Asymptotic behaviour of the sums of the digits for continued fraction algorithms

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Regular continued fractions and the Gauss map

For each $x \in (0, 1]$ there exists a unique sequence of digits $\{a_k\}_{k \geq 1} \subset \mathbb{N}$ such that

$$x = \lim_{k \to \infty} \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_k}}}} = \lim_{k \to \infty} \left[ a_1, a_2, a_3, \ldots, a_k \right]$$

Given the Gauss map $G : [0, 1] \to [0, 1]$, $G(x) := \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$ and $G(0) = 0$,

$$a_k = \left\lfloor \frac{1}{G^{k-1}(x)} \right\rfloor, \quad \forall k \geq 1,$$

or, given $A_n := (\frac{1}{n+1}, \frac{1}{n}]$, $n \geq 1$,

$$a_k = n \quad \text{if and only if} \quad G^{k-1}(x) \in A_n, \quad \forall k \geq 1.$$
For each \( x \in (0, 1] \) there exists a unique sequence of digits \( \{a_k\}_{k \geq 1} \subset \mathbb{N} \) such that

\[
x = \lim_{k \to \infty} \frac{1}{a_1} + \frac{1}{a_2 + \frac{1}{a_3 + \ldots + \frac{1}{a_k}}} = \lim_{k \to \infty} [a_1, a_2, a_3, \ldots, a_k]
\]

Given the Gauss map \( G : [0, 1] \to [0, 1] \), \( G(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \) and \( G(0) = 0 \), let

\[
f : (0, 1] \to \mathbb{R}^+ \quad \text{given by} \quad f(x) := \left\lfloor \frac{1}{x} \right\rfloor \quad \text{or} \quad f|_{A_n} \equiv n,
\]

then

\[
a_k = f(G^{k-1}(x)), \quad \forall k \geq 1.
\]
Regular continued fractions and the Gauss map

**Theorem (cfr. Khintchine)**

The Gauss map preserves the probability measure $d\mu(x) = \frac{1}{(1+x) \log 2} dx$ and it is ergodic. Since $\int_0^1 f d\mu = \infty$, by Birkhoff Ergodic Theorem

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^N a_k = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^N f(G^{k-1}(x)) = \infty \quad \text{a.s.}$$

**Question:** there exists $\gamma(N)$ such that $\gamma(N)/N \to \infty$ and

$$\lim_{N \to \infty} \frac{1}{\gamma(N)} \sum_{k=1}^N a_k = \lim_{N \to \infty} \frac{1}{\gamma(N)} \sum_{k=1}^N f(G^{k-1}(x)) = \text{const}(f) \in (0, +\infty) \quad \text{a.s.}?$$
Theorem (Khintchine weak law)

\[
\frac{1}{N \log_2 N} \sum_{k=1}^{N} a_k \rightarrow 1 \quad \text{in probability}
\]

Proposition (cfr. Aaronson ’81)

*For all* \( \gamma(N) \) *such that* \( \gamma(N)/N \rightarrow \infty \) *either*

\[
\limsup_{N \rightarrow \infty} \frac{1}{\gamma(N)} \sum_{k=1}^{N} a_k = \infty \quad \text{a.s.}
\]

*or*

\[
\liminf_{N \rightarrow \infty} \frac{1}{\gamma(N)} \sum_{k=1}^{N} a_k = 0 \quad \text{a.s.}
\]
Theorem (Diamond-Vaaler ’86)

\[
\lim_{N \to \infty} \frac{1}{N \log_2 N} \left( \sum_{k=1}^{N} a_k - \max_{1 \leq k \leq N} a_k \right) = 1 \quad \text{a.s.}
\]
Trimmed sums in ergodic theory

\( Y = (Y_k) \text{ r.v. on a probability space } (\Omega, \mathbb{P}). \) For \( \omega \in \Omega \) and \( N \in \mathbb{N} \) let \( \pi \) be a permutation of \( \{1, 2, \ldots, N\} \) such that

\[
Y_{\pi(1)}(\omega) \geq Y_{\pi(2)}(\omega) \geq \cdots \geq Y_{\pi(N)}(\omega).
\]

