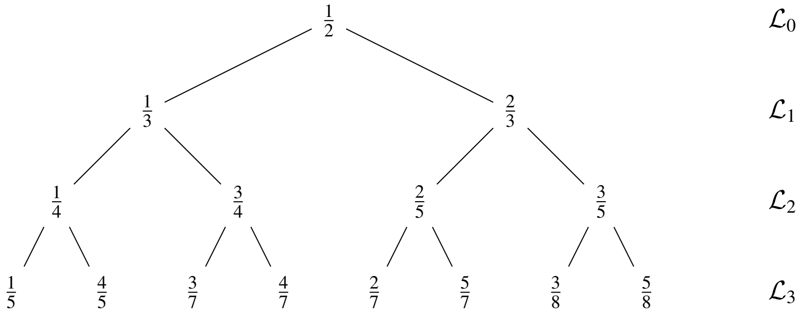


Infinite ergodic theory and the Farey tree

The *Farey tree* is a complete binary tree of fractions in $(0,1)$ generated by

- the Farey sum $\frac{a}{c} \oplus \frac{b}{d} := \frac{a+b}{c+d}$;
- the Farey map F;
- matrices L and R.

$$\mathcal{L}_n = F^{-n}\left(\frac{1}{2}\right)$$

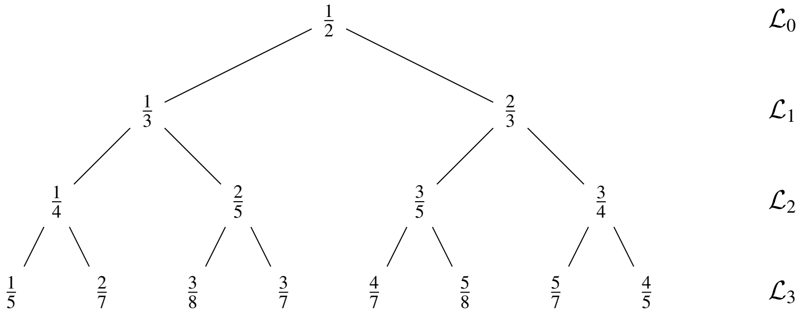


Infinite ergodic theory and the Farey tree

The *Farey tree* is a complete binary tree of fractions in $(0,1)$ generated by

- the Farey sum $\frac{a}{c} \oplus \frac{b}{d} := \frac{a+b}{c+d}$;
- the Farey map F;
- matrices L and R.

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \leftrightarrow \frac{1}{2},\ R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \frac{p}{q} = \frac{a}{c} \oplus \frac{b}{d} \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in L\{L,R\}^*$$



Infinite ergodic theory and the Farey tree

The *Farey tree* is a complete binary tree of fractions in $(0,1)$ generated by

- the Farey sum $\frac{a}{c} \oplus \frac{b}{d} := \frac{a+b}{c+d}$;
- the Farey map F;
- matrices L and R.

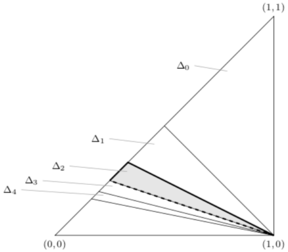
The *Farey coding* connects $\frac{p}{q} = [a_1, \ldots, a_n]$, with $a_n > 1$, to the tree and the ways it is generated. In particular

$$\frac{p}{q} = [a_1, \ldots, a_n] \leftrightarrow \begin{cases} LL^{a_1 -1} R^{a_2} \cdots L^{a_{n-1}} R^{a_n -1}, & \text{if } n \text{ is even} \\ LL^{a_1 -1} R^{a_2} \cdots R^{a_{n-1}} L^{a_n -1}, & \text{if } n \text{ is odd} \end{cases}$$

2d continued fractions and infinite ergodic theory

Let $\triangle := \{(x,y) \in \mathbb{R}^2 \,:\, 0 < y \le x \le 1\}$ and $T : \triangle \to \bar{\triangle}$ be the *Triangle map* (Garrity, 2001) given by

$$T(x,y) = \left(\frac{y}{x}, \frac{1-x-ky}{x} \right) \quad \text{where } k = \left\lfloor \frac{1-x}{y} \right\rfloor$$



$\triangle_k := \{(x,y) \in \triangle \,:\, ky \le 1-x < (k+1)y\}$ gives $T(\bar{\triangle}_k) = \bar{\triangle}$.

