Prevalence of matching for families of continued fraction algorithms. Old and new results

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Credits

Joint work with **Niels Langeveld** and **Wolfgang Steiner**

Carlo Carminati, Niels Langeveld, Wolfgang Steiner: *Tanaka-Ito α-continued fractions and matching*, arXiv:2004.14926

Aim: put it in perspective, reviewing some old results and looking towards new (and open) developments.

The Gauss map

The Gauss map $T : [0, 1] \to [0, 1]$ is defined by

$$T : x \mapsto \frac{1}{x} - c(x), \quad c(x) = \lfloor \frac{1}{x} \rfloor$$

$$x = \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{\cdots}}} = [0; c_1, c_2, \ldots]$$

$c_k = c(T^{k-1}(x))$.

Ergodic properties of RCF

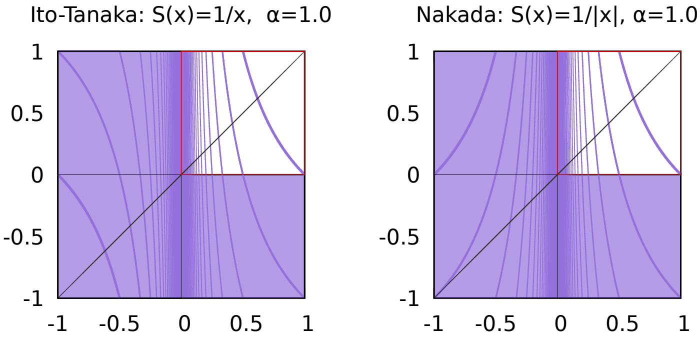
The Gauss map T has the folowing properties

- it has an invariant measure $d\mu(x) := \frac{dx}{(1+x)\log(2)}$;
- T is an exact map, hence it is ergodic;
- For almost every $x \in [0, 1]$:

$$\lim_{n \to +\infty} \frac{2}{n} \log q_n = h_\mu(T)$$

  where p_n/q_n is the n-th convergent of x and $h_\mu(T)$ is the entropy of T.
- $h_\mu(T) = \int_0^1 \log |T'(x)| d\mu(x) = \frac{\pi^2}{6 \log 2}$

The Gauss-like map $T_\alpha := S(x) - \lfloor S(x) + \alpha - 1 \rfloor$



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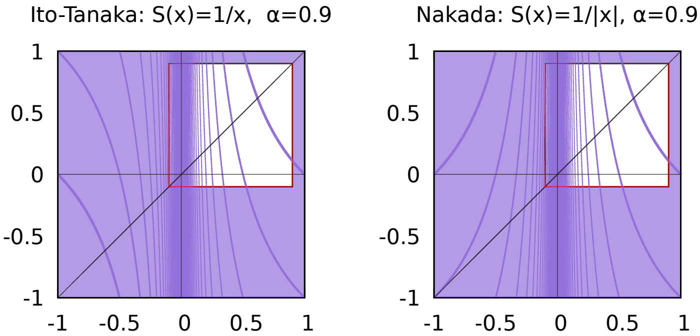

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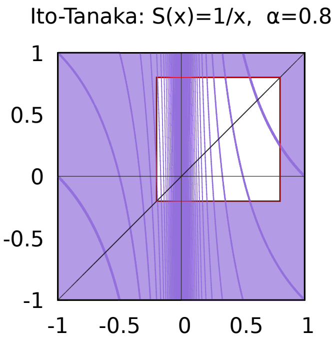

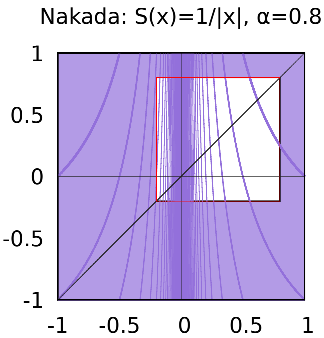

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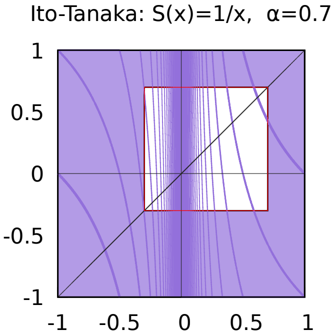