For a given \( r \in \mathbb{N}_0 \) the \textit{lightly trimmed sum} of \( Y \) is defined by

\[
S^r_N(Y, \omega) := \sum_{k=r+1}^{N} Y_{\pi(k)}(\omega).
\]

The \textit{intermediately trimmed sum} of \( (Y_k) \) is defined by

\[
S^{r_N}_N(Y, \omega) := \sum_{k=r_N+1}^{N} Y_{\pi(k)}(\omega)
\]

where \( r_N \to \infty \) and \( r_N/N \to 0 \).
Theorem (Diamond-Vaaler ’86)

Let $\mathcal{A} = (a_k)$ be r.v. on $([0, 1], Leb)$. Then

$$\lim_{N \to \infty} \frac{S^1_N(\mathcal{A}, x)}{N \log_2 N} = 1 \quad a.s.$$
Trimmed sums in ergodic theory

Let \( T : X \to X \) be a measurable transformation of \((X, \mathcal{B})\) and \( \mu \) a \( T \)-invariant probability. For a measurable observable \( f : X \to \mathbb{R}^+ \) define the **Birkhoff sums**

\[
S_N f(x) := \sum_{k=1}^{N} f(T^{k-1}(x))
\]

and the **lightly (intermediately) trimmed Birkhoff sum** \( S^r_N f \) for \( r \in \mathbb{N} \) \((r_N \to \infty \text{ with } r_N/N \to 0)\) as the lightly (intermediately) trimmed sum of \( \mathcal{F} = (f \circ T^{k-1}) \).

**Theorem (Diamond-Vaaler ’86)**

Let \([0, 1], G, \mu\) be the ergodic probability-preserving dynamical system with \( G \) the Gauss map and let \( f(x) = \lfloor 1/x \rfloor \). Then \( S_N f(x)/N \to \infty \) a.s. and

\[
\lim_{N \to \infty} \frac{S^1_N f(x)}{N \log_2 N} = 1 \quad \text{a.s.}
\]
Trimmed sums in ergodic theory

Let $T : X \to X$ be a measurable transformation of $(X, \mathcal{B})$ and $\mu$ a $T$-invariant probability. For a measurable observable $f : X \to \mathbb{R}^+$ define the Birkhoff sums

$$S_Nf(x) := \sum_{k=1}^{N} f(T^{k-1}(x))$$

and the lightly (intermediately) trimmed Birkhoff sum $S^r_Nf$ for $r \in \mathbb{N}$ ($r_N \to \infty$ with $r_N/N \to 0$) as the lightly (intermediately) trimmed sum of $\mathcal{F} = (f \circ T^{k-1})$.

Pointwise behaviour of trimmed Birkhoff sums for non-summable observables have been studied by Aaronson, Kesseböhmer, Nakada, Natsui, Schindler and many others for stochastic processes and deterministic systems with invariant finite distributions, in particular for the sums of the digits of $\alpha$-continued fractions.
Backward continued fractions

For each $x \in [0, 1]$ there exists a unique sequence of digits $\{b_k\}_{k \geq 1} \subset \mathbb{N}_{\geq 2}$ such that

$$x = \lim_{k \to \infty} 1 - \frac{1}{b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \cdots}}}$$

Let $T_{BCF} : [0, 1) \to [0, 1)$ given by $T_{BCF}(x) := \frac{1}{1-x} - \left\lfloor \frac{1}{1-x} \right\rfloor$
Backward continued fractions

For each $x \in [0, 1]$ there exists a unique sequence of digits $\{b_k\}_{k \geq 1} \subset \mathbb{N}_{\geq 2}$ such that

$$x = \lim_{k \to \infty} 1 - \frac{1}{b_1 - \frac{1}{b_2 - \frac{1}{\ddots - \frac{1}{b_k}}}$$

