Quasi-recognisability and continuous eigenvalues of torsion-free $\mathcal{S}$-adic shifts

Álvaro M. Bustos-Gajardo

Joint work with Neil Mañibo and Reem Yassawi

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Notation reference

We will deal with symbolic dynamical systems (*subshifts*):

- the phase space is a set $X \subseteq \mathcal{A}^{\mathbb{Z}}$ of sequences of symbols from some finite *alphabet* $\mathcal{A}$,
- the $\mathbb{Z}$-action (*shift map*) consists of translations: $\sigma((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$,
- X must be σ-closed and closed under the product topology.

In our situation, sequences from X are built combinatorially by iterating *morphisms*:

Definition

A *morphism* between two alphabets is a map $\theta\colon \mathcal{A} \longrightarrow \mathcal{B}^+$ which assigns a nonempty word with symbols from $\mathcal{B}$ to every symbol from $\mathcal{A}$; it extends to $\mathcal{A}^+$ by concatenation. If the word $\theta(a)$ has the same length $|\theta(a)|$ for every $a \in \mathcal{A}$, we say that θ is of *constant length* $|\theta| := |\theta(a)|$.

$$\theta\colon \begin{array}{ccc} a & \mapsto & ab \\ b & \mapsto & a \end{array} \Longrightarrow \theta(abaab) = \theta(a)\theta(b)\theta(a)\theta(a)\theta(b) = abaababa$$

Definition

A *directive sequence* is an infinite sequence $\boldsymbol{\theta} = (\theta^{(j)})_{j \geq 0}$ of morphisms $\theta^{(j)} \colon \mathcal{A}_{j+1} \longrightarrow \mathcal{A}_j^+$ between some sequence of alphabets $(\mathcal{A}_j)_{j \geq 0}$.

We will focus on sequences where every $\theta^{(j)}$ is of constant length $q_j = |\theta^{(j)}|$.

A directive sequence $\mathcal{A}_0 \xleftarrow{\theta^{(0)}} \mathcal{A}_1 \xleftarrow{\theta^{(1)}} \mathcal{A}_2 \xleftarrow{\theta^{(2)}} \cdots$ defines a sequence of shift spaces $X^{(j)} \subseteq \mathcal{A}_j^{\mathbb{Z}}$:

$$X^{(j)} := \{x \in \mathcal{A}_j^{\mathbb{Z}} : (\forall m < n \in \mathbb{Z})(\exists k > j)(\exists a \in \mathcal{A}_k) : x_{[m,n)} \sqsubseteq \underbrace{\theta^{[j,k)}(a) := \theta^{(j)} \circ \theta^{(j+1)} \circ \cdots \circ \theta^{(k-1)}(a)}_{\text{supertiles of } X^{(j)}}\}.$$

We call $X^{(0)} =: X_{\boldsymbol{\theta}}$ the *$\mathcal{S}$-adic shift* associated to the directive sequence $(\theta^{(j)})_{j \geq 0}$. If all $\theta^{(j)}$ are the same $\theta \colon \mathcal{A} \longrightarrow \mathcal{A}$, we say that $X_{\boldsymbol{\theta}} =: X_\theta$ is the *substitutive subshift* generated by θ.

The search of recognisability

Recognisability allows us to undo a substitution or morphism, by identifying the position of each supertile and the originating symbol for each:

$$\dots\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ \dots$$

$$\uparrow$$

$$\dots\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ \dots$$

$$\uparrow$$

$$\dots\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ \dots$$

$$\uparrow$$

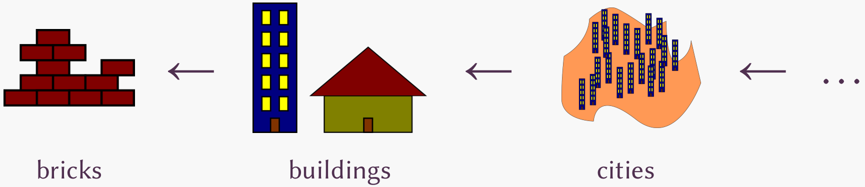
$$\dots\ 0\ 0\ 0\ 1\ \dots$$

Each sequence is obtained from the one below by applying the substitution $\theta\colon 0 \mapsto 10, 1 \mapsto 00$ and shifting. Recognisability amounts to being able to undo this process for any $x \in X_\theta$:

