Coboundaries and eigenvalues of finitary S-adic shifts.

Valérie Berthé, **Paulina Cecchi Bernales*** and Reem Yassawi.

*Center for Mathematical Modeling–Universidad de Chile.

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Framework: shift spaces

- Let $\mathcal{A}$ be a finite set of symbols, which we call an alphabet.
- The set $\mathcal{A}^{\mathbb{Z}}$ with the product topology of the discrete topology on $\mathcal{A}$ is a Cantor space.
- The shift map on $\mathcal{A}^{\mathbb{Z}}$ is $T : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ given by

$$T((x_n)_{n\in\mathbb{Z}}) := (x_{n+1})_{n\in\mathbb{Z}}$$

- Shift space: topological dynamical system given by the pair (X, T), where X is a closed T-invariant subset of $\mathcal{A}^{\mathbb{Z}}$.
- For any T-invariant probability Borel measure μ on X, (X, T, μ) is a measure-theoretic dynamical system.

Framework: eigenvalues

- Let μ be an invariant probability Borel measure of a shift (X, T).
- A complex $\lambda \in \mathbb{C}$ is an eigenvalue of (X, T, μ) iff $\exists f \in \mathcal{L}^2(X, \mu)$, $f \neq 0$, such that $f \circ T = \lambda f$.
- The function f is called the eigenfunction associated to λ.
- A complex $\lambda \in \mathbb{C}$ is a continuous eigenvalue of (X, T) if $\exists f \in \mathcal{C}(X)$, $f \neq 0$, such that $f \circ T = \lambda f$.
- The set of all (continuous) eigenvalues of (X, T) is called the (continuous) spectrum of (X, T).
- For every $\mu \in \mathcal{M}(X, T)$, the continuous spectrum of (X, T) is included in the spectrum of (X, T, μ).

Questions about eigenvalues

- Let (X, T) be a shift space.
- How to decide if a given $\lambda \in \mathbb{C}$ is an eigenvalue?
- If $\lambda \in \mathbb{C}$ is an eigenvalue, how to decide if it admits a continuous eigenfunction?
- Complete answers have been given in the case of substitution shifts and linearly recurrent shifts.
- We want to address these questions in the case of finitary S-adic shifts.

Substitution shifts

- Let $\mathcal{A}, \mathcal{B}$ be alphabets. Consider a non-erasing morphism $\sigma : \mathcal{A}^\star \to \mathcal{A}^\star$, *i.e.*
  - $\sigma(ab) = \sigma(a)\sigma(b)$ for all $a, b \in \mathcal{A}$,
  - $\forall a \in \mathcal{A}$, $\sigma(a)$ is non-empty.

- When $\mathcal{A} = \mathcal{B}$, σ is called a substitution.

- A substitution σ extends to a map from $\mathcal{A}^\mathbb{Z}$ to $\mathcal{A}^\mathbb{Z}$ by concatenation.
- The language of σ is

$$\mathcal{L}_\sigma = \{w \in \mathcal{A}^\star : w \prec \sigma^n(a) \text{ some } n \in \mathbb{N}, \text{ some } a \in \mathcal{A}\}.$$

- The shift generated by σ is the pair (X_σ, T), where

$$X_\sigma := \{x \in \mathcal{A}^\mathbb{Z} : \forall w, w \prec x \Rightarrow w \in \mathcal{L}_\sigma\}.$$

- A substitution σ is primitive if there exists a positive integer k such that $\forall a, b \in \mathcal{A}$, b occurs in $\sigma^k(a)$.