2d continued fractions and infinite ergodic theory

Let $\triangle := \{(x,y) \in \mathbb{R}^2 \,:\, 0 < y \le x \le 1\}$ and $T : \triangle \to \bar{\triangle}$ be the *Triangle map* (Garrity, 2001) given by

$$T(x,y) = \left(\frac{y}{x}, \frac{1-x-ry}{x} \right) \quad \text{where } r = \left\lfloor \frac{1-x}{y} \right\rfloor$$

For $(x,y) \in \triangle$, its *triangle sequence* is $\{\alpha_j\}_{j\ge 1}$ in $\mathbb{N}_0$ for which

$$T^n(x,y) \in \triangle_{\alpha_{n+1}}, \; \forall\, n \ge 0$$

Then we write

$$(x,y) = [\alpha_1, \alpha_2, \alpha_3, \dots] \longmapsto T(x,y) = [\alpha_2, \alpha_3, \alpha_4, \dots]$$

2d continued fractions and infinite ergodic theory

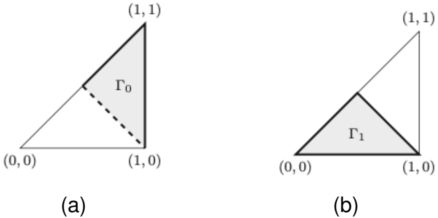
Properties of the Triangle map

- $(x, y) \in \mathbb{Q}^2 \cap \triangle$ implies $T^k(x, y) \in \{y = 0\}$ for some $k \geq 1$. The converse is *not true* (Garrity, 2001);
- if (x, y) has an eventually periodic triangle sequence, then x and y have degree at most 3. If x is an irrational solution in $(0, 1)$ of $t^3 + rt^2 + t - 1 = 0$ with $r \in \mathbb{N}_0$, then $(x, x^2) = [\overline{r}]$ (Garrity, 2001);
- the triangle sequence $\{\alpha_j(x, y)\}_{j \geq 1}$ is *weakly convergent* for a.e. $(x, y) \in \triangle$ (Messaoudi-Nogueira-Schweiger, 2009) ;
- $d\mu(x, y) = 12/(\pi^2\, x(1 + y))\, dxdy$ is an ergodic T-invariant *probability* measure (Messaoudi-Nogueira-Schweiger, 2009).

2d continued fractions and infinite ergodic theory

The Triangle map T has a "slow" version (B.-Del Vigna-Munday, 2021). Let $S : \bar{\triangle} \to \bar{\triangle}$ be the *Slow triangle map* given by

$$S(x,y) = \begin{cases} \left(\frac{y}{x}, \frac{1-x}{x}\right), & \text{if } (x,y) \in \Gamma_0 = \triangle_0 \\ \left(\frac{x}{1-y}, \frac{y}{1-y}\right), & \text{if } (x,y) \in \Gamma_1 = \bar{\triangle} \setminus \triangle_0 \end{cases}$$



then $S(\triangle_k) = \triangle_{k-1}$ for all $k \geq 1$ and

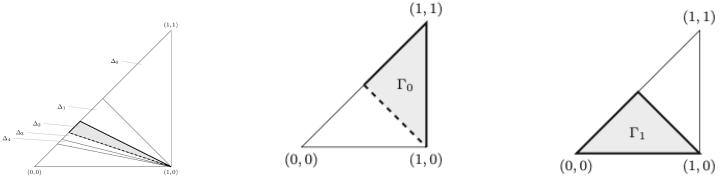
$$(x,y) = [\alpha_1, \alpha_2, \dots] \longmapsto S(x,y) = \begin{cases} [\alpha_1 - 1, \alpha_2, \dots], & \text{if } \alpha_1 \geq 1 \\ [\alpha_2, \alpha_3 \dots], & \text{if } \alpha_1 = 0 \end{cases}$$

2d continued fractions and infinite ergodic theory

T is the *jump transformation* of S on $\Gamma_0 = \Delta_0$, that is

$$T(x,y) = S^{\tau(x,y)}(x,y)$$

where $\tau(x,y) := 1 + \min\{k \geq 0 \,:\, S^k(x,y) \in \Gamma_0\} = \alpha_1(x,y) + 1$.



(c) (d) (e)

2d continued fractions and infinite ergodic theory

Properties of the Slow triangle map

- $d\nu(x,y) = 1/xy\, dxdy$ is the unique S-invariant *absolutely continuous* measure, is ergodic and $\nu(\bar{\triangle}) =$ *infinite*;
- $\{y=0\}$ is a set of neutral fixed points for S, and S has *intermittent* behaviour.

Theorem (B.-Del Vigna-Munday, 2021)

There exists a sequence $(a_N)_{N\geq 0}$ *which satisfies* $a_N \asymp N/\log^2 N$ *and such that, if it is regularly varying of index 1, for all* $\epsilon > 0$ *and all* $h \in L^1(\bar{\triangle}, \nu)$

$$\lim_{N\to\infty} \mathbb{P}\left(\left| \frac{1}{a_N} \sum_{j=0}^{N-1} h(S^j(x,y)) - \int_{\bar{\triangle}} h\, d\nu \right| > \epsilon \right) = 0.$$

Moreover, under the same assumption, there exists a sequence $(b_n)_{n\geq 0}$ *with* $b_n \asymp n \log^2 n$, *such that for all* $\epsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(\left| \frac{1}{b_n} \sum_{j=0}^{n-1} \alpha_j(x,y) - 1 \right| > \varepsilon \right) = 0.$$

Sketch of the proof.

