Asymptotic behaviour of the sums of the digits for continued fraction algorithms

Claudio Bonanno

joint work with Tanja I. Schindler

One World Numeration Seminar

November 28, 2023

Regular continued fractions and the Gauss map

For each $x \in (0,1]$ there exists a unique sequence of *digits* $\{a_k\}_{k\geq 1} \subset \mathbb{N}$ such that

$$x = \lim_{k\to\infty} \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}} = \lim_{k\to\infty} [a_1, a_2, a_3, \dots, a_k]$$

Given the *Gauss map* $G : [0,1] \to [0,1]$, $G(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$ and $G(0) = 0$,

$$a_k = \left\lfloor \frac{1}{G^{k-1}(x)} \right\rfloor, \quad \forall\, k \geq 1,$$

or, given $A_n := (\frac{1}{n+1}, \frac{1}{n}]$, $n \geq 1$,

$$a_k = n \quad \text{if and only if} \quad G^{k-1}(x) \in A_n, \quad \forall\, k \geq 1.$$

Regular continued fractions and the Gauss map

For each $x \in (0,1]$ there exists a unique sequence of *digits* $\{a_k\}_{k\geq 1} \subset \mathbb{N}$ such that

$$x = \lim_{k\to\infty} \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_k}}}} = \lim_{k\to\infty} [a_1, a_2, a_3, \dots, a_k]$$

Given the *Gauss map* $G : [0,1] \to [0,1]$, $G(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$ and $G(0) = 0$, let

$$f : (0,1] \to \mathbb{R}^+ \quad \text{given by} \quad f(x) := \left\lfloor \frac{1}{x} \right\rfloor \quad \text{or} \quad f|_{A_n} \equiv n,$$

then

$$a_k = f(G^{k-1}(x)), \quad \forall\, k \geq 1.$$

Regular continued fractions and the Gauss map

Theorem (cfr. Khintchine)

The Gauss map preserves the probability measure $d\mu(x) = \frac{1}{(1+x)\log 2}dx$ *and it is ergodic. Since* $\int_0^1 f\, d\mu = \infty$*, by Birkhoff Ergodic Theorem*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} a_k = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(G^{k-1}(x)) = \infty \quad \textit{a.s.}$$

Question: there exists $\gamma(N)$ such that $\gamma(N)/N \to \infty$ and

$$\lim_{N\to\infty} \frac{1}{\gamma(N)} \sum_{k=1}^{N} a_k = \lim_{N\to\infty} \frac{1}{\gamma(N)} \sum_{k=1}^{N} f(G^{k-1}(x)) = const(f) \in (0,+\infty) \quad \text{a.s.?}$$

Regular continued fractions and the Gauss map

Theorem (Khintchine weak law)

$$\frac{1}{N\,\log_2 N}\sum_{k=1}^{N} a_k \longrightarrow 1 \quad \textit{in probability}$$

Proposition (cfr. Aaronson '81)

For all $\gamma(N)$ *such that* $\gamma(N)/N \to \infty$ *either*

$$\limsup_{N\to\infty} \frac{1}{\gamma(N)}\sum_{k=1}^{N} a_k = \infty \quad \textit{a.s.}$$

or

$$\liminf_{N\to\infty} \frac{1}{\gamma(N)}\sum_{k=1}^{N} a_k = 0 \quad \textit{a.s.}$$

Regular continued fractions and the Gauss map

Theorem (Diamond-Vaaler '86)

$$\lim_{N\to\infty} \frac{1}{N\,\log_2 N} \left(\sum_{k=1}^{N} a_k - \max_{1\le k\le N} a_k \right) = 1 \quad \textit{a.s.}$$

Trimmed sums in ergodic theory

$Y = (Y_k)$ r.v. on a probability space $(\Omega, \mathbb{P})$. For $\omega \in \Omega$ and $N \in \mathbb{N}$ let π be a permutation of $\{1, 2, \dots, N\}$ such that

$$Y_{\pi(1)}(\omega) \geq Y_{\pi(2)}(\omega) \geq \cdots \geq Y_{\pi(N)}(\omega).$$

For a given $r \in \mathbb{N}_0$ the *lightly trimmed sum* of Y is defined by

$$S_N^r(Y, \omega) := \sum_{k=r+1}^{N} Y_{\pi(k)}(\omega).$$

The *intermediately trimmed sum* of (Y_k) is defined by

$$S_N^{r_N}(Y, \omega) := \sum_{k=r_N+1}^{N} Y_{\pi(k)}(\omega)$$

where $r_N \to \infty$ and $r_N/N \to 0$.