Substitution shifts

- If σ is primitive, the shift (X_σ, T) is minimal and uniquely ergodic.
- We consider minimal and shift-aperiodic substitutions: every element in X_σ is T-aperiodic.
- Suppose there exist $a, b \in \mathcal{A}$ such that,
  - $\sigma(a)$ ends with a, $\sigma(b)$ starts with b,
  - $|\sigma^n(a)|, |\sigma^n(b)|$ tend to ∞,
  - $ab \in \mathcal{L}_\sigma$

  Then, there exists a periodic point for σ,

$$u = \sigma^\infty(a) \cdot \sigma^\infty(b)$$

- If σ is primitive, for any $k \geq 1$, $(X_{\sigma^k}, T) = (X_\sigma, T)$, so we can assume that σ admits a fix point u.
- One notes that $X_\sigma = \overline{\mathcal{O}_T(u)}$.

Generalization: S-adic shifts

- Let $(\mathcal{A}_n)_{n\in\mathbb{N}}$ be a sequence of (possibly different) alphabets, and $\sigma_n : \mathcal{A}_{n+1}^\star \to \mathcal{A}_n^\star$ a non-erasing morphism.
- Denote $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ and $\sigma_{[n,N)} := \sigma_n \circ \sigma_{n+1} \circ \cdots \circ \sigma_{N-1}$.
- For $n \geq 0$, the language of order n of $\boldsymbol{\sigma}$ is

$$\mathcal{L}_{\boldsymbol{\sigma}}^{(n)} = \{w \in \mathcal{A}_n^\star, \exists N > n, a \in \mathcal{A}_N, w \prec \sigma_{[n,N)}(a)\}.$$

- For $n \geq 0$, the shift space of order n of $\boldsymbol{\sigma}$ is

$$X_{\boldsymbol{\sigma}}^{(n)} = \{x \in \mathcal{A}_n^{\mathbb{Z}}, w \prec x \Rightarrow w \in \mathcal{L}_{\boldsymbol{\sigma}}^{(n)}\}.$$

- Set $X_{\boldsymbol{\sigma}} := X_{\boldsymbol{\sigma}}^{(0)} \subseteq \mathcal{A}_0^{\mathbb{Z}}$ and define $(X_{\boldsymbol{\sigma}}, T)$ the $\mathcal{S}$-adic shift space associated to the directive sequence $\boldsymbol{\sigma}$.

S-adic vs substitutive

Substitutive

$\mathcal{A}_n = \mathcal{A}\ \forall n$

$\sigma_n = \sigma\ \forall n$

$\mathcal{L}_\sigma^{(n)} = \mathcal{L}_\sigma\ \forall n$

$X_\sigma^{(n)} = X_\sigma\ \forall n$

Primitivity.

S-adic

$(\mathcal{A}_n)_{n\in\mathbb{N}}$

$\sigma_n : \mathcal{A}_{n+1}^\star \to \mathcal{A}_n^\star$

$(\mathcal{L}_{\boldsymbol{\sigma}}^{(n)})_{n\in\mathbb{N}}$.

$(X_{\boldsymbol{\sigma}}^{(n)})_{n\in\mathbb{N}}$

Weak or strong primitivity.

- Weak primitivity of $\boldsymbol{\sigma}$: for all $n \in \mathbb{N}$, $\exists N > n$ such that $\sigma_{[n,N)}$ is primitive. This implies that every $(X_{\boldsymbol{\sigma}}^{(n)}, T)$ is minimal.
- Strong primitivity of $\boldsymbol{\sigma}$: there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $\sigma_{[n,n+N)}$ is primitive. If there exists S finite such that $\sigma_n \in S \quad \forall n$, this implies that $(X_{\boldsymbol{\sigma}}, T)$ is minimal and uniquely ergodic.

S-adic shifts

- Every substitutive shift is a stationary S-adic one and in the substitutive case weak primitivity=strong primitivity.
- Every minimal shift is S-adic.

Linearly recurrent shifts

- A shift space (X,T) is linearly recurrent (LR) if there exists a constant K such that $\forall w \in \mathcal{L}_X$, the length of any return word to w is bounded by $K|w|$.
- Every primitive substitution σ satisfies that (X_σ, T) is LR.
- LR shift spaces are minimal and uniquely ergodic.
- Necessary and sufficient conditions to be a measurable or continuous eigenvalue of a LR shift have been given in [Bressaud–Durand–Maass 05].
- A subshift (X,T) is LR if and only if it is a strongly primitive and proper S-adic shift [Durand 03].