Lemma (Nakada-Natsui, 2003)

Let $A \subset \mathbb{R}^d$ and $V : A \to A$ measurable. If (A, V) is a fibred system def-fs *then it admits an invariant probability measure with respect to which the system is continued fraction mixing* def-cfm.

Lemma (see Aaronson, 1997)

If there exists $A \subset \bar{\Delta}$ with $\nu(A) \in (0, \infty)$ such that the induced system (A, S_A, ν_A) is continued fraction mixing, then $(\bar{\Delta}, S, \nu)$ is pointwise dual ergodic def-pde *with sequence $(a_N)_{N\geq 0}$ given by*

$$a_N \asymp \frac{N}{\sum_{k=0}^{N-1} \nu(A \cap \{\varphi > k\})}$$

being $\varphi(x, y) = \min\{n \geq 1 \, : \, S^n(x, y) \in A\}$.

Darling-Kac Theorem and the fact that the Mittag-Leffler distribution of order 1 is constant imply the result.

A complete tree of rational pairs in $\bar{\Delta}$: the Triangle tree

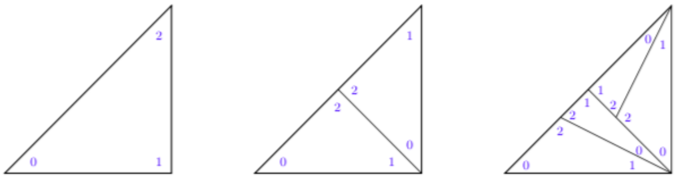
The Farey sum for rational pairs $(\frac{a}{c}, \frac{b}{d}) \oplus (\frac{a'}{c'}, \frac{b'}{d'}) = (\frac{a+a'}{c+c'}, \frac{b+b'}{d+d'})$. Let

$$\mathcal{S}_{-1} := \{v_0 = (0,0),\ v_1 = (1,0),\ v_2 = (1,1)\}$$

and $\mathcal{P}_0$ the partition of $\bar{\Delta}$ with vertices in $\mathcal{S}_{-1}$.
The level $\mathcal{S}_0$ is given by the Farey sums of pairs in $\mathcal{S}_{-1}$ close along sides in $\mathcal{P}_0$, that is

$$\mathcal{S}_0 := \left\{(0,0),\ \left(\frac{1}{2}, 0\right),\ (1,0),\ \left(1, \frac{1}{2}\right),\ (1,1),\ \left(\frac{1}{2}, \frac{1}{2}\right)\right\}$$

and $\mathcal{P}_1$ is the partition obtained by joining v_0, v_1, v_2 and $v_0 \oplus v_2$ from $\mathcal{S}_{-1}$ and by relabelling.



A complete tree of rational pairs in $\bar{\triangle}$: the Triangle tree

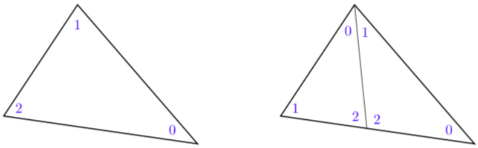
The Farey sum for rational pairs $(\frac{a}{c}, \frac{b}{d}) \oplus (\frac{a'}{c'}, \frac{b'}{d'}) = (\frac{a+a'}{c+c'}, \frac{b+b'}{d+d'})$. Let

$$\mathcal{S}_{-1} := \{v_0 = (0,0),\ v_1 = (1,0),\ v_2 = (1,1)\}$$

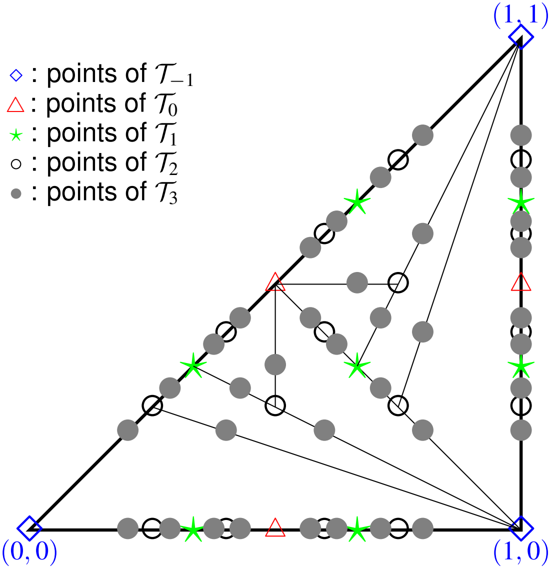
and $\mathcal{P}_0$ the partition of $\bar{\triangle}$ with vertices in $\mathcal{S}_{-1}$.
The level $\mathcal{S}_0$ is given by the Farey sums of pairs in $\mathcal{S}_{-1}$ close along sides in $\mathcal{P}_0$, that is

$$\mathcal{S}_0 := \left\{ (0,0),\ \left(\frac{1}{2}, 0\right),\ (1,0),\ \left(1, \frac{1}{2}\right),\ (1,1),\ \left(\frac{1}{2}, \frac{1}{2}\right) \right\}$$

and $\mathcal{P}_1$ is the partition obtained by joining v_0, v_1, v_2 and $v_0 \oplus v_2$ from $\mathcal{S}_{-1}$ and by relabelling.