Trimmed sums in ergodic theory

Theorem (Diamond-Vaaler '86)

Let $\mathcal{A} = (a_k)$ *be r.v. on* $([0,1], Leb)$. *Then*

$$\lim_{N\to\infty} \frac{S^1_N(\mathcal{A},x)}{N\,\log_2 N} = 1 \quad \textit{a.s.}$$

Trimmed sums in ergodic theory

Let $T : X \to X$ be a measurable transformation of $(X, \mathcal{B})$ and μ a T-invariant probability. For a measurable observable $f : X \to \mathbb{R}^+$ define the *Birkhoff sums*

$$S_N f(x) := \sum_{k=1}^{N} f(T^{k-1}(x))$$

and the *lightly (intermediately) trimmed Birkhoff sum* $S_N^r f$ for $r \in \mathbb{N}$ ($r_N \to \infty$ with $r_N/N \to 0$) as the lightly (intermediately) trimmed sum of $\mathcal{F} = (f \circ T^{k-1})$.

Theorem (Diamond-Vaaler '86)

Let $([0,1], G, \mu)$ *be the ergodic probability-preserving dynamical system with* G *the Gauss map and let* $f(x) = \lfloor \frac{1}{x} \rfloor$. *Then* $S_N f(x)/N \to \infty$ *a.s. and*

$$\lim_{N \to \infty} \frac{S_N^1 f(x)}{N \log_2 N} = 1 \quad \textit{a.s.}$$

Trimmed sums in ergodic theory

Let $T : X \to X$ be a measurable transformation of $(X, \mathcal{B})$ and μ a T-invariant probability. For a measurable observable $f : X \to \mathbb{R}^+$ define the *Birkhoff sums*

$$S_N f(x) := \sum_{k=1}^{N} f(T^{k-1}(x))$$

and the *lightly (intermediately) trimmed Birkhoff sum* $S_N^r f$ for $r \in \mathbb{N}$ ($r_N \to \infty$ with $r_N/N \to 0$) as the lightly (intermediately) trimmed sum of $\mathcal{F} = (f \circ T^{k-1})$.

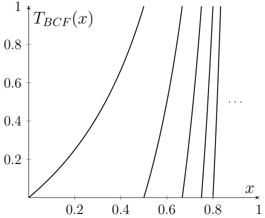
Pointwise behaviour of trimmed Birkhoff sums for non-summable observables have been studied by Aaronson, Kesseböhmer, Nakada, Natsui, Schindler and many others for stochastic processes and deterministic systems with invariant finite distributions, in particular for the sums of the digits of α-continued fractions.

Backward continued fractions

For each $x \in [0,1]$ there exists a unique sequence of *digits* $\{b_k\}_{k\geq 1} \subset \mathbb{N}_{\geq 2}$ such that

$$x = \lim_{k\to\infty} 1 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \frac{1}{b_k}}}}$$

Let $T_{BCF} : [0,1) \to [0,1)$ given by $T_{BCF}(x) := \frac{1}{1-x} - \left\lfloor \frac{1}{1-x} \right\rfloor$



Backward continued fractions

For each $x \in [0,1]$ there exists a unique sequence of *digits* $\{b_k\}_{k\ge 1} \subset \mathbb{N}_{\ge 2}$ such that

$$x = \lim_{k\to\infty} 1 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \frac{1}{b_k}}}}$$