- there is only one way to split x into “chunks” (*supertiles*) of the form 00 or 10, and
- there is only one “legal” sequence $y \in X_\theta$ such that y_j determines the j-th chunk of x,
- we can repeat this process for y and so on.

Definition

A directive sequence $\boldsymbol{\theta}$ is *recognisable* if, for every $x^{(0)} \in X^{(0)} = X_{\boldsymbol{\theta}}$ and every $n \geqslant 1$ there are unique $k_n \in \{0, 1, \ldots, p_n - 1\}$ (with $p_n = q_0 \cdots q_{n-1}$) and $x^{(n)} \in X^{(n)}$ such that $x^{(0)} = \sigma^{k_n} \circ \theta^{[0,n)}(x^{(n)})$.



$$X^{(0)} \xleftarrow{\theta^{(0)}} X^{(1)} \xleftarrow{\theta^{(1)}} X^{(2)} \xleftarrow{\theta^{(3)}} \ldots$$

"Recognisability" implies we can give the position of each brick that makes up a given building, each building that makes up a given city, and so on. If we do not know where one building ends and the next one starts, do we have recognisability?

Recognisability is hard

While recognisability is “essentially granted” for substitutive subshifts, in general it is impossible to guarantee that a directive sequence is recognisable. See the following example:

$$\theta\colon \begin{array}{rcl} 0 & \mapsto & 011 \\ 1 & \mapsto & 001 \\ \tilde{0} & \mapsto & 011 \\ \tilde{1} & \mapsto & 001 \end{array} \qquad \varrho\colon \begin{array}{rcl} 0 & \mapsto & 0\tilde{1}1 \\ 1 & \mapsto & 0\tilde{0}1 \\ \tilde{0} & \mapsto & \tilde{0}1\tilde{1} \\ \tilde{1} & \mapsto & \tilde{0}0\tilde{1} \end{array} \Longrightarrow \boldsymbol{\theta} = (\theta, \varrho, \varrho, \varrho, \dots).$$

A sequence $x \in X_{\boldsymbol{\theta}}$ can be split into supertiles from $\{001, 011\}$ in only one way, but it has two possible “ancestors” in $X^{(1)} = X_{\varrho}$, so it is not recognisable!

$$\dots\tilde{0}1\tilde{1}0\tilde{0}1.\tilde{0}1\tilde{1}0\tilde{1}1\tilde{0}0\tilde{1}0\tilde{1}1\dots \in X^{(1)}$$

$$\swarrow$$

$$X^{(0)} \ni \dots011|001|.011|011|001|011\dots$$

$$\nwarrow$$

$$\dots0\tilde{1}1\tilde{0}0\tilde{1}.0\tilde{1}1\tilde{0}1\tilde{1}0\tilde{0}1\tilde{0}1\tilde{1}\dots \in X^{(1)}$$

Eigenvalues, and what we need from recognisability

Definition

A *continuous eigenfunction* of a shift space X is a non-null map $f\colon X \longrightarrow \mathbb{C}$ which satisfies the condition $f \circ \sigma = \lambda f$ for some $\lambda \in \mathbb{C}$; this λ is called a *continuous eigenvalue*.