Linearly recurrent shifts

- Non-uniquely ergodic systems cannot be LR.
- There exist minimal, uniquely ergodic, non-LR S-adic shifts.
- Ex.: Consider $\mathcal{A} = \{a, b, c\}$ and $S = \{\sigma, \tau\}$, where

$$\begin{array}{cc} a \overset{\sigma}{\mapsto} acb & a \overset{\tau}{\mapsto} abc \\ b \overset{\sigma}{\mapsto} bab & b \overset{\tau}{\mapsto} acb \\ c \overset{\sigma}{\mapsto} cbc & c \overset{\tau}{\mapsto} aac \end{array}$$

- Consider the directive sequence $\boldsymbol{\sigma} = \sigma, \tau, \sigma, \sigma, \tau, \sigma, \sigma, \sigma, \cdots$
- The sequence $\boldsymbol{\sigma}$ is strongly primitive, which implies $(X_{\boldsymbol{\sigma}}, T)$ is minimal and uniquely ergodic, but is not linearly recurrent [Durand 03].

Eigenvalues of substitutive shifts

- How to characterize measurable and continuous eigenvalues? How to distinguish them?
- A complete answer has been given for substitutive subshifts.

Theorem (Host 86)

Let σ be a primitive, recognizable substitution on $\mathcal{A}$. Let (X_σ, T) be the associated shift. Let μ be the unique T-invariant probability measure on X_σ. Then,

- *A complex number $\lambda \in \mathbb{S}^1$ is an eigenvalue of (X_σ, T, μ) if and only if there exists a positive integer p such that for all $a \in \mathcal{A}$, the limit*

$$h(a) := \lim_{n \to \infty} \lambda^{|\sigma^{pn}(a)|}$$

  exists and defines a coboundary of σ.
- *Every eigenvalue of (X_σ, T, μ) is continuous.*

Eigenvalues of substitutive shifts

- The original statement of the previous theorem is for the one-sided shift,

$$\tilde{X}_\sigma := \{x \in \mathcal{A}^{\mathbb{N}} : \forall w, w \prec x \Rightarrow \exists a \in \mathcal{A}, \exists n \in \mathbb{N} : w \prec \sigma^n(a)\}.$$

Proposition

Let $(\tilde{X}, T)$ be a minimal one-sided shift, and let (X, T) be its natural extension. Let $\lambda \in \mathbb{S}^1$. Then,

- *λ is a continuous eigenvalue of $(\tilde{X}, T)$ if and only if it is a continuous eigenvalue of (X, T).*
- *Let $\tilde{\mu}$ be an invariant probability Borel measure of $(\tilde{X}, T)$ and let μ be the corresponding measure on (X, T). Then, λ is an eigenvalue of $(\tilde{X}, T, \tilde{\mu})$ if and only if it is an eigenvalue of (X, T, μ).*

Recognizability

- A substitution σ on $\mathcal{A}$ is recognizable if for all $y \in X_\sigma$, there exists a unique pair $(k,x) \in \mathbb{N} \times X_\sigma$ such that $y = T^k\sigma(x)$ and $0 \leq k < |\sigma(x_0)|$.
- Such a pair is called a centered σ-representation of y.
- If σ is primitive and aperiodic, then it is recognizable in X_σ [Mossé 96].
- If σ is aperiodic, then it is recognizable in X_σ [Bezuglyi–Kwiatkowski–Medynets 09].
- Any substitution on $\mathcal{A}$ is recognizable at aperiodic points [Berthé–Steiner–Thuswaldner–Yassawi 19].