Let $T_{BCF} : [0,1) \to [0,1)$ given by $T_{BCF}(x) := \frac{1}{1-x} - \left\lfloor \frac{1}{1-x} \right\rfloor$, then

$$b_k = \left\lfloor \frac{1}{1 - T_{BCF}^{j-1}(x)} \right\rfloor + 1, \quad \forall\, k \ge 1,$$

or, given $B_n := [1 - \frac{1}{n}, 1 - \frac{1}{n+1}),\ n \ge 1$,

$$b_k = n+1 \quad \text{if and only if} \quad T_{BCF}^{k-1}(x) \in B_n, \quad \forall\, k \ge 1.$$

Backward continued fractions

For each $x \in [0,1]$ there exists a unique sequence of *digits* $\{b_k\}_{k\geq 1} \subset \mathbb{N}_{\geq 2}$ such that

$$x = \lim_{k\to\infty} 1 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \frac{1}{b_k}}}}$$

Let $T_{BCF} : [0,1) \to [0,1)$ given by $T_{BCF}(x) := \frac{1}{1-x} - \left\lfloor \frac{1}{1-x} \right\rfloor$, let

$$g : [0,1) \to \mathbb{R}^+ \quad \text{given by} \quad g(x) := \left\lfloor \frac{1}{1-x} \right\rfloor + 1 \quad \text{or} \quad g|_{B_n} \equiv n+1,$$

then

$$b_k = g(T_{BCF}^{k-1}(x)), \quad \forall\, k \geq 1.$$

Backward continued fractions

Theorem (weak law - Aaronson '86)

$$\frac{1}{N}\sum_{k=1}^{N} b_k \longrightarrow 3 \quad \textit{in probability}$$

Theorem (Aaronson-Nakada '03)

For almost all $x \in [0,1)$

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} b_k = 2$$

and

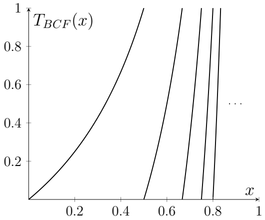
$$\limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} b_k = \infty$$

More statistical results in Takahasi '22.

Backward continued fractions

The map $T_{BCF} : [0,1) \to [0,1)$ preserves the ergodic measure $d\nu(x) = \frac{1}{x}dx$, which is infinite, and the observable $g(x) := \left\lfloor \frac{1}{1-x} \right\rfloor + 1$ satisfies

$$g \geq 2, \quad g \notin L^1(\nu), \quad g \notin L^\infty(\nu).$$



Trimmed sums in infinite ergodic theory

Let $T:(X,\mu)\to(X,\mu)$ be a conservative ergodic measure-preserving transformation with $\mu(X)=\infty$ and σ-finite.

Let $E\subset X$ measurable with $\mu(E)=1$ and $\varphi_{_E}:E\to\mathbb{N}$ the *first return time function* with level sets $\{F_s\}$ and super-level sets $F_{>s}=\cup_{\ell>s}F_\ell$ and $T_{_E}:E\to E$ the *induced map*. Let

$$w_n(E):=\sum_{s=0}^{n-1}\mu(F_{>s}) \quad \text{the } \textit{wandering rate,} \quad \alpha(n):=\frac{n}{w_n(E)}.$$

Let $R_{_{E,N}}(x)$ be the *number of visits to* E *up to time* N along the orbit of a point x

$$R_{_{E,N}}(x):=\sum_{k=1}^{N+1}(\chi_{_E}\circ T^{k-1})(x),$$

and

$$m(N,E,x)=\max\left\{(\varphi_{_E}\circ T_{_E}^{k-1})(x)\ :\ k=1,\dots,R_{_{E,N}}(x)\right\}$$

the *longest excursion out of* E *beginning in the first* N*-steps*.

Trimmed sums in infinite ergodic theory

Definition (ψ-mixing)

Let $Y = (Y_n)$ be r.v. on $(\Omega, \mathbb{P})$, and let $\mathcal{F}_h^k$, for $0 \leq h < k \leq \infty$, be the σ-field generated by $(Y_n)_{h \leq n \leq k}$. The sequence (Y_n) is *ψ-mixing* if

$$\psi(n) := \sup \left\{ \left| \frac{\mathbb{P}(B \cap C)}{\mathbb{P}(B)\mathbb{P}(C)} - 1 \right| \, : \, B \in \mathcal{F}_0^j, \, C \in \mathcal{F}_{j+n}^\infty, \, \mathbb{P}(B) > 0, \, \mathbb{P}(C) > 0, \, j \in \mathbb{N} \right\}$$

satisfies $\psi(n) \to 0$ as $n \to \infty$.