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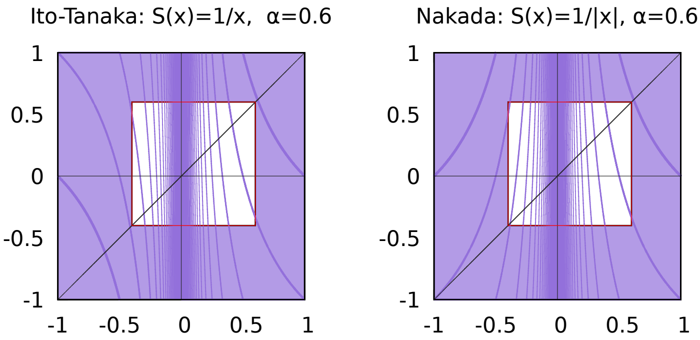

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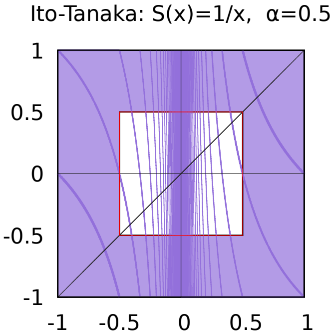

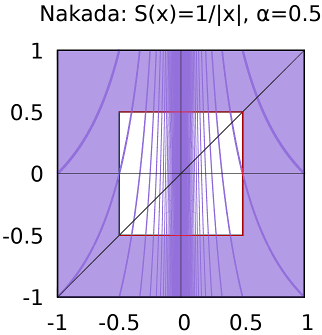

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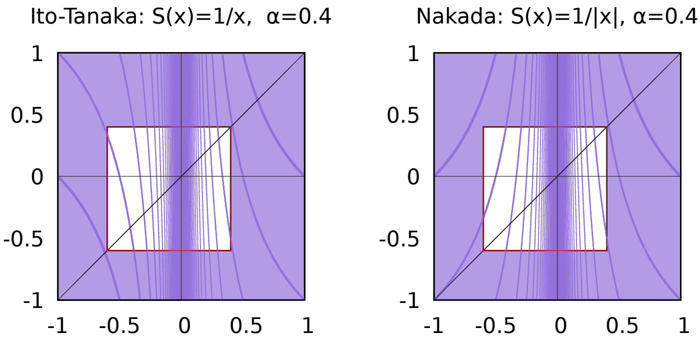

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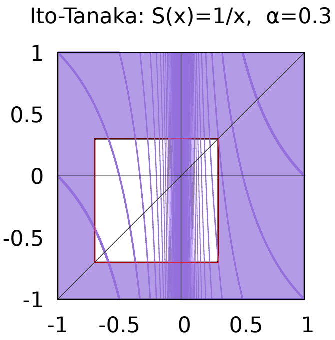

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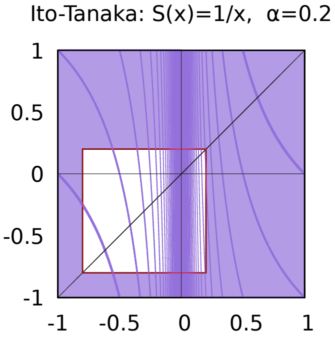

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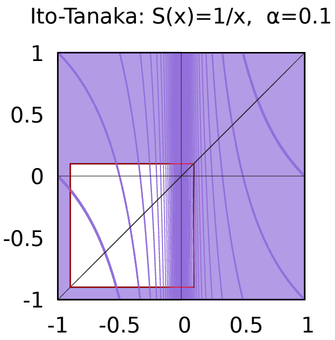

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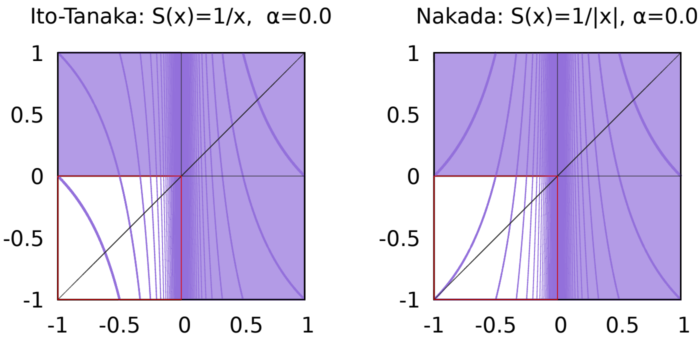


Invariant probability measures: classical examples

Back in the early '80s, the explicit expression of a a.c.i.p was found for α ranging in some intervals:
$\alpha \in [1/2, 1]$ for Nakada,
$\alpha \in [1/2, g]$ for Ito-Tanaka

Movie: natural extension for Nakada α-CF when α drops below $1/2$.