Recognizability

- Let σ be a substitution. Consider the following sequences of clopen covers of X_σ,

$$\mathcal{P}_n = \{T^j \sigma^n([a]), a \in \mathcal{A}, 0 \leq j < |\sigma^n(a)|\},$$

where, for each $a \in \mathcal{A}$, $[a] = \{x \in X_\sigma : x_0 = a\}$.

- If σ is recognizable, $(\mathcal{P}_n)_{n\in\mathbb{N}}$ is a nested sequence of clopen partitions of X_σ.

- If σ is primitive and recognizable and μ is an invariant probability measure on X_σ, then

$$\mu(\bigcap_{n\geq 0} \bigcup_{a\in\mathcal{A}} \sigma^n([a])) = 0$$

and $(\mathcal{P}_n)_{n\geq 0}$ is generating in measure [Berthé–Steiner–Thuswaldner–Yassawi 19].

Coboundaries

- Let (X, T) be a minimal shift space, let $w, w' \in \mathcal{L}_X$. A word $u \in \mathcal{L}_X$ is a transition word from w to w' if u starts with w, $uw' \in \mathcal{L}_X$ and uw' has exactly two occurrences of each w and w'.
- When $w = w'$ we say that u is a return word to w.
- Minimality implies that for all $w \in \mathcal{L}_X$, the following set is finite.

$$\mathcal{R}_w = \{u \in \mathcal{L}_X : u \text{ is a return word to } w\}$$

- Let σ be a primitive substitution. A map $h : \mathcal{L}_X \to \mathbb{S}^1$ is a coboundary if it is a morphism and for every $a \in \mathcal{A}$, $h(w) = 1$ whenever $w \in \mathcal{R}_a$.

Proposition (Host 86)

Let σ be a primitive substitution on $\mathcal{A}$. The map $h : \mathcal{A} \to \mathbb{S}^1$ is a coboundary if and only if there exists a function $f : \mathcal{A} \to \mathbb{S}^1$ verifying $f(b) = f(a)h(a)$ whenever ab belongs to $\mathcal{L}_\sigma$.

S-adic recognizability

- Let $\boldsymbol{\sigma} = (\sigma_n : \mathcal{A}_{n+1}^\star \to \mathcal{A}_n^\star)_{n\in\mathbb{N}}$ be a directive sequence. It is said to be recognizable at level n if σ_n is recognizable in $X_{\boldsymbol{\sigma}}^{(n+1)}$: for each $y \in X_{\boldsymbol{\sigma}}^{(n)}$ there exists a unique pair $(k, x) \in \mathbb{N} \times X_{\boldsymbol{\sigma}}^{(n+1)}$ such that $y = T^k\sigma_n(x)$ and $0 \leq k < |\sigma_n(x_0)|$.

- It is recognizable if it is recognizable at each level n.

- Consider the one-sided shift associated to each $X_{\boldsymbol{\sigma}}^{(n)}$,

$$\tilde{X}_{\boldsymbol{\sigma}}^{(n)} = \{x_{[0,\infty)} \in \mathcal{A}_n^{\mathbb{N}} : x \in X_{\boldsymbol{\sigma}}^{(n)}\}.$$

- The directive sequence $\boldsymbol{\sigma}$ is said to be one-sided recognizable if for each n, each $y \in \tilde{X}_{\boldsymbol{\sigma}}^{(n)}$ there exists a unique pair $(k, x) \in \mathbb{N} \times \tilde{X}_{\boldsymbol{\sigma}}^{(n+1)}$ such that $y = T^k\sigma_n(x)$ and $0 \leq k < |\sigma_n(x_0)|$.

S-adic recognizability

Theorem (Berthé–Steiner–Thuswaldner–Yassawi 19)

Let $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ be a primitive sequence of morphisms with $\sigma_n : \mathcal{A}_{n+1}^{\star} \to \mathcal{A}_n^{\star}$. If each σ_n satisfies at least one of the following conditions,

- *rk*$(M_{\sigma_n}) = |\mathcal{A}_{n+1}|$,
- $|\mathcal{A}_{n+1}| = 2$,
- σ_n *is (rotationally conjugate) to a left or right permutative morphism,*

then, $\boldsymbol{\sigma}$ is recognizable.