If $\boldsymbol{\theta}$ is constant-length, and we know where to cut an $x \in X_{\boldsymbol{\theta}}$, there is a natural eigenfunction:

$$\left.\begin{array}{rcl} f(\ldots 011|001|.011|011|001|011\ldots) & = & 1 \\ f(\ldots 011|001|0.11|011|001|011\ldots) & = & e^{2\pi i/3} \\ f(\ldots 011|001|01.1|011|001|011\ldots) & = & e^{4\pi i/3} \\ f(\ldots 011|001|011|.011|001|011\ldots) & = & e^{2\pi i} = 1 \end{array}\right\} \Longrightarrow f \circ \sigma = e^{2\pi i/3} \cdot f, \text{ where } |\theta^{(0)}| = 3.$$

In general, knowing how to make the cuts for $\theta^{[0,n)} = \theta^{(0)} \circ \cdots \circ \theta^{(n-1)}$ allows us to find an eigenfunction with eigenvalue $p_n = q_o \cdots q_{n-1}$. We only need the cuts for this, not the "ancestors"!

Quasi-recognisability (or how to make the cuts)

If $\boldsymbol{\theta}$ is a constant-length substitution, for any point in $X_{\boldsymbol{\theta}}$ we can find:

- $p_1 = q_0$ different positions where the 1-supertile overlapping the origin can be,
- $p_2 = q_0 q_1$ different positions for the 2-supertile at the origin,
- $p_3 = q_0 q_1 q_2$ different positions for the 3-supertile, and so on…

$$x^{(0)} = \ldots\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ .\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ \ldots \in X_{\mathrm{TM}}$$

$$x^{(0)} = \theta_{\mathrm{TM}}(x^{(1)}) = \sigma^2 \circ \theta_{\mathrm{TM}}^2(x^{(2)}) = \sigma^6 \circ \theta_{\mathrm{TM}}^3(x^{(3)}) = \cdots$$

$$\begin{array}{rccccccc} \text{where} & 0 & \equiv_2 & 2 & \equiv_4 & 6 & \equiv_8 & \cdots \\ & n_0 & \equiv_2 & n_1 & \equiv_4 & n_2 & \equiv_8 & \cdots \end{array}$$

The position of the j-supertile at the origen is given by some $n_j \in \mathbb{Z}/p_j\mathbb{Z}$. A $(j+1)$-supertile is made by pasting j-supertiles, and thus $n_{j+1} \equiv n_j \pmod{p_j}$, since, if we know how to place the $(j+1)$-th supertiles, we can split them into j-supertiles: we now know where they are as well.

Quasi-recognisability formally defined

We care about the situation where each n_j is uniquely determined, where we can encode the sequence $(n_j)_{j\geq 1}$ via a *(p_j)-adic group*:

Definition

A directive sequence $(\theta^{(n)})_{n\geq 0}$ is *quasi-recognisable* if there is a continuous factor map (the *tile factor map*) $\pi_{\text{tile}}\colon (X_{\boldsymbol{\theta}}, \sigma) \twoheadrightarrow (\mathbb{Z}_{(p_j)}, +1)$, where $\mathbb{Z}_{(p_j)} = \lim_{\leftarrow j} \mathbb{Z}/p_j\mathbb{Z}$ defines an *odometer*.

The (p_j)-adic group $\mathbb{Z}_{(p_j)}$ is the set of all sequences $(n_0, n_1, \ldots)$ with $n_j \in \mathbb{Z}/p_j\mathbb{Z}$, $n_{j+1} \equiv n_j \pmod{p_j}$, with an appropriate topological group structure.

Shifting some $x \in X_{\boldsymbol{\theta}}$ corresponds to moving every j-supertile one position to the left, and thus the corresponding n_j increases by one. Thus, it is natural to have:

$$\pi_{\text{tile}}(\sigma(x)) = \pi_{\text{tile}}(x) + 1, \quad \text{where } 1 = (1, 1, 1, 1, \ldots).$$

Does quasi-recognisability appear naturally?

The sequence $\boldsymbol{\theta} = (\theta, \varrho, \varrho, \dots)$ from before is not recognisable, but it is quasi-recognisable: we know how to split any $x \in X_{\boldsymbol{\theta}}$ into chunks of 3^n letters (n-supertiles) uniquely.

Note that every substitution involved is of length 3: does this relate to quasi-recognisability?