Letting $\mathcal{T}_{-1} := \mathcal{S}_{-1}$ and $\mathcal{T}_n := \mathcal{S}_n \setminus \mathcal{S}_{n-1}$ for all $n \geq 0$



A complete tree of rational pairs in $\bar{\triangle}$: the Triangle tree

Consider a modified Slow triangle map $\tilde{S} : \bar{\triangle} \to \bar{\triangle}$

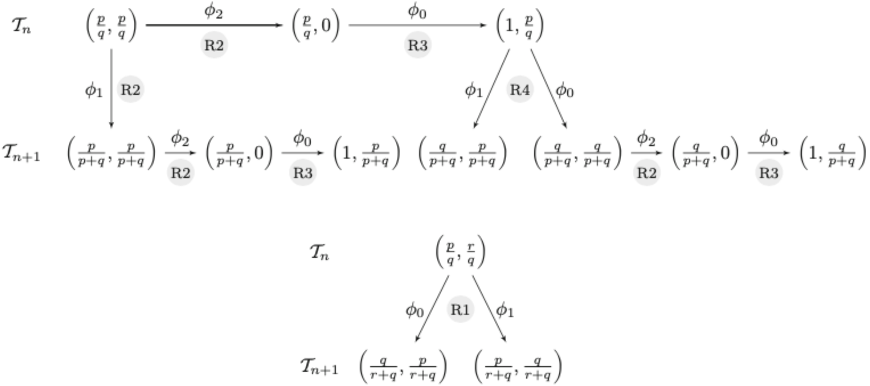
$$\tilde{S}(x,y) = \begin{cases} \left(\frac{y}{x}\,,\,\frac{1-x}{x}\right)\,, & \text{if } (x,y) \in \Gamma_0 = \triangle_0 \\ \left(\frac{x}{1-y}\,,\,\frac{y}{1-y}\right)\,, & \text{if } (x,y) \in \Gamma_1 \setminus \{y=0\} \\ (x\,,\,x)\,, & \text{if } (x,y) \in \{y=0\} \end{cases}$$

with local inverses

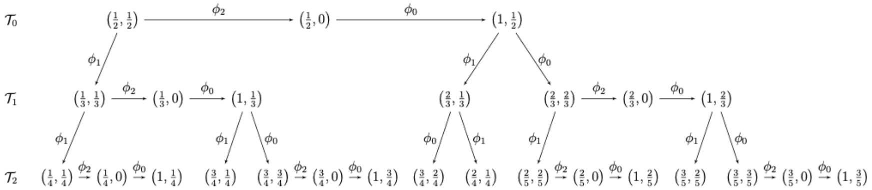
$$\phi_0 : \bar{\triangle} \setminus \{x=y\} \to \Gamma_0, \quad \phi_1 : \bar{\triangle} \setminus \{y=0\} \to \Gamma_1 \setminus \{y=0\},$$

$$\phi_2 : \bar{\triangle} \cap \{x=y\} \to \bar{\triangle} \cap \{y=0\}$$

A complete tree of rational pairs in $\bar{\Delta}$: the Triangle tree



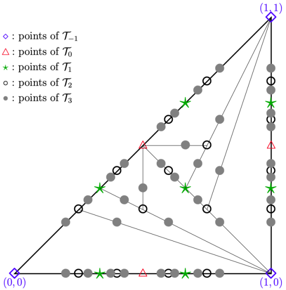
A complete tree of rational pairs in $\bar{\Delta}$: the Triangle tree



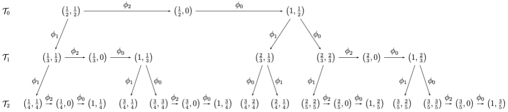
A complete tree of rational pairs in $\bar{\Delta}$: the Triangle tree

Theorem (B.-Del Vigna-Munday, 2021)

The two methods generates the same sets $(\mathcal{T}_n)_{n\geq 0}$.
The Triangle tree $\mathcal{T} = \cup \mathcal{T}_n$ *contains all pairs* $(x, y) \in \mathbb{Q}^2 \cap \bar{\Delta}$*, and each pair appears exactly once.*



(f)



(g)

Representation of pairs

We introduce a two-part *representation* (2d continued fraction expansion) of $(x, y) \in \bar{\triangle}$ (B.-Del Vigna, 2021).