- M_{σ_n}: incidence matrix of σ_n.
- Left- (right-) permutative: $\forall a, b \in \mathcal{A}$, $a \neq b \Rightarrow \sigma_n(a)_0 = \sigma_n(b)_0$ $(\sigma_n(a)_{|\sigma_n(a)|-1} = \sigma_n(b)_{|\sigma_n(n)|-1})$.
- Ex.: unimodular incidence matrices.

S-adic coboundaries

- From now on, every morphism σ_n is defined on the same alphabet: $\mathcal{A}_n = \mathcal{A}$ for all $n \in \mathbb{N}$.
- We say that $x \in \tilde{X}_{\boldsymbol{\sigma}}$ is a limit word if $x = \lim_{k\to\infty} \sigma_{[0,n_k)}(a)$ for some $a \in \mathcal{A}$, some sequence $(n_k)_{k\in\mathbb{N}}$.
- If $\boldsymbol{\sigma}$ is everywhere growing, then compactness guarantees the existence of limit words.
- For each letter $a \in \mathcal{A}$, define

$$\boldsymbol{a} := \{x \in \tilde{X}_{\boldsymbol{\sigma}}, \exists (n_k)_{k\in\mathbb{N}}, \forall k \in \mathbb{N}, x \in \sigma_{[0,n_k)}([a])\}.$$

- If $\boldsymbol{\sigma}$ is recognizable, each $\boldsymbol{a}$ is finite.
- The directive sequence is prefix-straight if for each letter $a \in \mathcal{A}$, $\boldsymbol{a}$ consists of a unique element, which we denote $\boldsymbol{a}$ as well.

S-adic coboundaries

- If every σ_n in $\boldsymbol{\sigma}$ is left-proper, then $\boldsymbol{\sigma}$ is prefix-straight.
- Recall that, in the substitutive case, $h : \mathcal{A} \to \mathbb{S}^1$ is a coboundary if and only if there exists a map $f : \mathcal{A} \to \mathbb{S}^1$ verifying $f(b) = f(a)h(a)$ whenever ab belongs to $\mathcal{L}_\sigma$.
- Note that $ab \in \mathcal{L}_\sigma$ is a transition word from a to b.
- This motivates the following definition: let $\boldsymbol{\sigma}$ be a prefix-straight directive sequence. A map $h : \mathcal{A} \to \mathbb{S}^1$ is a coboundary of $\boldsymbol{\sigma}$ if it is a morphism and there exists a map $f \mathcal{A} \to \mathbb{S}^1$ such that
  - $f(\boldsymbol{a}) = f(\boldsymbol{a'})$ whenever $\boldsymbol{a} = \boldsymbol{a'}$.
  - $f(b) = f(a) \lim_{k\to\infty} h(w_{n_k})$ whenever (w_{n_k}) is a sequence of transition words from a to b.

Coboundaries and continuous eigenvalues

Theorem (Berthé–CB–Yassawi 22)

Let $\boldsymbol{\sigma}$ be a prefix-straight, primitive directive sequence on $\mathcal{A}$. Suppose that for each $a \in \mathcal{A}$ there exists $\ell \in \mathcal{A}$ such that $a\ell$ is a fully essential word. If $\lambda \in \mathbb{S}^1$ is a continuous eigenvalue of $(X_{\boldsymbol{\sigma}}, T)$, then

$$h(a) := \lim_{n \to \infty} \lambda^{|\sigma_{[0,n)}(a)|}$$

exists and defines a coboundary.

- A word $w \in \mathcal{A}^\star$ is essential for $\boldsymbol{\sigma}$ if it occurs in $\mathcal{L}_{\boldsymbol{\sigma}}^{(n)}$ for infinitely many n.
- It is fully essential if it occurs in $\mathcal{L}_{\boldsymbol{\sigma}}^{(n)}$ for each n.