Let $T_{BCF} : [0, 1) \to [0, 1)$ given by $T_{BCF}(x) := \frac{1}{1-x} - \left\lfloor \frac{1}{1-x} \right\rfloor$, then

$$b_k = \left\lfloor \frac{1}{1 - T_{BCF}^{j-1}(x)} \right\rfloor + 1, \quad \forall k \geq 1,$$

or, given $B_n := [1 - \frac{1}{n}, 1 - \frac{1}{n+1}), n \geq 1$,

$$b_k = n + 1 \quad \text{if and only if} \quad T_{BCF}^{k-1}(x) \in B_n, \quad \forall k \geq 1.$$
Backward continued fractions

For each \( x \in [0, 1] \) there exists a unique sequence of digits \( \{b_k\}_{k \geq 1} \subset \mathbb{N}_{\geq 2} \) such that

\[
x = \lim_{k \to \infty} 1 - \frac{1}{b_1 - \frac{1}{b_2 - \frac{1}{\ldots - \frac{1}{b_k}}}}.
\]

Let \( T_{BCF} : [0, 1) \to [0, 1) \) given by \( T_{BCF}(x) := \frac{1}{1-x} - \left\lfloor \frac{1}{1-x} \right\rfloor \), let

\[g : [0, 1) \to \mathbb{R}^+ \quad \text{given by} \quad g(x) := \left\lfloor \frac{1}{1-x} \right\rfloor + 1 \quad \text{or} \quad g|_{B_n} \equiv n + 1,\]

then

\[b_k = g(T_{BCF}^{k-1}(x)), \quad \forall \, k \geq 1.\]
Backward continued fractions

Theorem (weak law - Aaronson ’86)

\[ \frac{1}{N} \sum_{k=1}^{N} b_k \longrightarrow 3 \quad \text{in probability} \]

Theorem (Aaronson-Nakada ’03)

For almost all \( x \in [0, 1) \)

\[ \lim \inf_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} b_k = 2 \]

and

\[ \lim \sup_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} b_k = \infty \]

More statistical results in Takahasi ’22.
The map $T_{BCF} : [0, 1) \rightarrow [0, 1)$ preserves the ergodic measure $d\nu(x) = \frac{1}{x}dx$, which is infinite, and the observable $g(x) := \left\lfloor \frac{1}{1-x} \right\rfloor + 1$ satisfies

$$g \geq 2, \quad g \not\in L^1(\nu), \quad g \not\in L^\infty(\nu).$$
Trimmed sums in infinite ergodic theory

Let $T : (X, \mu) \to (X, \mu)$ be a conservative ergodic measure-preserving transformation with $\mu(X) = \infty$ and $\sigma$-finite.

Let $E \subset X$ measurable with $\mu(E) = 1$ and $\varphi_E : E \to \mathbb{N}$ the first return time function with level sets $\{F_s\}$ and super-level sets $F_{>s} = \bigcup_{\ell>s} F_{\ell}$ and $T_E : E \to E$ the induced map. Let

$$w_n(E) := \sum_{s=0}^{n-1} \mu(F_{>s}) \quad \text{the wandering rate,} \quad \alpha(n) := \frac{n}{w_n(E)}.$$

Let $R_{E,N}(x)$ be the number of visits to $E$ up to time $N$ along the orbit of a point $x$

$$R_{E,N}(x) := \sum_{k=1}^{N+1} (\chi_E \circ T^{k-1})(x),$$

and

$$m(N, E, x) = \max \{ (\varphi_E \circ T_E^{k-1})(x) : k = 1, \ldots, R_{E,N}(x) \}$$

the longest excursion out of $E$ beginning in the first $N$-steps.
Trimmed sums in infinite ergodic theory

**Definition (ψ-mixing)**

Let $Y = (Y_n)$ be r.v. on $(\Omega, \mathbb{P})$, and let $\mathcal{F}_h^k$, for $0 \leq h < k \leq \infty$, be the $\sigma$-field generated by $(Y_n)_{h \leq n \leq k}$. The sequence $(Y_n)$ is **ψ-mixing** if

$$
\psi(n) := \sup \left\{ \left| \frac{\mathbb{P}(B \cap C)}{\mathbb{P}(B)\mathbb{P}(C)} - 1 \right| : B \in \mathcal{F}_0^j, C \in \mathcal{F}_{j+n}^\infty, \mathbb{P}(B) > 0, \mathbb{P}(C) > 0, j \in \mathbb{N} \right\}
$$

satisfies $\psi(n) \to 0$ as $n \to \infty$.