Infinite triangle sequence. $(x, y) = [\alpha_1, \alpha_2, \dots], \alpha_j \in \mathbb{N}_0$.

$$rep(x, y) := \Big([\alpha_1, \alpha_2, \dots], [2] \Big)$$

since: in the *convergent case*

$$(x, y) = \lim_{n \to \infty} \phi_1^{\alpha_1} \phi_0 \phi_1^{\alpha_2} \phi_0 \dots \phi_1^{\alpha_n} \phi_0 \phi_2 \left(\frac{1}{2}, \frac{1}{2} \right) ;$$

in the *non-convergent case* (x, y) lies on a line $[P, Q]$ of points for which

$$P = \lim_{n \to \infty} \phi_1^{\alpha_1} \phi_0 \phi_1^{\alpha_2} \phi_0 \dots \phi_1^{\alpha_{2n}} \phi_0 \phi_2 \left(\frac{1}{2}, \frac{1}{2} \right)$$

$$Q = \lim_{n \to \infty} \phi_1^{\alpha_1} \phi_0 \phi_1^{\alpha_2} \phi_0 \dots \phi_1^{\alpha_{2n+1}} \phi_0 \phi_2 \left(\frac{1}{2}, \frac{1}{2} \right) .$$

Representation of pairs

We introduce a two-part *representation* (2d continued fraction expansion) of $(x,y) \in \bar{\triangle}$ (B.-Del Vigna, 2021).

Finite triangle sequence. $(x,y) = [\alpha_1, \alpha_2, \dots, \alpha_k]$, $\alpha_j \in \mathbb{N}_0$.

If (x,y) is in the *interior of* $\bar{\triangle}$ then there exists a unique $\xi = [a_1, a_2, \dots] \in (0,1)$ such that

$$(x,y) = \phi_1^{\alpha_1}\phi_0\phi_1^{\alpha_2}\phi_0 \dots \phi_1^{\alpha_k}\phi_0\phi_2\,(\xi,\xi)$$

with $\xi \in \mathbb{Q}$ if and only if $(x,y) \in \mathbb{Q}^2$, then

$$rep(x,y) := \Big([\alpha_1, \alpha_2, \dots, \alpha_k], [a_1, a_2, \dots]\Big)$$

If (x,y) is in the *boundary of* $\bar{\triangle}$ an analogous argument works.

The Triangular coding for rational pairs on $\mathcal{T}$

Let $(x, y) \in \mathbb{Q}^2$ be in the interior of $\bar{\triangle}$ with

$$rep(x,y) = \Big([\alpha_1, \alpha_2, \ldots, \alpha_k], [a_1, a_2, \ldots, a_n] \Big)$$

Then

$$(x,y) \in \phi_1^{\alpha_1} \phi_0 \phi_1^{\alpha_2} \phi_0 \ldots \phi_1^{\alpha_k} \phi_0 \phi_2 \Big(\{x = y\} \cap \bar{\triangle} \Big) =: \mathcal{L}$$

How to reach (x, y) by motions on the Triangular tree on $\bar{\triangle}$?

The Triangular coding for rational pairs on $\mathcal{T}$

How to reach (x, y) by motions on the Triangular tree on $\bar{\triangle}$?

1) We start from $(\frac{1}{2}, \frac{1}{2})$ and reach

$$(\alpha, \beta) := \phi_1^{\alpha_1} \phi_0 \phi_1^{\alpha_2} \phi_0 \dots \phi_1^{\alpha_k} \phi_0 \phi_2 \left(\frac{1}{2}, \frac{1}{2} \right)$$

2) We move on $\mathcal{L}$ from (α, β) as in the Farey coding.

The Triangular coding for rational pairs on $\mathcal{T}$

Let us introduce the following motions:

- *Motion L*. It means moving on an oriented line by taking the Farey sum of a pair and its left parent.
- *Motion R*. It means moving on an oriented line by taking the Farey sum of a pair and its right parent.
- *Motion I*. It means taking the Farey sum of a pair $\phi_\omega(\frac{1}{2}, \frac{1}{2})$ and $\phi_\omega(1, 0)$ (it means moving to another line).

The Triangular coding for rational pairs on $\mathcal{T}$

Theorem (B.-Del Vigna, 2021)

Let $(x, y) \in \mathbb{Q}^2$ *be in the interior of* $\bar{\triangle}$ *with*

$$rep(x,y) = \Big([\alpha_1, \alpha_2, \dots, \alpha_k], [a_1, a_2, \dots, a_n] \Big)$$

then

$$(x,y) = \begin{cases} \left(\frac{1}{2}, \frac{1}{2}\right) L^{\alpha_1} I \cdots L^{\alpha_{k-1}} I L^{\alpha_k - 1} I \, L^{a_1 - 1} R^{a_2} \cdots L^{a_{n-1}} R^{a_n - 1}, & \text{if } n \text{ is even} \\ \left(\frac{1}{2}, \frac{1}{2}\right) L^{\alpha_1} I \cdots L^{\alpha_{k-1}} I L^{\alpha_k - 1} I \, L^{a_1 - 1} R^{a_2} \cdots R^{a_{n-1}} L^{a_n - 1}, & \text{if } n \text{ is odd} \end{cases}$$