Ergodic properties common to both families T_α

The maps T_α ($\alpha > 0$) have the folowing properties

- α-expansion and α-convergents can be defined;
- T_α has an invariant probability measure $\mu_\alpha(x) := \rho_\alpha(x)dx$ with ρ_α of bounded variation;
- T_α is an exact map, hence it is ergodic;
- For almost every $x \in [0,1]$:

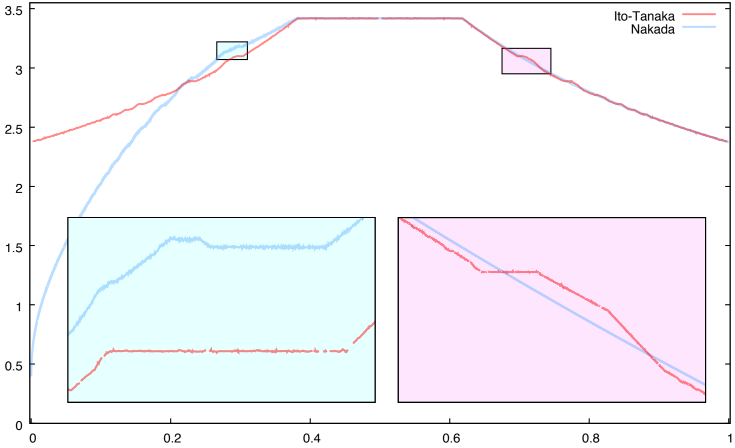
$$\lim_{n\to+\infty} \frac{1}{n} \log q_{n,\alpha}^2 = h_{\mu_\alpha}(T_\alpha)$$

  where $p_{n,\alpha}/q_{n,\alpha}$ is the n-th convergent of the α-expansion of x and $h_{\mu_\alpha}(T_\alpha)$ is the entropy of T_α.
- The entropy $h(T_\alpha)$ can be computed using Rohlin formula:

$$h_{\mu_\alpha}(T_\alpha) = \int_{\alpha-1}^{\alpha} \log |T'_\alpha(x)| d\mu_\alpha(x);$$

References: [KSS2012] [T2014] for case (N); [NS2020] [Lan2019] for case (IT)

Entropy function $\alpha \mapsto h_{\mu_\alpha}(T_\alpha)$



Matching intervals - in general

Definition (Matching)

Let $J \subset [0, 1]$ be a non-empty open interval in parameter space. We say that J is a **matching interval** for T_α (with **exponents** M, N) if

1. $T_\alpha^M(\alpha - 1) = T_\alpha^N(\alpha)$ for all $\alpha \in J$,
2. $T_\alpha^{M-1}(\alpha - 1) \neq T_\alpha^{N-1}(\alpha)$ for almost all $\alpha \in J$,
3. J is not contained in a larger open interval with properties 1 and 2 above.

The difference $\Delta := M - N$ is called **matching index**.

First discovered by Nakada-Natsui, Nonlinearity [NN2008]
Other names: cycle property, synchronization property.

Algebraic nature

Lemma

Let M, M', N, N' be such that $M - N \neq M' - N'$. Then there are at most countably many $\alpha \in [0, 1]$ such that $T_\alpha^M(\alpha - 1) = T_\alpha^N(\alpha)$ and $T_\alpha^{M'}(\alpha - 1) = T_\alpha^{N'}(\alpha)$.

Consequences:

- the matching index is well defined;
- matching intervals with different exponents do not intersect.

For more details on algebraic features: [Lan2019]

Matching index and monotonicity of entropy

Let J be a matching interval for T_α with matching exponents (M, N), then the entropy function $\alpha \mapsto h(T_\alpha)$

- is increasing if $N > M$;
- is decreasing if $N < M$;
- is constant if $N = M$.

[NN2008] for Nakada's CF
[Lan2019] for Ito-Tanaka CF

When matching fails

Bifurcation set = set of points which do not belong to any matching interval.

a.k.a. **exceptional set**, and usually denoted by the symbol $\mathcal{E}$.