Coboundaries and continuous eigenvalues

Theorem (Berthé–CB–Yassawi 22)

Let $\boldsymbol{\sigma}$ be a prefix-straight, one-sided recognizable, primitive directive sequence on $\mathcal{A}$. Suppose that each σ_n appears infinitely often in $\boldsymbol{\sigma}$. Let $\lambda \in \mathbb{S}^1$ be a rational. If

$$h(a) := \lim_{n\to\infty} \lambda^{|\sigma_{[0,n)}(a)|}$$

exists and is a constant coboundary, then λ is a continuous eigenvalue of $(X_{\boldsymbol{\sigma}}, T)$.

Theorem (Berthé–CB–Yassawi 22)

Let $\boldsymbol{\sigma}$ be a finitary, prefix-straight, one-sided recognizable, primitive directive sequence on $\mathcal{A}$. Suppose that each σ_n appears infinitely often in $\boldsymbol{\sigma}$. Let $\lambda \in \mathbb{S}^1$. If h is a coboundary of $\boldsymbol{\sigma}$ and

$$\sum_{n\geq 1} |\lambda^{|\sigma_{[0,n)}(a)|} - h(a)| < \infty$$

for each $a \in \mathcal{A}$, then λ is a continuous eigenvalue of $(X_{\boldsymbol{\sigma}}, T)$.

Speed of convergence

- If σ is a primitive recognizable subtitution, we have the following.

Lemma (Host 86)

Let $\lambda \in \mathbb{S}^1$. If $h(a) = \lim_{n\to\infty}$ exists for each $a \in \mathcal{A}$, then there exist $C > 0$ and $0 < r < 1$ such that

$$|\lambda^{|\sigma^n(a)|} - h(a)| < Cr^n.$$

- This property is used in the proof of

$\lambda^{|\sigma^n(a)|}$ converges and defines a coboundary for each a

$\implies$ λ is a continuous eigenvalue.

- This is why we need a *fast convergence type* condition in the previous theorem.

The constant-length finitary S-adic case

- Let $\boldsymbol{\sigma}$ be a constant-length directive sequence on $\mathcal{A}$: for each $n \in \mathbb{N}$, σ_n is a constant-length substitution. Let $(q_n)_{n\in\mathbb{N}}$ be the sequence of lengths.
- Concerning one-sided recognizability, we have the following,

Lemma

Let $\boldsymbol{\sigma}$ be a primitive, constant-length directive sequence on $\mathcal{A}$, with length sequence $(q_n)_{n\in\mathbb{N}}$. Suppose that each σ_n is right-permutative. If $\boldsymbol{\sigma}$ is recognizable, then it is one-sided recognizable.

- The directive sequence is finitary if there is a finite set $\mathcal{S}$ such that $\sigma_n \in \mathcal{S}$ for each n.

The constant-length finitary S-adic case

Theorem (Berthé–CB–Yassawi 22)

Let $\boldsymbol{\sigma}$ be a primitive, constant-length directive sequence on $\mathcal{A}$, with length sequence $(q_n)_{n\geq 0}$, where each $q_n \geq 2$. Suppose that $\boldsymbol{\sigma}$ has a fully essential word of length 2. If $\lambda \in \mathbb{S}^1$ is a continuous eigenvalue of $(X_{\boldsymbol{\sigma}}, T)$, then $h := \lim_{n\to\infty} \lambda^{q_0\cdots q_n}$ exists and defines a constant coboundary.
If in addition $\boldsymbol{\sigma}$ is finitary and prefix-straight and one-sided recognizable, then λ is a continuous eigenvalue if and only if $h := \lim_{n\to\infty} \lambda^{q_0\cdots q_n}$ exists and defines a coboundary.
Moreover, each continuous eigenvalue λ is rational, and there exists $\tilde{h} \in \mathbb{N}$ such that

- *$\tilde{h}$ is coprime to each q_n,*
- *$\tilde{h}$ divides $p := \prod_{q\in\{q_n : n\geq 0\}} (q-1)$, where the product is over the set of distinct values of elements in $\{q_n : n \geq 0\}$,*

and the maximal equicontinuous factor of $(X_{\boldsymbol{\sigma}}, T)$ is $(\mathbb{Z}_{\tilde{h},(q_n)}, +1)$.