The Triangular coding for rational pairs on $\mathcal{T}$

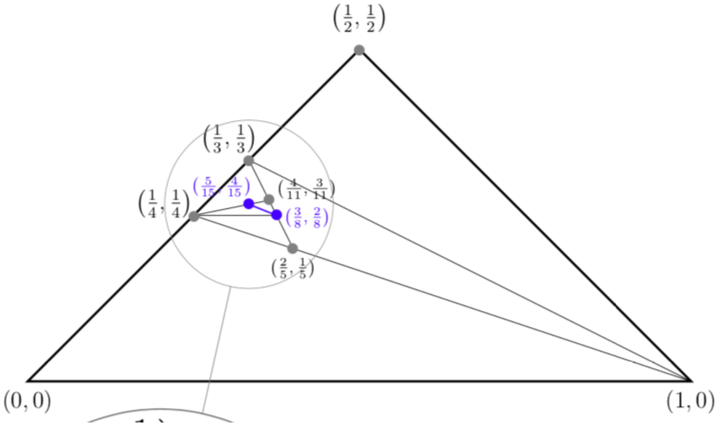
$$(x,y) = \begin{cases} \left(\frac{1}{2},\frac{1}{2}\right) L^{\alpha_1} I \cdots L^{\alpha_{k-1}} I L^{\alpha_k - 1} I \, L^{a_1 - 1} R^{a_2} \cdots L^{a_{n-1}} R^{a_n - 1}\,, & \text{if } n \text{ is even} \\ \left(\frac{1}{2},\frac{1}{2}\right) L^{\alpha_1} I \cdots L^{\alpha_{k-1}} I L^{\alpha_k - 1} I \, L^{a_1 - 1} R^{a_2} \cdots R^{a_{n-1}} L^{a_n - 1}\,, & \text{if } n \text{ is odd} \end{cases}$$

Ex. $rep(\frac{19}{54},\frac{14}{54}) = ([2,0,1,1],[2,2])$ then

$$\left(\frac{19}{54},\frac{14}{54}\right) = \left(\frac{1}{2},\frac{1}{2}\right) L\,L\,I\ I\ L\,I\ I\ \ L\ R$$

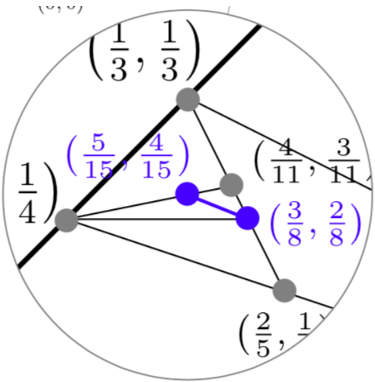
The Triangular coding for rational pairs on $\mathcal{T}$

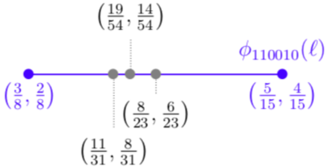
$$\left(\frac{19}{54}, \frac{14}{54}\right) = \left(\frac{1}{2}, \frac{1}{2}\right) L\,L\,I\;I\;L\,I\;I\;\;L\;R$$



The Triangular coding for rational pairs on $\mathcal{T}$

$$\left(\frac{19}{54},\frac{14}{54}\right) = \left(\frac{1}{2},\frac{1}{2}\right) L\,L\,I\;I\;L\,I\;I\;\;L\,R$$





The Triangular coding for rational pairs on $\mathcal{T}$

Matrices in L, R, and I.

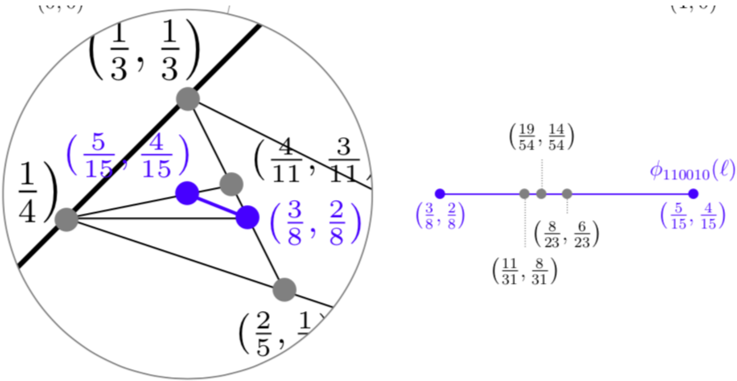
$$L := \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leftrightarrow \left(\frac{1}{2}, \frac{1}{2}\right), \quad R := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad I := \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let $rep(x,y) = ([\alpha_1, \alpha_2, \dots, \alpha_k], [a_1, a_2, \dots, a_n])$, then

$$(x,y) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} L^{\alpha_1} I \cdots L^{\alpha_{k-1}} I L^{\alpha_k - 1} I \, L^{a_1 - 1} R^{a_2} \cdots L^{a_{n-1}} R^{a_n - 1}, & \text{if } n \text{ is even} \\ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} L^{\alpha_1} I \cdots L^{\alpha_{k-1}} I L^{\alpha_k - 1} I \, L^{a_1 - 1} R^{a_2} \cdots R^{a_{n-1}} L^{a_n - 1}, & \text{if } n \text{ is odd} \end{cases}$$