Definition

Let's assume that $T_{_E}(F_s) = E$ a.s. for all $s \in \mathbb{N}$. We say that E *induces rapid ψ-mixing* if for any sequence $(\phi_n)_{n \in \mathbb{N}}$ of functions from E to $\mathbb{R}$ such that each ϕ_n is piecewise constant on the partition $\mathcal{P}_{_E}$ of E induced by $T_{_E}$, i.e. the finest partition such that for each element $P \in \mathcal{P}_{_E}$ we have $T_{_E}(P) = E$ a.s., the sequence $(\phi_n \circ T_{_E}^{n-1})_{n \geq 1}$ is ψ-mixing with coefficient $\psi(n)$ fulfilling $\sum_{n \geq 1} \psi(n)/n < \infty$.

Trimmed sums in infinite ergodic theory

Theorem (B.-Schindler '23)

Let E induce rapid ψ-mixing and $g : X \to \mathbb{R}_{\geq 0}$ be a measurable observable. Assume that:

(i) *$w_n(E)$ is slowly varying and*

$$\sum_{s \geq 1} \frac{s(\mu(F_{>s}))^2}{w_s(E)^2} < \infty.$$

(ii) *There exists a constant $\kappa > 0$ such that $\mu(g > s) \sim \kappa \mu(F_{>s})$.*

(iii) *The function g is locally constant on the partition $\mathcal{P}_E$ of E, and $g \notin L^1(E, \mu)$.*

(iv) *There exists $c \in \mathbb{R}$ such that $g \equiv c$ on $X \setminus E$.*

Then, for μ-a.e. $x \in X$ we have

$$\lim_{N \to \infty} \frac{S^1_{N+m(N,E,x)} g(x) - c\, m(N, E, x)}{N} = c + \kappa.$$

Trimmed sums in infinite ergodic theory

In B.-Schindler '22 we have considered the cases:

- $g \in L^1(X,\mu)$ for which

$$S_{N+m(N,E,x)}g(x) \sim \alpha(N) \int_X g\, d\mu, \quad \mu - a.s.$$

- $g \notin L^1(X,\mu)$, $g \geq 0$, and some more specific assumptions (e.g. g locally constant on the level sets of the hitting time function) but $g \in L^\infty(E,\mu)$, for which there exists $G(n)$ s.t.

$$S_N g(x) \sim G(N), \quad \mu - a.s.$$

Trimmed sums of digits of backward cf

$T_{BCF} : ([0,1],\nu) \to ([0,1],\nu)$ is conservative and ergodic with $d\nu(x) = \frac{1}{x \log 2} dx$.

$E = [\frac{1}{2}, 1)$ has measure 1 and $T_{_E}$ has full branches wrt $\mathcal{P}_{_E} = \{F_s \cap B_n\}$ for $s \geq 1$ and $n \geq 2$. The set E induces rapid ψ-mixing.

$g : [0,1) \to \mathbb{N}_{\geq 2}$, $g(x) = \left\lfloor \frac{1}{1-x} \right\rfloor + 1$ is locally constant on $\{B_n\}$, hence on $\mathcal{P}_{_E}$.

- $\nu(F_{>s}) \sim \frac{1}{s \log 2}$ and $w_n(E) \sim \frac{\log s}{\log 2}$ is slowly varying. Then

$$\sum_{s \geq 1} \frac{s(\nu(F_{>s}))^2}{w_s(E)^2} \sim \sum_{s \geq 1} \frac{1}{s\,(\log s)^2} < \infty.$$

- $\nu(g > s) = \sum_{n>s} \nu(B_n) \sim \frac{1}{s \log 2} \sim \nu(F_{>s})$, hence $\kappa = 1$.
- $[0,1) \setminus E = [0, \frac{1}{2})$ and $g|_{[0,\frac{1}{2})} \equiv 2$, hence $c = 2$.