**Definition**

Let’s assume that $T_E(F_s) = E$ a.s. for all $s \in \mathbb{N}$. We say that $E$ induces rapid **ψ-mixing** if for any sequence $(\phi_n)_{n \in \mathbb{N}}$ of functions from $E$ to $\mathbb{R}$ such that each $\phi_n$ is piecewise constant on the partition $\mathcal{P}_E$ of $E$ induced by $T_E$, i.e. the finest partition such that for each element $P \in \mathcal{P}_E$ we have $T_E(P) = E$ a.s., the sequence $(\phi_n \circ T_{E}^{n-1})_{n \geq 1}$ is ψ-mixing with coefficient $\psi(n)$ fulfilling

$$
\sum_{n \geq 1} \psi(n)/n < \infty.
$$
Theorem (B.-Schindler ’23)

Let $E$ induce rapid $\psi$-mixing and $g : X \to \mathbb{R}_{\geq 0}$ be a measurable observable. Assume that:

(i) $w_n(E)$ is slowly varying and

$$\sum_{s \geq 1} \frac{s(\mu(F_s))^2}{w_s(E)^2} < \infty.$$ 

(ii) There exists a constant $\kappa > 0$ such that $\mu(g > s) \sim \kappa \mu(F_s)$.

(iii) The function $g$ is locally constant on the partition $\mathcal{P}_E$ of $E$, and $g \notin L^1(E, \mu)$.

(iv) There exists $c \in \mathbb{R}$ such that $g \equiv c$ on $X \setminus E$.

Then, for $\mu$-a.e. $x \in X$ we have

$$\lim_{N \to \infty} \frac{S_{N+m(N,E,x)}^1 g(x) - c m(N, E, x)}{N} = c + \kappa.$$
In B.-Schindler '22 we have considered the cases:

- $g \in L^1(X, \mu)$ for which

  $$S_{N+m(N,E,x)}g(x) \sim \alpha(N) \int_X g \, d\mu, \quad \mu - a.s.$$ 

- $g \notin L^1(X, \mu)$, $g \geq 0$, and some more specific assumptions (e.g. $g$ locally constant on the level sets of the hitting time function) but $g \in L^\infty(E, \mu)$, for which there exists $G(n)$ s.t.

  $$S_N g(x) \sim G(N), \quad \mu - a.s.$$
Trimmed sums of digits of backward cf

$T_{BCF} : ([0, 1], \nu) \to ([0, 1], \nu)$ is conservative and ergodic with $d\nu(x) = \frac{1}{x \log 2} dx$.

$E = [\frac{1}{2}, 1)$ has measure 1 and $T_E$ has full branches wrt $\mathcal{P}_E = \{F_s \cap B_n\}$ for $s \geq 1$ and $n \geq 2$. The set $E$ induces rapid $\psi$-mixing.

$g : [0, 1) \to \mathbb{N}_{\geq 2}, g(x) = \left\lfloor \frac{1}{1-x} \right\rfloor + 1$ is locally constant on $\{B_n\}$, hence on $\mathcal{P}_E$.

- $\nu(F_{>s}) \sim \frac{1}{s \log 2}$ and $w_n(E) \sim \frac{\log s}{\log 2}$ is slowly varying. Then

\[
\sum_{s \geq 1} s \left( \frac{\nu(F_{>s})}{w_s(E)} \right)^2 \sim \sum_{s \geq 1} \frac{1}{s (\log s)^2} < \infty.
\]

- $\nu(g > s) = \sum_{n > s} \nu(B_n) \sim \frac{1}{s \log 2} \sim \nu(F_{>s})$, hence $\kappa = 1$.