The Triangular coding for rational pairs on $\mathcal{T}$

$$\left(\frac{19}{54}, \frac{14}{54}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} L\,L\,I\;I\;L\,I\;I\;\;L\,R = \begin{pmatrix} 23 & 31 & 11 \\ 8 & 11 & 4 \\ 6 & 8 & 3 \end{pmatrix}$$



Approximation of non-rational pairs

The representation of $(x, y) \in \bar{\triangle} \setminus \mathbb{Q}^2$ induces the definition of an infinite word

$$\mathcal{W}(x,y) = \begin{cases} L^{\alpha_1} I \cdots L^{\alpha_{k-1}} I L^{\alpha_k - 1} I \, L^{a_1 - 1} R^{a_2} L^{a_3} \dots , & \text{finite triangle seq.} \\ L^{\alpha_1} I \, L^{\alpha_2} I \, L^{\alpha_3} I \dots , & \text{infinite triangle seq.} \end{cases}$$

Approximations can be constructed by the finite sub-words of $\mathcal{W}(x, y)$.

Approximation of non-rational pairs

Ex.: finite triangle sequence.

Let $(x,y) = (\frac{1}{2}, \sqrt{2}-1)$. Then $rep(x,y) = ([1,1],[4,\overline{1}])$ and $\mathcal{W}(x,y) = LIILLL\,\overline{RL^4}$.
Then

$\left(\frac{1}{2},\frac{1}{2}\right)$	$\mathcal{W}_{(0)} = \varepsilon$	$([1],[2])$
$\left(\frac{1}{3},\frac{1}{3}\right)$	$\mathcal{W}_{(1)} = L$	$([2],[3])$
$\left(\frac{2}{4},\frac{1}{4}\right)$	$\mathcal{W}_{(2)} = LI$	$([2],[2])$
$\left(\frac{3}{6},\frac{2}{6}\right)$	$\mathcal{W}_{(3)} = LII$	$([1,1],[2])$
$\left(\frac{4}{8},\frac{3}{8}\right)$	$\mathcal{W}_{(4)} = LIIL$	$([1,1],[3])$
$\left(\frac{5}{10},\frac{4}{10}\right)$	$\mathcal{W}_{(5)} = LIILL$	$([1,1],[4])$
$\left(\frac{6}{12},\frac{5}{12}\right)$	$\mathcal{W}_{(6)} = LIILLL$	$([1,1],[5])$
$\left(\frac{11}{22},\frac{9}{22}\right)$	$\mathcal{W}_{(7)} = LIILLLR$	$([1,1],[4,2])$
$\left(\frac{17}{34},\frac{14}{34}\right)$	$\mathcal{W}_{(8)} = LIILLLRL$	$([1,1],[4,1,2])$
$\left(\frac{23}{46},\frac{19}{46}\right)$	$\mathcal{W}_{(9)} = LIILLLRLL$	$([1,1],[4,1,3])$

$\vdots$

Approximation of non-rational pairs

Rem.: infinite non-convergent triangle sequence.

Let $(x,y) \in \bar{\triangle}$ with $rep(x,y) = ([\alpha_1, \alpha_2, \dots], [2])$ and $(x,y) \in [P,Q]$. Let

$$(\xi_j, \eta_j) := T^j(x,y) \in \triangle_{\alpha_j} \quad \Rightarrow \quad \eta_j \xrightarrow[j\to\infty]{} 0$$

Then using $\xi_j = [a_1(j), a_2(j), \dots]$ we construct the approximations $(\frac{p_j}{r_j}, \frac{q_j}{r_j})$ for which

$$rep\left(\frac{p_j}{r_j}, \frac{q_j}{r_j}\right) = \Big([\alpha_1, \alpha_2, \dots, \alpha_j], [a_1(j), a_2(j), \dots, a_j(j)]\Big)$$

Future directions of research

- The Slow triangle map in higher dimensions;
- properties of the Triangle tree and other trees (?);
- connections with statistical properties of dynamical systems with two parameters;
- connections with other number theoretic problems (e..g. integer partitions, see B.-Del Vigna-Garrity-Isola, arxiv).

Thank you!