Definition

A directive sequence $(\theta^{(n)})_{n\geq 0}$ is *torsion-free* if:

1. the associated shift $X_{\boldsymbol{\theta}}$ is minimal and aperiodic,
2. whenever a prime p divides some q_j, it must divide infinitely many $q_{j'}$ with $j' > j$.

(2) is equivalent to the group $\mathbb{Z}_{(p_j)}$ having no elements of finite order (hence the name). This is a natural setting for us, especially because of the following:

Proposition *(B–Mañibo–Yassawi)*

A torsion-free directive sequence is always quasi-recognisable.

We are not that far from recognisability

Torsion-free directive sequences are pretty close to being recognisable: we know how to place the cuts, so we only need to determine the letters $a \in \mathcal{A}_j$ each supertile $\theta^{[0,j)}(a)$ comes from.

$$\begin{array}{lccccccccc}
x^{(0)} = & 0100 & 0100 & 0001 & 0001 & 0001 & 0001 & 0100 & 0100 & 0100 \\
 & & & & & \uparrow \theta^{(0)} & & & & \\
x^{(1)} = & 01 & 01 & 10 & 10 & 10 & 10 & 01 & 01 & 01 \\
 & & & & & \uparrow \theta^{(1)} & & & & \\
x^{(2)} = & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}$$

Idea: after we cut $x^{(0)}$ into 1-supertiles, we can use the fact that $\theta^{(0)}$ is injective on letters to retrieve $x^{(1)}$. As $\theta^{(1)}$ is injective, we can repeat this to recover $x^{(2)}$...

Problem: in general, why would any of the $\theta^{(j)}$ be injective?

Injectivity

Theorem *(B–Mañibo–Yassawi)*

From a constant-length directive sequence $\boldsymbol{\theta} = (\theta^{(j)})_{j\geq 0}$ with bounded alphabet size, we can always obtain another directive sequence $\hat{\boldsymbol{\theta}} = (\hat{\theta}^{(j)})_{j\geq 0}$ with bounded alphabet size and same length sequence such that $X_{\boldsymbol{\theta}} = X_{\hat{\boldsymbol{\theta}}}$, and where every $\hat{\theta}^{(j)}$ is injective.

For $\boldsymbol{\theta} = (\theta, \varrho, \varrho, \varrho, \dots)$, $\theta = \theta^{(0)}$ was not injective, as $\theta(a) = \theta(\tilde{a})$ for $a \in \{0, 1\}$. We can write:

$$\theta^{(0)} = \hat{\theta}^{(0)} \circ \tau^{(0)}, \text{ where } \hat{\theta}^{(0)} \text{ is injective and } \tau^{(0)}(a) = \tau^{(0)}(\tilde{a}) = a, a \in \{0, 1\}.$$

The new directive sequence $\boldsymbol{\theta}' = (\hat{\theta}^{(0)}, \tau^{(0)} \circ \varrho, \varrho, \varrho, \dots)$ generates the same shift, but now the first morphism is injective! We can repeat this with the second morphism, then the third, and so on, "pushing" the problem away.

Our knowledge so far:

$$\begin{array}{ccccc} \text{Torsion-free} & \Longrightarrow & \text{Quasi-recognisable} & & \\ \wr & & + & & \\ \text{can be made} & \rightsquigarrow & \text{Injective} & \Longrightarrow & \text{Recognisable} \end{array}$$

that is, an $\mathcal{S}$-adic shift defined by a torsion-free directive sequence can always be described via (possibly another) directive sequence which is “truly” recognisable.

The TF/QR hypothesis is important! Without it, we may get less than stellar results, e.g.:

$$\left.\begin{array}{rrcl} \alpha\colon & A & \mapsto & 00 \\ & B & \mapsto & 01 \\ & C & \mapsto & 10 \end{array} \quad \begin{array}{rrcl} \beta\colon & A & \mapsto & ACABA \\ & B & \mapsto & ACAAA \\ & C & \mapsto & AAABA \end{array}\right\} \boldsymbol{\theta} = (\alpha, \beta, \beta, \beta, \dots)$$

Both α and β are injective, but $\boldsymbol{\theta}$ is not recognisable[1], so it cannot be quasi-recognisable.