Bifurcation set $\mathcal{E}_\mathcal{N}$ for Nakada's α-CF

Characterization of $\mathcal{E}_\mathcal{N}$ using Gauss map T_1:

$$\mathcal{E}_\mathcal{N} = \{x \in [0,1] \ : \ T_1^k(x) \geq x \quad \forall k \in \mathbb{N}\}$$

Consequences:

1. $\mathbb{Q} \cap \mathcal{E}_\mathcal{N} = \{0\}$;
2. $\text{meas}(\mathcal{E}_\mathcal{N}) = 0$;
3. $\dim_H(\mathcal{E}_\mathcal{N}) = 1$;
4. for all $t > 0$,

$$\dim_H(\mathcal{E}_\mathcal{N} \cap [0,t]) = 1, \quad \dim_H(\mathcal{E}_\mathcal{N} \cap [t,1]) < 1.$$

Relation with bounded type numbers

$$E_n := \{x \in [0,1] \ : \ x = [0; a_1, a_2, a_3, ...], \quad a_k \le n \ \ \forall k\}$$

Properties:

- $\dim_H(E_n) < 1 \ \forall n$;
- $\sup \dim_H(E_n) = 1$

For $a \in \mathbb{N}$ set $\phi_a(x) := \dfrac{1}{a+x}$; we have:

- $\mathcal{E}_{\mathcal{N}} \cap [\dfrac{1}{n+1}, 1] \subset E_n$;
- $\mathcal{E}_{\mathcal{N}} \cap [\frac{1}{a+1}, \frac{1}{a}] \supset \phi_a(E_{n-1})$ for all $a \in \mathbb{N}$, $a \ge n$.

Why "bifurcation set"?

Let us set $\mathcal{B}(t) := \{x \in [0,1] \ : T^k(x) \geq t\}$.

Note: $E_n = \mathcal{B}(\frac{1}{n+1})$

In [CT202*] it is proved that

- $\mathcal{E}_{\mathcal{N}}$ is the **bifurcation set** of the function $t \mapsto \mathcal{B}(t)$: in particular this function is constant outside $\mathcal{E}_{\mathcal{N}}$;
- $t \mapsto \dim_H(\mathcal{E}_{\mathcal{N}} \cap [t,1])$ is a continuous function and

$$\dim_H(\mathcal{E}_{\mathcal{N}} \cap [t,1]) = \dim_H(\mathcal{B}(t))$$

See [CT202*]: arXiv:1109.0516

Ito-Tanaka α-expansions

Ito-Tanaka case $T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]$ is defined by

$$T : x \mapsto \frac{1}{x} - c_\alpha(x), \qquad c_\alpha(x) = \lfloor \frac{1}{x} + 1 - \alpha \rfloor$$

$$x = [0; c_{\alpha,1}, c_{\alpha,2}, \ldots] \qquad c_{\alpha,k} = c_\alpha(T_\alpha^{k-1}(x))$$

$$\frac{p_{\alpha,n}}{q_{\alpha,n}} := [0; c_{\alpha,1}, c_{\alpha,2}, \ldots, c_{\alpha,n}]$$

Formally identical to the classical case, the only difference being that $c_{\alpha,n}, p_{\alpha,n}, q_{\alpha,n}$ need no more to be positive.
For "tipical" x

$$h_{\mu_\alpha}(T_\alpha) = \lim_{n\to+\infty} \frac{1}{n} \log q_{n,\alpha}^2$$

Ito-Tanaka α-CF: parameter reduction

The following diagram 'almost' commutes, with only countable many exceptions ($\tau(x) := -x$)

$$\begin{array}{rccc} T_\alpha : & [\alpha-1,\alpha) & \xrightarrow{T_\alpha} & [\alpha-1,\alpha) \\ & \downarrow \tau & & \downarrow \tau \\ T_{1-\alpha} : & [-\alpha,\alpha-1) & \xrightarrow{T_{1-\alpha}} & [-\alpha,\alpha-1) \end{array}$$

Therefore the two dynamical systems are conjugated.

WLOG: $\alpha \in [1/2, 1]$.

Unexplored region: $[g, 1]$
($g = \frac{\sqrt{5}-1}{2}$).