References

- C. Bonanno, A. Del Vigna, S. Munday, *A slow triangle map with a segment of indifferent fixed points and a complete tree of rational pairs*, Monatsh. Math. **194** (2021), 1–40
- C. Bonanno, A. Del Vigna, *Representation and coding of rational pairs on a Triangular tree and Diophantine approximation in* $\mathbb{R}^2$, Acta Arith. **200** (2021), no. 4, 389–427
- J. Aaronson, "An introduction to infinite ergodic theory". Mathematical Surveys and Monographs, 50. American Mathematical Society, Providence, RI, 1997
- T. Garrity, *On periodic sequences for algebraic numbers*, J. Number Theory **88** (2001), no. 1, 86–103
- A. Messaoudi, A. Nogueira, F. Schweiger, *Ergodic properties of triangle partitions*, Monatsh. Math. **157** (2009), no. 3, 283–299
- H. Nakada, R. Natsui, *On the metrical theory of continued fraction mixing fibred systems and its application to Jacobi-Perron algorithm*, Monatsh. Math. **138** (2003), no. 4, 267–288

A fibred system $V : A \to A$

(h1) There exists a finite or countable measurable partition $\mathcal{C} = \{C_i\}_{i \in \mathcal{I}}$ of A such that the restriction of V to C_i is injective for all $i \in \mathcal{I}$.

(h2) The map V is differentiable and non-singular.

(h3) There exists a sequence $(\sigma(n))_{n \geq 0}$ with $\sigma(n) \to 0$ as $n \to \infty$ and such that

$$\sup_{(i_1, \ldots, i_n) \in \mathcal{I}^n} \text{diam}\, C_{i_1, \ldots, i_n} \leq \sigma(n).$$

(h4) There exist a finite number of measurable subsets $U_1, \ldots, U_N$ of A such that for any cylinder $C_{i_1, \ldots, i_n}$ of positive measure, there exists U_j with $1 \leq j \leq N$ such that $V^n(C_{i_1, \ldots, i_n}) = U_j$ up to measure-zero sets.

(h5) There exists a constant $\lambda \geq 1$ such that for $\psi_{i_1, \ldots, i_n} := (V^n|_{C_{i_1, \ldots, i_n}})^{-1}$

$$\sup_{V^n(C_{i_1, \ldots, i_n})} |J\psi_{i_1, \ldots, i_n}| \leq \lambda \inf_{V^n(C_{i_1, \ldots, i_n})} |J\psi_{i_1, \ldots, i_n}|$$

where $J\psi_{i_1,\ldots,i_n}$ denotes the Jacobian determinant of $\psi_{i_1, \ldots, i_n}$.

(h6) For any $1 \leq j \leq N$, U_j contains a proper cylinder.

back

A fibred system $V : A \to A$

(h7) There is a constant $r_1 > 0$ such that

$$|J\psi_{i_1,\dots,i_n}(p_1) - J\psi_{i_1,\dots,i_n}(p_2)| \le r_1 m(C_{i_1,\dots,i_n})\|p_1 - p_2\|$$

for any $p_1, p_2 \in U_j$ and all j.

(h8) There is a constant $r_2 > 0$ such that

$$\|\psi_{i_1,\dots,i_n}(p_1) - \psi_{i_1,\dots,i_n}(p_2)\| \le r_2 \sigma(n)\|p_1 - p_2\|$$

for any $p_1, p_2 \in U_j$ and all j.

(h9) Let $\mathcal{F}$ be a finite partition generated by $U_1, \dots, U_N$ and denote by $\mathcal{F}_m^c$ the cylinders in $\mathcal{C}^m$ that are not contained in any element of $\mathcal{F}$. Then, as $m \to \infty$

$$\gamma(m) := \sum_{C(i_1,\dots,i_m) \in \mathcal{F}_m^c} m(C(i_1, \dots, i_m)) \to 0.$$

back

Continued fraction mixing

Let $V : A \to A$ and μ be a V-invariant probability on the Borel σ-algebra $\mathcal{B}$. Let $\mathcal{C}$ be a countable measurable generating partition for V. The system $(A, \mathcal{B}, \mu, V, \mathcal{C})$ is said to be *continued fraction mixing* if

$$\psi_n := \sup_{\substack{C \in \mathcal{C}^k, k \geq 1, \mu(C) > 0 \\ B \in \mathcal{B},\ \mu(B) > 0}} \frac{\left| \mu\left(C \cap V^{-(k+n)} B\right) - \mu(C)\mu(B) \right|}{\mu(C)\mu(B)} \quad \xrightarrow[n \to \infty]{} \quad 0$$

back

Pointwise dual ergodicity

The system $(\bar{\triangle}, S, \nu)$ is *pointwise dual ergodic* if there exists a sequence $(a_N)_{N\geq 0}$ such that

$$\lim_{N\to\infty} \frac{1}{a_N} \sum_{j=0}^{N-1} (P^j h)(x,y) = \int_{\bar{\triangle}} h \, d\nu$$

for ν-almost every $(x,y) \in \bar{\triangle}$ and for all $h \in L^1(\bar{\triangle}, \nu)$, where P is the transfer operator of the system.

back