Trimmed sums of digits of backward cf

$T_{BCF} : ([0,1],\nu) \to ([0,1],\nu)$ is conservative and ergodic with $d\nu(x) = \frac{1}{x \log 2} dx$.

$E = [\frac{1}{2}, 1)$ has measure 1 and $T_{_E}$ has full branches wrt $\mathcal{P}_{_E} = \{F_s \cap B_n\}$ for $s \geq 1$ and $n \geq 2$. The set E induces rapid ψ-mixing.

$g : [0,1) \to \mathbb{N}_{\geq 2}$, $g(x) = \left\lfloor \frac{1}{1-x} \right\rfloor + 1$ is locally constant on $\{B_n\}$, hence on $\mathcal{P}_{_E}$.

Corollary (B.-Schindler '23)

$$\lim_{N\to\infty} \frac{S^1_{N+m(N,E,x)} g(x) - 2\, m(N,E,x)}{N} =$$

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N+m(N,E,x)} b_k - \max_{1\leq k \leq N+m(N,E,x)} b_k - 2\, m(N,E,x)}{N} = 3.$$

Trimmed sums of digits of backward cf

We can extend our main result to different asymptotics for $\nu(g > s)$ wrt $\nu(F_{>s})$. In particular for the sums of b_k^ρ.

Proposition

If $\rho \in (0,1)$ *then*

$$\lim_{N\to\infty} \frac{\sum_{k=1}^N b_k^\rho}{N} = 2.$$

If $\rho > 1$ *then for* $u > 1$

$$\lim_{N\to\infty} \frac{S_{N+m(N,E,x)}^{(\log\log N)^u}(b_k^\rho)}{\gamma_N} = 1,$$

where

$$\gamma_n \sim \frac{(\log 2)^\rho}{\rho - 1} \left(\frac{n}{\log n}\right)^\rho (\log\log n)^{(1-\rho)u}.$$

Proof of the main result

Theorem (B.-Schindler '23)

Let E induce rapid ψ-mixing and $g : X \to \mathbb{R}_{\geq 0}$ be a measurable observable. Assume that:

(i) $w_n(E)$ *is slowly varying and*

$$\sum_{s\geq 1} \frac{s(\mu(F_{>s}))^2}{w_s(E)^2} < \infty.$$

(ii) *There exists a constant* $\kappa > 0$ *such that* $\mu(g > s) \sim \kappa\mu(F_{>s})$.

(iii) *The function* g *is locally constant on the partition* $\mathcal{P}_E$ *of* E, *and* $g \notin L^1(E, \mu)$.

(iv) *There exists* $c \in \mathbb{R}$ *such that* $g \equiv c$ *on* $X \setminus E$.

Then, for μ*-a.e.* $x \in X$ *we have*

$$\lim_{N\to\infty} \frac{S^1_{N+m(N,E,x)}g(x) - c\, m(N,E,x)}{N} = c + \kappa.$$

Proof of the main result

Use the induced map. Write

$$S_N g(x) = \sum_{n=1}^{R_{E,N}(x)-1} (g^E \circ T_E^{n-1})(x) + \sum_{k=\tau_{E,x}(R_{E,N}(x)-1)+1}^{N} (g \circ T^{k-1})(x)$$

where $\tau_{E,x}(R_{E,N}(x) - 1)$ is the time of the last visit to E up to N and

$$g^E(x) := \sum_{k=1}^{\varphi_E(x)} (g \circ T^{k-1})(x).$$

Proof of the main result

Divide and conquer (case $g|_{X\setminus E}$ bounded and non-constant).

Let $c := \sup_{X\setminus E} g$ and wlog $c \le \inf_E g$ (using $R_{N+m(N,E,x)} \sim \alpha(N) = o(N)$). Let

$$g_c(x) := \begin{cases} g(x), & \text{if } x \in X \setminus E; \\ c, & \text{if } x \in E. \end{cases} \quad \text{for which} \quad g_c \in L^\infty(X,\mu)$$

($g_c \equiv c$ under assumption (iv)), and

$$h := g - g_c \ge 0, \quad \text{for which} \quad h|_{X\setminus E} \equiv 0, \quad h \notin L^1(E,\mu).$$

Then

$$S_N g(x) = S_N g_c(x) + S_N h(x) = S_N g_c(x) + \sum_{n=1}^{R_{E,N}(x)-1} (h^E \circ T_E^{n-1})(x).$$

Proof of the main result

Control the infinitude of the invariant measure.