$\mathcal{E}_{IT} \cap [g, 1]$: characterization

Theorem

The set $\mathcal{E}_{IT} \cap [g, 1]$ can be characterized as

1. $\{\alpha \in [g, 1] : \; T_\alpha^n(\alpha - 1) \leq \frac{1}{\alpha+1}$ *and* $T_\alpha^n(\frac{1}{\alpha} - 1) \leq \frac{1}{\alpha+1} \;\; \forall n \geq 1\}$
2. $\left\{\alpha \in [g, 1] : \; T_g^n(\alpha - 1) \geq \alpha - 1\right.$ *and* $T_g^n(\frac{1}{\alpha} - 1) \geq \alpha - 1 \;\; \left.\forall \, n \geq 1\right\}$
3. $$\left\{\alpha \in [g, 1] : T_1^n(\alpha) \notin \left(\frac{1}{\alpha+1}, \alpha\right) \forall \, n \geq 2 \text{ and } T_1^n(\alpha) \notin \left(1 - \alpha, \frac{\alpha}{\alpha+1}\right) \forall \, n \geq 2 \text{ such that } P_n(\alpha) \text{ is odd}\right\},$$

   where
   $P_n(\alpha) = \min\{k \geq 0 : \; T_1^{n-k}(\alpha) \leq 1 - \alpha$ *or* $T_1^{n-k-1}(\alpha) \geq \alpha\}$.

Consequences: properties of $\mathcal{E}_{IT}$.

Theorem

We have that $\mathcal{E}_{IT}$ is a Lebesgue measure zero set and

$$\dim_H(\mathcal{E}_{IT}) = 1.$$

Moreover, for all $\delta > 0$

$$\dim_H\left(\mathcal{E}_{IT} \cap (g, g+\delta)\right) = 1 \quad \textit{and} \quad \dim_H\left(\mathcal{E}_{IT} \cap (g+\delta, 1)\right) < 1.$$

Unexpected features: neighbourhoods of $r_0 \in \mathcal{E}_{IT} \cap \mathbb{Q}$

Theorem

- *$\mathcal{E}_{IT}$ contains infinitely many rational values (a.k.a. bad rationals);*
- *the set of rational bifurcation parameters $\mathcal{E}_{IT} \cap \mathbb{Q}$ has no isolated points;*
- *for all $r \in \mathcal{E}_{IT} \cap \mathbb{Q}$ and for all $\delta > 0$ we have that*

$$\dim_H(\mathcal{E}_{IT} \cap (r - \delta, r + \delta)) > 1/2.$$

Bad rationals: $2/3$, the simplest example

$r_0 := 2/3 = [0; 1, 2]$ is accumulated (on both sides) by elements of $\mathcal{E}_{IT}$ (hence belongs to $\mathcal{E}_{IT}$).
The value $\alpha = [0; 1, 2, a_3, a_4, ...]$ is such that

- $\alpha > r_0$
- if $a_k \geq 4\ \forall k \geq 3$ then $\alpha \in \mathcal{E}_{IT}$

$$\frac{1}{\alpha + 1} = [0; 1, 1, 2, a_3, a_4, ...], \quad \alpha - 1 = -[0; 3, a_3, a_4, ...]$$

$$\forall k \geq 1 \quad \begin{cases} T_\alpha^k(\alpha) & = T_1^k(\alpha) < \frac{1}{\alpha+1} \\ T_\alpha^k(\alpha - 1) & = -T_1^k(\alpha) \end{cases}$$

Injecting high type numbers

$$H_n := \{x \in [0,1] \ : \ x = [0; a_1, a_2, a_3, ...], \quad a_k \geq n \ \ \forall k\}$$

$$\dim_H H_n > \frac{1}{2} \quad \forall n.$$

We can inject a lipschitz copy of H_4 in any neighbourhood of $2/3$.

An analogous result holds for all bad rationals (choosing a suitable H_n).

Mode locking ...

$$x_n := T_\alpha^n(\alpha - 1), \quad y_n := T_\alpha^n(\tfrac{1}{\alpha} - 1)$$

Lemma

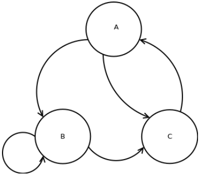
Let $\alpha \in [g, 1]$, $m \in \mathbb{N}$ be such that

$$x_n \le \frac{1}{\alpha+1} \quad \textit{and} \quad y_n \le \frac{1}{\alpha+1} \quad \textit{for all } 0 \le n < m. \qquad (1)$$

Then for all $0 \le n \le m$ the pair (x_n, y_n) satisfies one of the following relations:

$$\begin{array}{ll} \text{(A)} & (x_n + 1)(y_n + 1) = 1, \\ \text{(B)} & x_n + y_n = 0, \\ \text{(C)} & x_n + y_n = 1. \end{array}$$

If $x_m > \frac{1}{\alpha+1}$ or $y_m > \frac{1}{\alpha+1}$, then $x_m + y_m = 1$.