- $[0, 1) \setminus E = [0, \frac{1}{2})$ and $g|_{[0, \frac{1}{2})} \equiv 2$, hence $c = 2$. 

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Trimmed sums of digits of backward cf

\[ T_{BCF} : ([0, 1], \nu) \to ([0, 1], \nu) \] is conservative and ergodic with \( d\nu(x) = \frac{1}{x \log 2} dx \).

\( E = [\frac{1}{2}, 1) \) has measure 1 and \( T_E \) has full branches wrt \( \mathcal{P}_E = \{F_s \cap B_n\} \) for \( s \geq 1 \) and \( n \geq 2 \). The set \( E \) induces rapid \( \psi \)-mixing.

\[ g : [0, 1) \to \mathbb{N}_{\geq 2}, \quad g(x) = \left\lfloor \frac{1}{1-x} \right\rfloor + 1 \] is locally constant on \( \{B_n\} \), hence on \( \mathcal{P}_E \).

**Corollary (B.-Schindler ’23)**

\[
\lim_{N \to \infty} \frac{S_{N+m(N,E,x)}^1 g(x) - 2 m(N, E, x)}{N} = 3
\]

\[
\lim_{N \to \infty} \frac{\sum_{k=1}^{N+m(N,E,x)} b_k - \max_{1 \leq k \leq N+m(N,E,x)} b_k - 2 m(N, E, x)}{N} = 3.
\]
Trimmed sums of digits of backward cf

We can extend our main result to different asymptotics for $\nu(g > s)$ wrt $\nu(F_{>s})$. In particular for the sums of $b_k^\rho$.

Proposition

If $\rho \in (0, 1)$ then

$$\lim_{N \to \infty} \frac{\sum_{k=1}^{N} b_k^\rho}{N} = 2.$$  

If $\rho > 1$ then for $u > 1$

$$\lim_{N \to \infty} \frac{S_{N+m(N,E,x)}^{(\log \log N)^u}(b_k^\rho)}{\gamma_N} = 1,$$

where

$$\gamma_n \sim \frac{(\log 2)^\rho}{\rho - 1} \left(\frac{n}{\log n}\right)^\rho (\log \log n)^{(1-\rho)u}.$$
Proof of the main result

Theorem (B.-Schindler ’23)

Let $E$ induce rapid $\psi$-mixing and $g : X \rightarrow \mathbb{R}_{\geq 0}$ be a measurable observable. Assume that:

(i) $w_n(E)$ is slowly varying and

$$\sum_{s \geq 1} \frac{s(\mu(F > s))^2}{w_s(E)^2} < \infty.$$ 

(ii) There exists a constant $\kappa > 0$ such that $\mu(g > s) \sim \kappa \mu(F > s)$.

(iii) The function $g$ is locally constant on the partition $\mathcal{P}_E$ of $E$, and $g \notin L^1(E, \mu)$.

(iv) There exists $c \in \mathbb{R}$ such that $g \equiv c$ on $X \setminus E$.

Then, for $\mu$-a.e. $x \in X$ we have

$$\lim_{N \rightarrow \infty} \frac{S^1_{N + m(N, E, x)} g(x) - c m(N, E, x)}{N} = c + \kappa.$$
Proof of the main result

Use the induced map. Write

\[
S_N g(x) = \sum_{n=1}^{R_{E,N}(x)-1} (g^E \circ T_{E}^{n-1})(x) + \sum_{k=\tau_{E,x}(R_{E,N}(x)-1)+1}^{N} (g \circ T^{k-1})(x)
\]

where \(\tau_{E,x}(R_{E,N}(x) - 1)\) is the time of the last visit to \(E\) up to \(N\) and

\[
g^E(x) := \sum_{k=1}^{\varphi_E(x)} (g \circ T^{k-1})(x).
\]
Proof of the main result

Divide and conquer (case \( g|_{X\setminus E} \) bounded and non-constant).