1. Berthé, V., Steiner, W., Thuswaldner, J. M. & Yassawi, R. Recognizability for sequences of morphisms, *ETDS*, 2019.

The failure of recognisability

$$u = \ldots\ A\ B\ A\ A\ C\ A\ A\ A\ A\ .\ C\ A\ B\ A\ A\ C\ A\ B\ A\ \ldots \in X^{(1)}$$

$$\downarrow$$

$$x = \ldots\ 00\ 01\ 00\ 00\ 10\ 00\ 00\ 00\ 00\ .\ 10\ 00\ 01\ 00\ 00\ 10\ 00\ 01\ 00\ \ldots \in X^{(0)}$$

$$\wr$$

$$\sigma^{-1}(x) = \ldots\ 00\ 00\ 10\ 00\ 01\ 00\ 00\ 00\ 00\ .\ 01\ 00\ 00\ 10\ 00\ 01\ 00\ 00\ 10\ \ldots \in X^{(0)}$$

$$\uparrow$$

$$v = \ldots\ A\ A\ C\ A\ B\ A\ A\ A\ A\ .\ B\ A\ A\ C\ A\ B\ A\ A\ C\ \ldots \in X^{(1)}$$

We can see that $x = \alpha(u) = \sigma \circ \alpha(v)$, where $u \neq v \in X_\beta = X^{(1)}$. There is no "correct" way to split x into supertiles of length 2, as both ways are valid, so n_1 is not well-defined (not unique!). That is, we do not know how to place the bars in a point from this shift space: this example fails to be quasi-recognisable. As well, this point has two different "ancestors" in $X^{(1)}$.

Back to eigenvalues

The set of continuous eigenvalues of (X,σ) is deeply connected to its *maximal equicontinuous factor* (MEF), which is a factor map $\pi\colon(X,\sigma)\twoheadrightarrow(Z,R_\alpha)$ onto another t.d.s. with nice properties:

- in our scenario, Z is a *monothetic group*: there is some element $\alpha\in Z$ such that $\overline{\langle\alpha\rangle}=Z$,
- the $\mathbb{Z}$-action in Z is given by $R_\alpha(x)=x+\alpha$ (a *minimal group rotation*),
- the Pontryagin dual of Z can be identified with the set of all continuous eigenvalues of X.

When we have quasi-recognisability, the odometer $\mathbb{Z}_{(q_j)}$ and the function π_{tile} are very close to being a MEF. How far away are we from the true MEF?

Theorem *(B–Mañibo–Yassawi)*

If (X,T) is a compact minimal t.d.s. where there is a factor map $\pi_G\colon(X,T)\longrightarrow(G,R_\beta)$ over a minimal group rotation such that, for some $g\in G$, $|\pi_G^{-1}[\{g\}]|=c<\infty$, and (Z,R_α) is the MEF of the system, then Z is a group extension of G by a finite group K, with $|K|\leqslant c$.

Column numbers

For torsion-free shifts, π_{tile} is indeed *somewhere c-to-one* for some $c < \infty$. What is c?

$$\begin{array}{rcccccccccccccccc}
\theta\colon a & \mapsto & a & b & a & \mapsto & a & b & a & b & a & c & a & b & a & \mapsto & \cdots \\
b & \mapsto & b & a & c & \mapsto & b & a & c & a & b & a & c & a & b & \mapsto & \cdots \\
c & \mapsto & c & a & b & \mapsto & c & a & b & a & b & a & b & a & c & \mapsto & \cdots \\
\text{\# of symbols in each column:} & & 3 & 2 & 3 & & 3 & 2 & 3 & 2 & 2 & 2 & 3 & 2 & 3 & & \cdots
\end{array}$$

We can show that $\pi_{\text{tile}}\colon X_\theta \twoheadrightarrow \mathbb{Z}_3$ satisfies $\pi_{\text{tile}}^{-1}[\{z\}] \in \{2, 3\}$ for every $z \in \mathbb{Z}_3$; 2 is called the *naïve column number* of θ. Thus, $\mathbb{Z}_3$ is "close" to the actual MEF, which is the group $\mathbb{Z}_3 \times (\mathbb{Z}/2\mathbb{Z})$.