... leads to matching

$$\tilde{\mathcal{E}} := \{\alpha \in [g, 1] : \ x_n \le \tfrac{1}{\alpha+1} \text{ and } y_n \le \tfrac{1}{\alpha+1} \text{ for all } n \ge 1\}$$

Let $\alpha \in (g, 1]$, $m \in \mathbb{N}$ and assume that

$$x_n \le \tfrac{1}{\alpha+1} \quad \text{and} \quad y_n \le \tfrac{1}{\alpha+1} \quad \text{for all } 0 \le n < m. \tag{2}$$

but

$$x_m > \tfrac{1}{\alpha+1} \quad \text{or} \quad y_m > \tfrac{1}{\alpha+1} \tag{3}$$

Then α belongs to some matching interval J

Other directions

Matching pops up also in other families of CF algorithms

- Rosen α-continued fractions [DKS2009];
- Katok-Ugarcovici α-continued fractions ($S(x) = -1/x$): [KU2010] [KU2012] [CIT2018]
- α-continued fractions associated to distinct triangle Fuchsian groups [CKS2017]

Meta-question: why does this happen?

A simpler meta-question

For $\beta > 1$ a fixed value, let us define

$$T_\alpha : \begin{array}{ll} [0,1] \to & [0,1] \\ x \mapsto & T_\alpha(x) := \beta x + \alpha \pmod 1 \end{array}$$

In [BCK2017] it is shown that:

- if $\beta > 1$ is a quadratic irrational number, matching holds iff β is Pisot.
  In such a case matching intervals cover a.e. $\alpha \in [0,1]$
- if α belongs to a matching interval, the invariant density for T_α is constant on the complement of a finite set

$$\rho_\alpha(x) := \frac{d\mu(x)}{dx} = \sum_{T_\alpha^n(1) < x} \beta^{-n} - \sum_{T_\alpha^n(0) < x} \beta^{-n},$$

- if there is a matching interval for T_α then the slope β must be an algebraic integer.

How density changes

with the parameter α for $\beta = \frac{3+\sqrt{5}}{2}$

Show movie

... and yet not so simple!

- Which values of β give rise to matching intervals?
- For which values of β matching intervals cover almost all parameter space?

The end.

Some more open questions

- Using the techniques of [T] one can prove that the entropy is Hölder continuous also for the family (TI); it is natural to ask whether it is Lipschitz continuous.
- Is the entropy weakly decreasing on $[g, 1]$? We believe the answer is affirmative, but we still cannot rule out some devil staircase pathology.
- Can one characterize the isolated points of $\mathcal{E}$?
- Is there some countable chain of adjacent intervals as it was observed for the family (N)?
- Are there non isolated points of $\mathcal{E}_{IT}$ at which the local Hausdorff dimension of $\mathcal{E}_{IT}$ falls in the open interval $(0, 1/2)$?

α–continued fractions and the maps T_α

The maps $T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]$ are defined as follows:

$$T_\alpha(x) := \frac{1}{|x|} - c_\alpha(x), \quad c_\alpha(x) := \lfloor \frac{1}{|x|} + 1 - \alpha \rfloor.$$

Inverting the first equation above we get

$$x = \frac{\epsilon(x)}{c_\alpha(x) + T_\alpha(x)}, \quad \epsilon(x) = \mathrm{sign}(x)$$

Iterating this procedure we recover the infinite α-continued fractional expansion:

$$x = \cfrac{\epsilon_{1,\alpha}}{c_{1,\alpha} + \cfrac{\epsilon_{2,\alpha}}{c_{2,\alpha} + \ddots}}$$

which is sometimes written as

$$x = [0; (\epsilon_{\alpha,1}, c_{\alpha,1}), (\epsilon_{\alpha,2}, c_{\alpha,2}), (\epsilon_{\alpha,3}, c_{\alpha,3}), ...]$$