Let \( c := \sup_{X\setminus E} g \) and wlog \( c \leq \inf_E g \) (using \( R_{N+m(N,E,x)} \sim \alpha(N) = o(N) \)). Let

\[
g_c(x) := \begin{cases} 
g(x), & \text{if } x \in X \setminus E; \\
c, & \text{if } x \in E. \end{cases}
\]

for which \( g_c \in L^\infty(X, \mu) \) \((g_c \equiv c \text{ under assumption (iv)}\)), and

\( h := g - g_c \geq 0 \), for which \( h|_{X\setminus E} \equiv 0 \), \( h \notin L^1(E, \mu) \).

Then

\[
S_N g(x) = S_N g_c(x) + S_N h(x) = S_N g_c(x) + \sum_{n=1}^{R_{E,N}(x)-1} (h^E \circ T_{E}^{n-1})(x).
\]
Proof of the main result

Control the infinitude of the invariant measure.

If $g_c \equiv c$ then

$$S_N g_c(x) = c N.$$

If $g_c \in L^\infty$ is non-constant then we apply Theorem 2.7 from B.-Schindler ’22 to obtain

$$S_N g_c(x) \sim g_c^E|_{F_N}$$

under suitable assumptions.
Proof of the main result

Control the unboundedness of the observable.

Lemma (Aaronson-Nakada ’03)

Let \( Y = (Y_k) \) i.d. \( \psi \)-mixing r.v. \((\Omega, \mathbb{P})\) with \( Y_k \geq 0 \) and \( \sum_{n \geq 1} \psi(n)/n < \infty \). Let \( P(y) = \mathbb{P}(Y_1 \leq y) \), and let for some \( y_0 > 1 \)

\[
W := \min \left\{ r \in \mathbb{N} : \int_{y_0}^{\infty} \left( \frac{y (1 - P(y))}{\int_{y_0}^{y} (1 - P(t)) \, dt} \right)^{r+1} \frac{1}{y} \, dy < \infty \right\}. \tag{1}
\]

Then there exists a sequence \((b(n))\) such that

\[
\lim_{N \to \infty} \frac{S^W_N(Y, \omega)}{b(N)} = 1, \quad \mathbb{P} - \text{a.e.}
\]

If we set \( a(y) := y/\int_{y_0}^{y} (1 - P(t)) \, dt \), then \( b(n) \) is the asymptotic inverse function of \( a(n) \).
Proof of the main result

\[ H = (h^E \circ T^{k-1}_E) \] satisfies the lemma with

\[
1 - P(y) = \mu(h^E > y) = \mu(h > y) \sim \mu(g > y) \sim \kappa \mu(F_{>s}),
\]

\[
\sum_{s \geq 1} \frac{s(\mu(F_{>s}))^2}{w_s(E)^2} < \infty \quad \Rightarrow \quad W = 1
\]

and \( b(N) \) is the asymptotic inverse of

\[
a(y) = \frac{y}{\int_1^y (1 - P(t)) dt} \sim \frac{y}{\int_1^y \kappa \mu(F_{>t}) dt} \sim \frac{y}{\kappa w_y(E)} \sim \frac{\alpha(y)}{\kappa}.
\]

Then, by Lemma 4.4 in B.-Schindler ’22 and since \( w_n(E) \) is slowly varying,

\[
b(R_{E,N,m}(x)) \sim b(\alpha(N)) \sim b(\kappa a(N)) \sim \kappa N.
\]
Proof of the main result

Hence we apply the lemma to get $\mu$-a.e.

$$
\sum_{n=1}^{R_{E,N,m}(x)-1} (h^E \circ T_{E}^{n-1})(x) - \max_{1 \leq k \leq R_{E,N,m}(x)-1} (h^E \circ T_{E}^{k-1})(x) \sim b(R_{E,N,m}(x)) \sim \kappa N.
$$