Proposition *(B–Mañibo–Yassawi)*

For any quasi-recognisable, injective (in particular, any torsion-free) constant length directive sequence $\boldsymbol{\theta} = (\theta^{(j)})_{j\geq 0}$ over a sequence of bounded alphabets $(\mathcal{A}_j)_{j\geq 0}$, the naïve column number $\bar{c}(\boldsymbol{\theta})$ is well-defined in terms of the number of symbols seen in each "column" of $\theta^{[m,n)}$, bounded by $\max_{j\geq 0}|\mathcal{A}_j|$, and $|\pi_{\text{tile}}^{-1}[\{z\}]| \geq \bar{c}(\boldsymbol{\theta})$, with equality for some $z \in \mathbb{Z}_{(p_j)}$.

The true MEF and eigenvalues

Theorem *(B–Mañibo–Yassawi)*

For a torsion-free directive sequence $\boldsymbol{\theta} = (\theta^{(j)})_{j\geq 0}$, there is a finite number $h \geq 1$ called the *height*, which is coprime to all q_j's and can be described combinatorially, such that the maximal equicontinuous factor of $(X_{\boldsymbol{\theta}}, \sigma)$ is the group rotation $(\mathbb{Z}_{(p_j)} \times \mathbb{Z}/h\mathbb{Z}, +(1,1))$.

Using Pontryagin duality, we obtain the following important result[2] (a version of which also applies for the quasi-recognisable case):

Corollary *(B–Mañibo–Yassawi)*

For a torsion-free directive sequence $\boldsymbol{\theta} = (\theta^{(j)})_{j\geq 0}$, every eigenvalue λ of $(X_{\boldsymbol{\theta}}, \sigma)$ must be of the form $\mathrm{e}^{2\pi \mathrm{i} k / h q_0 q_1 \cdots q_{n-1}}$ for some $n \in \mathbb{N}$ and $0 \leq k < h q_0 q_1 \cdots q_{n-1}$. Thus, all eigenvalues of a torsion-free $\mathcal{S}$-adic shift are rational.

2. See also Berthé, V., Cecchi-Bernales, P. and Yassawi, R., Coboundaries and eigenvalues of finitary $\mathcal{S}$-adic systems.

Height and (naïve) column number are generalisations of well-known notions from the theory of substitutions in this much more general context, and relate combinatorics to dynamics for the much larger class of torsion-free $\mathcal{S}$-adic systems. For example:

Proposition *(B–Mañibo–Yassawi)*

If a torsion-free directive sequence $\boldsymbol{\theta} = (\theta^{(j)})_{j \geq 0}$ has height h, then for sufficiently large j the alphabet $\mathcal{A}_j$ partitions into h disjoint sets $\mathcal{A}_{j,k}$, so that for any $x \in X^{(j)}$ a symbol from $\mathcal{A}_{j,k}$ always is followed by a symbol from $\mathcal{A}_{j,k+1 \bmod h}$.

Under the appropriate hypotheses, this becomes a combinatorial definition of height, allowing us to give estimates of h from the behaviour of the shift spaces $X^{(j)}$, $j \geq 0$.