If $g_c \equiv c$ then

$$S_N g_c(x) = c\,N.$$

If $g_c \in L^\infty$ is non-constant then we apply Theorem 2.7 from B.-Schindler ’22 to obtain

$$S_N g_c(x) \sim g_c^E|_{F_N}$$

under suitable assumptions.

Proof of the main result

Control the unboundedness of the observable.

Lemma (Aaronson-Nakada '03)

Let $Y = (Y_k)$ *i.d.* ψ*-mixing r.v.* $(\Omega, \mathbb{P})$ *with* $Y_k \geq 0$ *and* $\sum_{n\geq 1} \psi(n)/n < \infty$*. Let* $P(y) = \mathbb{P}(Y_1 \leq y)$*, and let for some* $y_0 > 1$

$$W := \min \left\{ r \in \mathbb{N} \colon \int_{y_0}^{\infty} \left(\frac{y\,(1 - P(y))}{\int_{y_0}^{y} (1 - P(t))\,\mathrm{d}t} \right)^{r+1} \frac{1}{y}\,\mathrm{d}y < \infty \right\}. \quad (1)$$

Then there exists a sequence $(b(n))$ *such that*

$$\lim_{N\to\infty} \frac{S_N^W(Y,\omega)}{b(N)} = 1, \quad \mathbb{P} - a.e.$$

If we set $a(y) := y / \int_{y_0}^{y} (1 - P(t))\mathrm{d}t$*, then* $b(n)$ *is the asymptotic inverse function of* $a(n)$*.*

Proof of the main result

$\mathcal{H} = (h^E \circ T_E^{k-1})$ satisfies the lemma with

$$1 - P(y) = \mu(h^E > y) = \mu(h > y) \sim \mu(g > y) \sim \kappa\mu(F_{>s}),$$

$$\sum_{s\geq 1} \frac{s(\mu(F_{>s}))^2}{w_s(E)^2} < \infty \quad \Rightarrow \quad W = 1$$

and $b(N)$ is the asymptotic inverse of

$$a(y) = \frac{y}{\int_1^y (1 - P(t))\mathrm{d}t} \sim \frac{y}{\int_1^y \kappa\mu(F_{>t})\mathrm{d}t} \sim \frac{y}{\kappa\, w_y(E)} \sim \frac{\alpha(y)}{\kappa}.$$

Then, by Lemma 4.4 in B.-Schindler '22 and since $w_n(E)$ is slowly varying,

$$b(R_{E,N,m}(x)) \sim b(\alpha(N)) \sim b(\kappa\, a(N)) \sim \kappa\, N.$$

Proof of the main result

Hence we apply the lemma to get μ-a.e.

$$\sum_{n=1}^{R_{E,N,m}(x)-1} (h^E \circ T_E^{n-1})(x) - \max_{1\le k\le R_{E,N,m}(x)-1} (h^E \circ T_E^{k-1})(x) \sim b(R_{E,N,m}(x)) \sim \kappa N.$$

Since

$$S_{N+m(N,E,x)}\, g(x) = c\,(N + m(N,E,x)) + \sum_{n=1}^{R_{E,N,m}(x)-1} (h^E \circ T_E^{n-1})(x),$$

$$\max_{1\le k\le R_{E,N,m}(x)-1} (h^E \circ T_E^{k-1})(x) = \max_{1\le k\le N+m(N,E,x)} (g \circ T^{k-1})(x) - c,$$

then

$$S_{N+m(N,E,x)}\, g(x) - \max_{1\le k\le N+m(N,E,x)} (g \circ T^{k-1})(x) - c\, m(N,E,x) \sim (c+\kappa)\, N.$$