Since

$$
S_{N+m(N,E,x)}g(x) = c(N + m(N,E,x)) + \sum_{n=1}^{R_{E,N,m}(x)-1} (h^E \circ T_{E}^{n-1})(x),
$$

$$
\max_{1 \leq k \leq R_{E,N,m}(x)-1} (h^E \circ T_{E}^{k-1})(x) = \max_{1 \leq k \leq N+m(N,E,x)} (g \circ T_{E}^{k-1})(x) - c,
$$

then

$$
S_{N+m(N,E,x)}g(x) - \max_{1 \leq k \leq N+m(N,E,x)} (g \circ T_{E}^{k-1})(x) - c m(N,E,x) \sim (c + \kappa) N.
$$
Corollary

If additionally we have that

$$
\mu \left( \{ g > N \} \cap \{ \varphi_E > M \} \right) \approx \mu \left( g > N \right) \mu \left( \varphi_E > M \right)
$$

for $M, N \in \mathbb{N}$, then for $\mu$-a.e. $x \in X$

$$
S_{N+m(N,E,x)} g(x) - \max \left\{ \max_{1 \leq k \leq N+m(N,E,x)} (g \circ T^{k-1})(x), c m(N, E, x) \right\} \sim (c + \kappa) N.
$$
Theorem

Under the same assumptions on \((X, T, \mu)\) and \(E\) with the sets \(F_{>s}\) and the wandering rate \(w_n(E)\) satisfying assumptions (i), (iii) and (iv) of the main result, if we furthermore assume

\[\mu(g > n) = o(\mu(F_{>n})),\]

then, for \(\mu\)-a.e. \(x \in X\) we have

\[
\lim_{N \to \infty} \frac{S_N g(x) - \max_{1 \leq k \leq N} (g \circ T^{k-1})(x)}{N} = c.
\]

If additionally we assume

\[\mu(g > n) = o(\mu(F_{>n})) \text{ and } \sum_{n=1}^{\infty} \mu(g > \epsilon \beta(n)) < \infty \text{ for all } \epsilon > 0\]

where \(\beta(n)\) be the asymptotic inverse of \(\alpha(n)\), then, for \(\mu\)-a.e. \(x \in X\) we have

\[
\lim_{N \to \infty} \frac{S_N g(x)}{N} = c.
\]
Theorem

Under the same assumptions on \((X, T, \mu)\) and \(E\) with the sets \(F_{>s}\) and the wandering rate \(w_n(E)\) satisfying assumptions (i), (iii) and (iv) of the main result, if we furthermore assume

\[(\text{iib}) \quad \mu(g > n)/\mu(F_{>n}) \to \infty,\]

and for \(P(y) = 1 - \mu(g > y)\) assume that the quantity \(W\) defined in the lemma is finite, then, we have for \(\mu\)-a.e. \(x \in X\)

\[
\lim_{N \to \infty} \frac{S_{N+m(N,E,x)}^W g(x) - c m(N, E, x)}{b(\alpha(N))} = 1,
\]

where \(b(n)\) is the asymptotic inverse function of \(a(y) := y/ \int_{y_0}^y \mu(g > t) \, dt\).

If \(W = \infty\), we need intermediate trimming.
Other results from B.-Schindler ’23

Let us consider the *Even-Integer Continued Fraction* expansion

\[
x = \frac{1}{2h_1 + \frac{\varepsilon_1}{2h_2 + \frac{\varepsilon_2}{2h_3 + \cdots}}} \in [0, 1]
\]

with \( h_j \in \mathbb{N} \) and \( \varepsilon_j \in \{-1, +1\} \). The main result applies and

**Corollary**

For a.e. \( x \in [0, 1] \), the digits \( \{h_j(x)\} \) satisfy

\[
\lim_{N \to \infty} \frac{1}{N} \left[ \sum_{j=1}^{N+m(N,E,x)} 2h_j(x) - \max \left\{ 2m(N, E, x), \max_{1 \leq k \leq N+m(N,E,x)} 2h_k(x) \right\} \right] = 3.
\]

The same result holds with \( 2h_j(x) + \frac{1}{2}(\varepsilon_j(x) - 1) \) instead of \( 2h_j(x) \).
References