A torsion-free $\mathcal{S}$-adic shift with height 1 (that is, where $\mathbb{Z}_{(q_j)}$ is the true MEF) is called *pure*. We can often disregard non-pure shifts, since:

Theorem *(B–Mañibo–Yassawi)*

Given an injective torsion-free directive sequence $\boldsymbol{\theta} = (\theta^{(j)})_{j\geq 0}$ of height h, we can find another sequence $\boldsymbol{\vartheta} = (\vartheta^{(j)})_{j\geq 0}$ with the same length sequence $(q_j)_{j\geq 0}$ such that:

- $X_{\boldsymbol{\vartheta}}$ is pure (we call it the *pure base* of $X_{\boldsymbol{\theta}}$), and
- $(X_{\boldsymbol{\theta}}, \sigma) \cong (X_{\boldsymbol{\vartheta}} \times \mathbb{Z}/h\mathbb{Z}, T)$ where $T(x,n) = \begin{cases} (x, n+1) & \text{if } 0 \leq n < h-1, \\ (\sigma(x), 0) & \text{if } n = h-1. \end{cases}$

The *(true) column number* of $\boldsymbol{\theta}$ is the naïve column number of $\boldsymbol{\vartheta}$. This number determines the least amount of preimages of some $z \in Z$ under π_{MEF}.

Measurable eigenvalues and the spectrum

Often, we can glean more information from a system if we allow more freedom in our definitions. In particular, ergodic theory focuses on the "typical behaviour" of a dynamical system by incorporating a (probability) measure; one example is a more general notion of eigenvalue:

Definition

The *Koopman operator* of a measurable dynamical system (X, T, μ) is the linear functional $U_T \colon L^2(X,\mu) \longrightarrow L^2(X,\mu)$ defined by $U_T(f) = f \circ T$; the *spectrum* of (X, T, μ) is the spectrum of U_T, i.e. the set $\mathrm{sp}(U_T)$ of all $\lambda \in \mathbb{C}$ such that $U_T - \lambda I$ is not a bijection. This includes, in particular, the *measurable eigenvalues* of X, which satisfy $f \circ T = \lambda f$ for a nonzero $f \in L^2(X,\mu)$.

Evidently, continuous eigenvalues are measurable in our context; the reciprocal is often false. What can we say about them in our context?

The spectrum $\mathrm{sp}(U_\sigma)$ is often classified as one of two kinds:

- *pure point spectrum*, when all its elements are eigenvalues of U_T, or equivalently, the span of the set of all eigenfunctions of U_T is the whole of $L^2(X,\mu)$,
- *mixed spectrum* otherwise.

We can give a partial characterisation of the spectrum in terms of the column number:

Proposition *(B–Mañibo–Yassawi)*

For a torsion-free directive sequence $\boldsymbol{\theta}$ defined over a sequence of bounded alphabets, provided with an invariant ergodic measure μ, if $c(\boldsymbol{\theta}) = 1$ and

$$\mu(\{x \in X_{\boldsymbol{\theta}} : |\pi_{\mathrm{MEF}}^{-1}(\{\pi_{\mathrm{MEF}}(x)\})| > 1\}) = 0,$$

then $(X_{\boldsymbol{\theta}}, \sigma)$ is *uniquely ergodic*, and $(X_{\boldsymbol{\theta}}, \sigma, \mu)$ has pure point spectrum, with all eigenfunctions being continuous.

We can obtain some conclusions in the case of column number greater than 1, as well:

Proposition *(B–Mañibo–Yassawi)*

For a uniquely ergodic torsion-free directive sequence $\boldsymbol{\theta}$ defined over a sequence of bounded alphabets, if $c(\boldsymbol{\theta}) > 1$, then either:

- $(X_{\boldsymbol{\theta}}, \sigma, \mu)$ has mixed spectrum, or
- $(X_{\boldsymbol{\theta}}, \sigma, \mu)$ has discrete spectrum, but there exists an eigenfunction that is not continuous.

The previous results are partial generalisations of known results by Dekking[3] on the theory of substitutions. Torsion-free and quasi-recognisable shifts allow for in-depth study under similar principles to constant length substitutions, while exhibiting much more variety and including several previously studied situations such as finitary directive sequences.

3. Dekking, F. M., The spectrum of dynamical systems arising from substitutions of constant length. *Zeitschrift für Wahrscheinlichkeitstheorie*, 1978.

Thank you for your attention!