□

Other results from B.-Schindler '23

Corollary

If additionally we have that

$$\mu\left(\{g > N\} \cap \{\varphi_{_E} > M\}\right) \asymp \mu\left(g > N\right) \mu\left(\varphi_{_E} > M\right)$$

for $M, N \in \mathbb{N}$*, then for* μ*-a.e.* $x \in X$

$$S_{N+m(N,E,x)}\, g(x) - \max\left\{ \max_{1 \le k \le N+m(N,E,x)} (g \circ T^{k-1})(x), c\, m(N,E,x) \right\} \sim (c+\kappa)\, N.$$

Other results from B.-Schindler '23

Theorem

Under the same assumptions on (X, T, μ) and E with the sets $F_{>s}$ and the wandering rate $w_n(E)$ satisfying assumptions (i), (iii) and (iv) of the main result, if we furthermore assume

(iia) $\mu(g > n) = o(\mu(F_{>n}))$,

then, for μ-a.e. $x \in X$ we have

$$\lim_{N\to\infty} \frac{S_N g(x) - \max_{1\leq k\leq N} (g \circ T^{k-1})(x)}{N} = c.$$

If additionally we assume

(iiã) $\mu(g > n) = o(\mu(F_{>n}))$ *and* $\sum_{n=1}^{\infty} \mu(g > \epsilon\beta(n)) < \infty$ *for all* $\epsilon > 0$

where $\beta(n)$ be the asymptotic inverse of $\alpha(n)$, then, for μ-a.e. $x \in X$ we have

$$\lim_{N\to\infty} \frac{S_N g(x)}{N} = c.$$

Other results from B.-Schindler '23

Theorem

Under the same assumptions on (X, T, μ) and E with the sets $F_{>s}$ and the wandering rate $w_n(E)$ satisfying assumptions (i), (iii) and (iv) of the main result, if we furthermore assume

(iib) $\mu(g > n)/\mu(F_{>n}) \to \infty$,

and for $P(y) = 1 - \mu(g > y)$ assume that the quantity W defined in the lemma is finite, then, we have for μ-a.e. $x \in X$

$$\lim_{N\to\infty} \frac{S^W_{N+m(N,E,x)} g(x) - c\, m(N,E,x)}{b(\alpha(N))} = 1,$$

where $b(n)$ is the asymptotic inverse function of $a(y) := y/\int_{y_0}^{y} \mu(g > t)\,\mathrm{d}t$.

If $W = \infty$, we need intermediate trimming.

Other results from B.-Schindler '23

Let us consider the *Even-Integer Continued Fraction* expansion

$$x = \cfrac{1}{2h_1 + \cfrac{\varepsilon_1}{2h_2 + \cfrac{\varepsilon_2}{2h_3 + \dots}}} \in [0,1]$$

with $h_j \in \mathbb{N}$ and $\varepsilon_j \in \{-1,+1\}$. The main result applies and

Corollary

For a.e. $x \in [0,1]$, *the digits* $\{h_j(x)\}$ *satisfy*

$$\lim_{N\to\infty} \frac{1}{N} \left[\sum_{j=1}^{N+m(N,E,x)} 2h_j(x) - \max\left\{ 2m(N,E,x), \max_{1\le k\le N+m(N,E,x)} 2h_k(x) \right\} \right] = 3.$$

The same result holds with $2h_j(x) + \frac{1}{2}(\varepsilon_j(x) - 1)$ *instead of* $2h_j(x)$.

References

- B.-Schindler '22. Discrete Contin. Dyn. Syst. 42 (2022), no. 11, 5541–5576
- B.-Schindler '23. arXiv:2304.01132 [math.DS]
- Aaronson-Nakada '03. Stochastic Process. Appl. 104 (2003), no. 2, 173–192
- Diamond-Vaaler '86. Pacific J. Math. 122 (1986), no. 1, 73–82
- Kesseböhmer-Schindler '19. Stochastic Process. Appl. 129 (2019), no. 10, 4163–4207
- Nakada-Natsui '02. J. Number Theory 97 (2002), no. 2, 287–300
- Takahasi '22. Stochastic Process. Appl. 144 (2022), 153–172