The constant-length finitary S-adic case

Corollary (Cobham's type result)

Let $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ be two finitary, primitive, constant-length directive sequences on $\mathcal{A}$, with length sequences $(q_n)_{n\geq 0}$ and $(\tilde{q}_n)_{n\geq 0}$, where $q_n, \tilde{q}_n \geq 2$ for each n.
Suppose that each length occurs infinitely often and each directive sequence possesses a fully essential word of length 2.
Suppose that $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ are prefix-straight and one-sided recognizable.
If there is a prime factor of some $q \in \{q_n : n \geq 0\}$ that is not a prime factor of any $\tilde{q} \in \{\tilde{q}_n : n \geq 0\}$, and if $(X_{\boldsymbol{\sigma}}, T)$ and $(X_{\tilde{\boldsymbol{\sigma}}}, T)$ have the same maximal equicontinuous factor, then they are both finite.

- In particular, if $(X_{\boldsymbol{\sigma}}, T)$ is infinite, then it cannot be both $\boldsymbol{\sigma}$-adic and $\tilde{\boldsymbol{\sigma}}$-adic.

Measurable eigenvalues

- Let $\boldsymbol{\sigma}$ be a recognizable directive sequence on $\mathcal{A}$. For each $a \in \mathcal{A}$ and each n, let a_n be the first letter of $\sigma_n(a)$. Suppose that for each $a \in \mathcal{A}$ and each n, $\sigma_n(a_{n+1})$ starts with a_n. Then we say that $\boldsymbol{\sigma}$ is strongly prefix-straight.

Theorem (Berthé–CB–Yassawi 22)

Let $\boldsymbol{\sigma}$ be a finitary, recognizable, strongly prefix-straight and strongly primitive directive sequence on $\mathcal{A}$. Let mu be its unique invariant measure. If $\lambda \in \mathbb{S}^1$ is a measurable eigenvalue of $(X_{\boldsymbol{\sigma}}, T, \mu)$, then

$$\sum_{n=1}^{\infty} |\lambda^{h_n(w_n(a,b))} - 1|^2 < \infty$$

for any a, b such that $\boldsymbol{a} = \boldsymbol{b}$ and any transition word form a to b $w_n(a,b) \in \mathcal{L}_{\boldsymbol{\sigma}}^{(n)}$.

Measurable eigenvalues

- In the case of constant-length directive sequences, we can avoid the condition on strong prefix-straightness.

Theorem (Berthé–CB–Yassawi)

Let $\boldsymbol{\sigma}$ be a finitary, one-sided recognizable, strongly primitive constant-length directive sequence on $\mathcal{A}$, with length sequence $(q_n)_{n\geq 0}$. Suppose that there is a unique right-infinite limit word $\boldsymbol{a}$, and $\sigma_n(a)$ starts with a for each n. If $\lambda \in S^1$ is a measurable eigenvalue of $(X_{\boldsymbol{\sigma}}, T, \mu)$, then $\sum_{n=1}^{\infty} |\lambda^{q_0 \cdots q_n} - 1|^2 < \infty$.

Corollary

Let $\boldsymbol{\sigma}$ be a finitary, recognizable and strongly primitive constant-length directive sequence on $\mathcal{A}$. Suppose that either

- *$\boldsymbol{\sigma}$ is strongly prefix-straight, or*
- *there is a unique right-infinite limit word $\boldsymbol{a}$, and such that $\sigma_n(a)$ starts with a for each n.*

Then every measurable eigenvalue is continuous.